Find maximum and minimum values of a function over a closed interval

Facts: Let $f(x)$ be a function on $[a, b]$ and c is a point in the interval $[a, b]$.

(1) If for any point x in $[a, b]$, $f(x) \geq f(c)$ (respectively, $f(x) \leq f(c)$), then $f(c)$ is the **absolute (or global) minimum value** (respectively, **absolute (or global) local maximum value**) of $f(x)$ on $[a, b]$.

(2) If $a < c < b$, and for any point x in an open interval containing c, $f(x) \geq f(c)$ (respectively, $f(x) \leq f(c)$), then $f(c)$ is a **local minimum value** of $f(x)$ (respectively, **local maximum value**) on $[a, b]$.

(3) If $f(x)$ is continuous on $[a, b]$ and differentiable in (a, b), a point c in $[a, b]$ is a **critical point** of $f(x)$ if either $f'(c)$ does not exist, or $f'(x) = 0$.

(4) **Important**: If $f(x)$ is continuous on $[a, b]$ and differentiable in (a, b), and if for some c in (a, b), $f(c)$ is a local maximum or local minimum, then c must be a critical point. Any absolute maximum or minimum must take place at critical points inside the interval or at the boundaries point a or b.

Example 1 State whether the function $f(x) = |x - 2|$ attains a maximum value or a minimum value in the interval $(1, 4]$.

Solution: Apply the definition of absolute value to get

$$f(x) = \begin{cases} x - 2 & \text{if } 2 \leq x \leq 4, \\ 2 - x & \text{if } 1 < x < 2. \end{cases}$$

Thus the graph of this function consists of two pieces of lines, and so the minimum value $f(2) = 0$ @ $x = 2$, and the maximum value is $f(4) = 2$ @ $x = 4$.

Example 2 Find the maximum value and the minimum value attained by $f(x) = \dfrac{1}{x(1-x)}$ in the interval $[2, 3]$.

Solution: Note that the domain of $f(x)$ does not contain $x = 0$ and $x = 1$, and these points are not in the interval $[2, 3]$.

(Step 1) Find critical points. Compute

$$f'(x) = -\frac{1 - 2x}{x^2(1-x)^2} = \frac{2x - 1}{x^2(1-x)^2}.$$

Therefore, the only possible critical point is $x = \frac{1}{2}$. As this point is not in the interval $[2, 3]$, it is not a critical point.
(Step 2) Compute $f(x)$ at the critical point(s) and at the boundaries of the closed interval.

$$\begin{aligned} f(2) &= \frac{1}{2(1-2)} = -\frac{1}{2}, \\ f(3) &= \frac{1}{3(1-3)} = -\frac{1}{6}. \end{aligned}$$

(Step 3) Compare the data resulted in Step 2 to make conclusions.

$f(x)$ attains its absolute maximum value $f(3) = -\frac{1}{6}$ @ $x = 3$ and $f(x)$ attains its absolute minimum value $f(2) = -\frac{1}{2}$ @ $x = 2$.

Example 3 Find the maximum value and the minimum value attained by $f(x) = x^2 + frac16x$ in the interval $[1, 3]$.

Solution: Note that the domain of $f(x)$ does not contain $x = 0$, and this point is not in the interval $[1, 3]$.
(Step 1) Find critical points. Compute

$$f'(x) = 2x - \frac{16}{x^2} = \frac{2x^3 - 16}{x^2}.$$

Set $f'(x) = 0$. As a fraction equals zero if and only if its numerator equals zero, we have $2x^3 - 16 = 0$, and so the only possible critical point is $x = 2$. As this point is in the interval $[1, 3]$, it is a critical point.
(Step 2) Compute $f(x)$ at the critical point(s) and at the boundaries of the closed interval.

$$\begin{aligned} f(1) &= 1 + \frac{16}{1} = 17, \\ f(2) &= 2^2 + \frac{16}{2^2} = 8, \\ f(3) &= 3^2 + \frac{16}{3^2} = 9 + \frac{16}{9} = \frac{97}{9}. \end{aligned}$$

(Step 3) Compare the data resulted in Step 2 to make conclusions.

Note that $8 < \frac{97}{9} < 17$, and so $f(x)$ attains its absolute maximum value $f(1) = 17$ @ $x = 1$ and $f(x)$ attains its absolute minimum value $f(2) = 8$ @ $x = 2$.

Example 4 Find the maximum value and the minimum value attained by $f(x) = x^{\frac{1}{2}} - x^{\frac{3}{2}}$ in the interval $[0, 4]$.

Solution: Note that the domain of $f(x)$ does not contain any negative number, and so the function is continuous on $[0, 4]$.
(Step 1) Find critical points. Compute

$$f'(x) == \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{2\sqrt{x}} - \frac{3\sqrt{x}}{2} = \frac{1}{2\sqrt{x}} - \frac{3x}{2\sqrt{x}} = \frac{1-3x}{2\sqrt{x}}.$$

Set $f'(x) = 0$. As a fraction equals zero if and only if its numerator equals zero, we have $1 - 3x = 0$, and so the $x = \frac{1}{3}$ is a critical point. Since $f'(x)$ does not exist at $x = 0$, but $f(x)$ is (right) continuous at $x = 0$, both $\frac{1}{3}$ and 0 are critical points.
(Step 2) Compute $f(x)$ at the critical point(s) and at the boundaries of the closed interval.

$$\begin{aligned} f(0) &= 0 - 0 = 0, \\ f(\frac{1}{3}) &= \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}} = \frac{1}{2\sqrt{3}}, \\ f(4) &= 4^{\frac{1}{2}} - 4^{\frac{3}{2}} = 2 - 8 = -6. \end{aligned}$$

(Step 3) Compare the data resulted in Step 2 to make conclusions.

Note that $-6 < 0 < \frac{1}{2\sqrt{3}}$, and so $f(x)$ attains its absolute maximum value $f(\frac{1}{3}) = \frac{1}{2\sqrt{3}}$ @ $x = \frac{1}{3}$ and $f(x)$ attains its absolute minimum value $f(4) = -6$ @ $x = 4$.