Compute derivatives of trigonometric functions

Facts: Let $u = u(x)$ be a differentiable function.

$$\frac{d}{dx}\sin u(x) = \cos u(x)\frac{du}{dx} \qquad \frac{d}{dx}\cos u(x) = -\sin u(x)\frac{du}{dx}$$
$$\frac{d}{dx}\tan u(x) = \sec^2 u(x)\frac{du}{dx} \qquad \frac{d}{dx}\cot u(x) = -\csc^2 u(x)\frac{du}{dx}$$
$$\frac{d}{dx}\sec u(x) = \sec u(x)\tan u(x)\frac{du}{dx} \qquad \frac{d}{dx}\csc u(x) = -\csc u(x)\cot u(x)\frac{du}{dx}$$

Example 1 Given $f(x) = \cos(5x)\sin(7x)$, find $f'(x)$.

Solution: Apply product rule, and then use the differentiation formulas for sine and cosine functions.

$$f'(x) = \underline{\frac{d\cos(5x)}{dx}}\sin(7x) + \cos(5x)\underline{\frac{d\sin(7x)}{dx}} = -5\sin(5x)\sin(7x) + 7\cos(5x)\cos(7x).$$

Example 2 Given $f(x) = (\tan x)^7$, find $f'(x)$.

Solution: Apply generalized power rule, and then use the differentiation formula for $\tan x$.

$$f'(x) = 7(\tan x)^{7-1}\frac{d\tan x}{dx} = 7(\tan x)^6\sec^2 x.$$

Example 3 Given $f(x) = \sec x \sin x$, find $f'(x)$.

Solution: Apply product rule, and then use the differentiation formulas for sine and secant functions.

$$f'(x) = \underline{\sec x\tan x}\sin x + \sec x\underline{\cos x}.$$

Example 4 Given $f(x) = \sec(\sin x)$, find $f'(x)$.

Solution: Compare this with Example 3. This is a composition function, not a product. Therefore, we apply Chain Rule to view $f(x) = \sec u(x)$ with $u = \sin x$.

$$f'(x) = \sec u(x)\tan u(x)\frac{du(x)}{dx} = \sec(\sin x)\tan(\sin x)\cos x.$$

Example 5 Given $f(x) = x\cos x$, find an equation of the line tangent to the curve $y = f(x)$ at the point where $x = \pi$.

Solution: Note that $f(\pi) = \pi \cos \pi = -\pi$. Thus the line passes through $(\pi, -\pi)$. To find the slope of the line, we compute $f'(x)$ using product rule.

$$f'(x) = \cos x - x \sin x.$$

Thus the slope $m = f'(\pi) = \cos \pi - \pi \sin \pi = -1$, and so the answer is

$$y - (-\pi) = (-1)(x - \pi) \text{ or } x + y = 0.$$

Example 6 Given $f(x) = x - 2\sin x$, find all points on the curve $y = f(x)$ where the tangent line is horizontal.

Solution: This amounts to find the points at which $f'(x) = 0$. Note that

$$f'(x) = 1 - 2\cos x.$$

Thus $f'(x) = 0$ if and only if $\cos x = \frac{1}{2}$, which means that $x = \pm\frac{\pi}{3} + \pm 3n\pi$, for any integer n. Thus the points at which $y = f(x)$ has horizontal tangent line are $(\pm\frac{\pi}{3} + \pm 2n\pi, f(\pm\frac{\pi}{3} + \pm 2n\pi))$, for any integer n. (There are infinitely many of such places).

Example 7 A water trough is to be made from a long strip of tin 6 ft wide by bending up at an angle θ a 2-ft strip on each side. What angle θ would maximize the cross section area, and thus the volume, of the trough?

Solution: We first express the area in turns of the angle θ. We notice that the cross section is a trapezoid. From geometry, we know that the area of this trapezoid is

$$A(\theta) = \frac{2 + (2 + 4\cos\theta)}{2}(2\sin\theta) = 4(1 + \cos\theta)\sin\theta.$$

As θ ranges from 0 to $\frac{\pi}{2}$, we are maximizing $A(\theta)$ over the interval $[0, \frac{\pi}{2}]$. Note that $\sin^2\theta + \cos^2\theta = 1$, or $-\sin^2\theta = \cos^2\theta - 1$. We have

$$A'(\theta) = 4[-\sin^2\theta + (1 + \cos\theta)\cos\theta] = 4(\cos\theta + 2\cos^2\theta - 1).$$

To solve the equation $\cos\theta + 2\cos^2\theta - 1 = 0$, we first factor the left side to get $(2\cos\theta - 1)(\cos theta + 1) = 0$. Therefore, either $\cos\theta = \frac{1}{2}$, whence in $[0, \frac{\pi}{2}]$, $\theta = \frac{\pi}{3}$; or $\cos\theta = 1$, whence in $[0, \frac{\pi}{2}]$, $\theta = 0$.

Computing the corresponding values of A, we have $A(0) = 0$, $A(\frac{\pi}{3}) = 4(1+\frac{1}{2})\frac{\sqrt{3}}{2} = 3\sqrt{3}$ and $A(\frac{\pi}{2}) = 4$. Since $\sqrt{3} > 1.7$, we conclude that $3\sqrt{3} > 4$, and so $\theta = \frac{\pi}{3}$ maximizes the cross sectional area.