Substitution? Which one?

Substitution technique

1. Antiderivatives: Set $u = u(x)$ (and so $du = u'(x)dx$) to get $\displaystyle\int f(u(x))u'(x)dx = \int f(u)du$. This is useful when $\int f(u)du$ is easy to find.

2. Integrals: Suppose $u(x)$ has a continuous derivative on $[a, b]$ and $f(u)$ is continuous on the set $u([a, b])$. Set $u = u(x)$ to get

$$\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(u)du.$$

3. Common consideration for selecting u

(3.1) For $\displaystyle\int f(ax^{n+1} + b)x^n dx$, try $u = ax^{n+1} + b$.

Example 1: Compute $\displaystyle\int x(x^2 - 3)^4 dx$.

Solution: Note that the integral has the form $\int f(x^2 - 3)xdx$ where $f(u) = u^4$. Choose $u = x^2 - 3$. Then $du = 2xdx$ or $\frac{1}{2}du = xdx$. Therefore,

$$\int x(x^2 - 3)^4 dx = \int \frac{1}{2}u^4 du = \frac{u^5}{10} + C = \frac{(x^2 - 3)^5}{10} + C.$$

Example 2: Compute $\displaystyle\int_0^2 x\sqrt{x^2 + 1}dx$.

Solution: Note that the integral has the form $\int f(x^2 + 1)xdx$ where $f(u) = u^{\frac{1}{2}}$. Choose $u = x^2 + 1$. Then $du = 2xdx$ or $\frac{1}{2}du = xdx$. Moreover, $u(0) = 1$ and $u(1) = 2$. Therefore,

$$\int_0^2 x\sqrt{x^2 + 1}dx = \int_1^2 \frac{1}{2} \cdot u^{\frac{1}{2}} du = \left[\frac{u^{\frac{3}{2}}}{3}\right]_1^2 = \left[\frac{2^{\frac{3}{2}}}{3} - \frac{1}{3}\right] = \frac{2^{\frac{3}{2}} - 1}{3}$$

(3.2) For $\displaystyle\int f(\sin x)\cos x dx$, try $u = \sin x$; for $\displaystyle\int f(\cos x)\sin x dx$, try $u = \cos x$.

Example 3: Compute $\displaystyle\int \cos x\sqrt{\sin x + 1}dx$.

Solution: Note that the integral has the form $\int \cos x f(\sin x)dx$ where $f(u) = \sqrt{u + 1}$. Choose $u = \sin x + 1$. Then $du = \cos x dx$. Therefore,

$$\int \cos x\sqrt{\sin x + 1}dx = \int \sqrt{u}du = \frac{2u^{\frac{3}{2}}}{3} + C = \frac{2(\sin x + 1)^{\frac{3}{2}}}{3} + C.$$

Example 4: Compute $\displaystyle\int \frac{\sin x}{\sqrt{\cos x}}dx$.

Solution: Note that the integral has the form $\int f(\cos x)\sin x dx$ where $f(u) = u^{-\frac{1}{2}}$. Choose $u = \cos x$. Then $du = -\sin x dx$ or $-du = \sin x dx$. Therefore,

$$\int \frac{\sin x}{\sqrt{\cos x}} dx = -\int u^{-\frac{1}{2}} du = -2u^{\frac{1}{2}} + C = -2(\cos x)^{\frac{1}{2}} + C.$$

(3.3) For $\displaystyle\int f(\tan x)\sec^2 x dx$, try $u = \tan x$; for $\displaystyle\int f(\sec x)\sec x tan x dx$, try $u = \sec x$. (The corresponding cofunctions can be teated similarly).

Example 5: Compute $\displaystyle\int \sec^2 x \cos(\tan x) dx$.

Solution: Note that the integral has the form $\int f(\tan x) sec^2 x dx$ where $f(u) = \cos u$. Choose $u = tan x$. Then $du = sec^2 x dx$. Therefore,

$$\int \sec^2 x \cos(\tan x) dx = \int \cos u du = \sin u + C = \sin(\tan x) + C.$$

(3.4) For $\displaystyle\int f(\ln x)\frac{1}{x} dx$, try $u = \ln x$; for $\displaystyle\int \frac{1}{\sqrt{x}} f(\sqrt{x}) dx$, try $u = \sqrt{x}$.

Example 6: Compute $\displaystyle\int \frac{(\ln x + 2)^2}{x} dx$.

Solution: Note that the integral has the form $\displaystyle\int f(\ln x + 2)\frac{1}{x} dx$, where $f(u) = u^2$. Choose $u = \ln x + 2$. Then $du = \frac{1}{x} dx$. Therefore,

$$\int \frac{(\ln x + 2)^2}{x} dx = \int u^2 du = \frac{u^3}{3} + C = \frac{(\ln x + 2)^3}{3} + C.$$

(3.4) Other examples of substitutions:

Example 7: Compute $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$.

Solution: Note that $(e^x + e^{-x})' = e^x - e^{-x}$. Choose $u = e^x + e^{-x}$. Then $du = e^x - e^{-x} dx$. Therefore,

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \int \frac{1}{u} du = \ln|u| + C = \ln|e^x + e^{-x}| + C.$$

Example 8: Compute $\displaystyle\int \frac{x^2}{\sqrt[3]{x+3}} dx$.

Solution: Choose $u = x + 3$. Then $x = u - 3$ and so $du = dx$ and $x^2 = (u-3)^2 = u^2 - 6u + 9$. Therefore,

$$\int \frac{x^2}{\sqrt[3]{x+3}} dx = \int \frac{u^2 - 6u + 9}{u^{\frac{1}{3}}} du = \int (u^{\frac{5}{3}} - 6u^{\frac{2}{3}} + 9u^{-\frac{1}{3}}) du = \frac{3(x+3)^{\frac{7}{3}}}{7} - 6 \cdot \frac{3(x+3)^{\frac{5}{3}}}{5} + 9 \cdot \frac{3(x+3)^{\frac{2}{3}}}{2} + C.$$

Example 9: Compute $\int_0^{\frac{\pi}{4}} \sec x dx$.

Solution: Note that $\sec x' = \sec x \tan x$ and $\tan x' = \sec^2 x$. Choose $u = \tan x + \sec x$. Then $du = \sec x(\sec x + \tan x)dx$. Moreover, $u(0) = \tan 0 + \sec 0 = 1$ and $u(\frac{\pi}{4}) = \tan(\frac{\pi}{4}) + \sec(\frac{\pi}{4}) = 1 + \sqrt{2}$. Therefore,

$$\int_0^{\frac{\pi}{4}} \sec x dx = \int_0^{\frac{\pi}{4}} \frac{\sec x(\sec x + \tan x)}{\tan x + \sec x} dx = \int_1^{1+\sqrt{2}} \frac{1}{u} du = [\ln |u|]_1^{1+\sqrt{2}} = \ln(1+\sqrt{2}) = \ln 1 = \ln(1+\sqrt{2}).$$