Find vertical tangent lines

Fact:

The curve $y = f(x)$ has a **vertical tangent line** at the point $(a, f(a))$ if
(i) $f(x)$ is a continuous at $x = a$.
(ii) $\lim_{x\to a} |f'(x)| = \infty$ or equivalently, $\lim_{x\to a} \frac{1}{|f'(x)|} = 0$. (When a is an end point of the domain of $f(x)$, the limit should be an appropriate side limit. See Example 1).
When both (i) and (ii) are satisfied, the vertical line $x = a$ is a tangent line of the curve $y = f(x)$ at the point $(a, f(a))$.

Example 1 Find all the points on the graph $y = x^{1/2} - x^{3/2}$ where the tangent line is either horizontal or vertical.

Solution: We first observe the domain of $f(x) = x^{1/2} - x^{3/2}$ is $[0, \infty)$. Since horizontal tangent lines occur when $y' = 0$ and vertical tangent lines occur when (i) and (ii) above are satisfied, we should compute the derivative.

$$y' = \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{2\sqrt{x}} - \frac{3\sqrt{x}}{2} = \frac{1}{2\sqrt{x}} - \frac{3x}{2\sqrt{x}} = \frac{1-3x}{2\sqrt{x}}.$$

Therefore, when $x = \frac{1}{3}$, $y' = 0$ and so $y = f(x)$ has a horizontal tangent line at $(\frac{1}{3}, f(\frac{1}{3}))$; and as $f(x)$ is (right) continuous at 0, and $\lim_{x\to 0^+} |f'(x)| = \infty$, $y = f(x)$ has a vertical tangent line at $(0, 0)$.

Example 2 Find all the points on the graph $y = \frac{x}{\sqrt{1-x^2}}$ where the tangent line is either horizontal or vertical.

Solution: We first observe the domain of $f(x) = \frac{x}{\sqrt{1-x^2}}$ is $(-1, 1)$. Since horizontal tangent lines occur when $y' = 0$ and vertical tangent lines occur when (i) and (ii) above are satisfied, we should compute the derivative. Write $f(x) = x(1-x^2)^{-\frac{1}{2}}$.

$$\begin{aligned} y' &= (1-x^2)^{-\frac{1}{2}} + x\left(\frac{-1}{2}\right)(1-x^2)^{-\frac{3}{2}}(-2x) \\ &= \frac{1}{(1-x^2)^{\frac{1}{2}}} + \frac{x^2}{(1-x^2)^{\frac{3}{2}}} \\ &= \frac{1-x^2}{(1-x^2)^{\frac{3}{2}}} + \frac{x^2}{(1-x^2)^{\frac{3}{2}}} \end{aligned}$$

$$= \frac{1-x^2+x^2}{(1-x^2)^{\frac{3}{2}}} = \frac{1}{(1-x^2)^{\frac{3}{2}}}.$$

One can also use quotient rule to compute the derivative:

$$y' = \frac{\sqrt{1-x^2} - x\frac{-2x}{2\sqrt{1-x^2}}}{1-x^2} = \frac{1}{(1-x^2)^{\frac{3}{2}}}.$$

Therefore, for any x, $f'(x) \neq 0$, and so the graph does not have a horizontal tangent line. When $x = 1$, $\lim_{x\to 1^-} |f'(x)| = \infty$, and $x = -1$, $\lim_{x\to -1^+} |f'(x)| = \infty$, but as these points are not in the domain of $f(x)$, $y = f(x)$ does not have a vertical tangent line either.

Example 3 Find all the points on the graph $y = x\sqrt{4-x^2}$ where the tangent line is either horizontal or vertical.

Solution: We first observe the domain of $f(x) = x\sqrt{4-x^2}$ is $[-2, 2]$. Since horizontal tangent lines occur when $y' = 0$ and vertical tangent lines occur when (i) and (ii) above are satisfied, we should compute the derivative. View $f(x) = x(4-x^2)^{\frac{1}{2}}$.

$$\begin{aligned} y' &= (4-x^2)^{\frac{1}{2}} + x\frac{1}{2}(4-x^2)^{\frac{-1}{2}}(-2x) = \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}} \\ &= \frac{4-x^2}{\sqrt{4-x^2}} - \frac{x^2}{\sqrt{4-x^2}} = \frac{4-x^2-x^2}{\sqrt{4-x^2}} \\ &= \frac{2(2-x^2)}{\sqrt{4-x^2}}. \end{aligned}$$

Therefore, when $x = \pm\sqrt{2}$, $y' = 0$ and so $y = f(x)$ has a horizontal tangent line at $(-\sqrt{2}, f(-\sqrt{2}))$ and at $(\sqrt{2}, f(\sqrt{2}))$; and as $f(x)$ is (right) continuous at $x = -2$, and (left) continuous at $x = 2$, and as $\lim_{x\to -2^+} |f'(x)| = \infty$ and as $\lim_{x\to 2^-} |f'(x)| = \infty$, $y = f(x)$ has a vertical tangent line at both $(-2, f(-2))$ and $(2, f(2))$.