MATH 251 - Worksheet 8

NAME:

I.D.:

Instruction: Circle your answers and show all your work CLEARLY. Solutions with answer only and without supporting procedures will have little credit.

1. Compute the value of the triple integral $\int\int\int_T f(x,y,z)dV$, where $f(x,y,z) = x^2$, and T is the tetrahedron bounded by the coordinate planes and the first octant part of the plane with equation $x+y+z=1$.

Solution As the solid is in the first octant, we observe that $0 \le x \le 1$. For fixed x, $0 \le y \le 1-x$. For fixed x and y, $0 \le z \le 1-x-y$. Thus the answer is

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^2 dzdydx = \int_0^1 \int_0^{1-x} x^2(1-x-y)dydx$$

$$= \int_0^1 \left[x^2(1-x)^2 - \frac{x^2(1-x)^2}{2} \right] dx = \frac{1}{2}\int_0^1 (x^2 - 2x^4 + x^4)dx = \frac{1}{60}.$$

2. Compute the value of the triple integral $\int\int\int_T f(x,y,z)dV$, where $f(x,y,z) = xyz$, and T lies below the surface $z = 1-x^2$ and above the rectangle $-1 \le x \le 1$, $0 \le y \le 2$ in the $z=0$ plane.

Solution The rectangle region in the xy-plane suggests the following integration bounds. The answer is (you may use the integration property of an odd function over a symmetric interval).

$$\int_{-1}^1 \int_0^2 \int_0^{1-x^2} xyzdzdydx = \frac{1}{2}\int_{-1}^1 x(1-x^2)^2 dx \int_0^2 ydy = \frac{1}{2}\int_{-1}^1 (x - 2x^3 + x^5)dx = 0.$$

3. Compute the value of the triple integral $\int\int\int_T f(x,y,z)dV$, where $f(x,y,z) = 2y+z$, and T lies below the surface $z = 4-y^2$ and above the rectangle $-1 \le x \le 1$, $-2 \le y \le 2$ in the xy-plane.

Solution Following the bounds given, we set up the integral as follows. (One could use the properties of odd and even functions to simplify the last step of integration for dy, in which case both $16y$ and $-4y^3$ disappear).

$$\int_{-1}^1 \int_{-2}^2 \int_0^{4-y^2} (2y+z)dzdydx = \int_{-1}^1 dx \int_{-2}^2 \left(2y(4-y^2) + \frac{(4-y^2)^2}{2} \right) dy$$

$$= \int_{-2}^2 \left(16 + 16y - 8y^2 - 4y^3 + y^4\right) dy = 2\left[16y - \frac{8y^3}{3} + \frac{y^5}{5}\right]_0^2 = \frac{512}{15}.$$

4. Find the volume of the solid bounded by the surfaces $y + z = 4$, $y = 4 - x^2$, $y = 0$ and $z = 0$ by triple integration.

Solution The solid lies between $z = 0$ and $z = 4 - y$ over a region R on the xy-plane bounded by $y = 4 - x^2$ and $y = 0$. The bounds for R will then be $-2 \leq x \leq 2$ and $0 \leq y \leq 4 - x^2$. Thus the volume is

$$\begin{aligned} V &= \int_{-2}^{2} \int_{0}^{4-x^2} \int_{0}^{4-y} dz dy dx = \int_{-2}^{2} \int_{0}^{4-x^2} (4-y) dy dx = \int_{-2}^{2} \left[4y - \frac{y^2}{2} \right]_{0}^{4-x^2} dx \\ &= \int_{-2}^{2} \left[8 - \frac{x^4}{2} \right] dx = \left[8x - \frac{x^5}{10} \right]_{-2}^{2} = \frac{128}{5}. \end{aligned}$$

5. Find the volume of the solid bounded by the surfaces $z = x^2$, $y + z = 4$, $y = 0$ and $z = 0$ by triple integration.

Solution View the y-axis as the vertical axis. Then T lies between $y = 0$ and $y = 4 - z$. The region R on the xz-plane is bounded by $z = x^2$ and $z = 4$ (obtained by substituting $y = 0$ in $y + z = 4$). Therefore, the volume is

$$\begin{aligned} V &= \int\int_{R} \left(\int_{0}^{4-z} dy \right) dA = \int_{-2}^{2} \int_{x^2}^{4} (4-z) dz dx = \int_{-2}^{2} \left[4z - \frac{z^2}{2} \right]_{x^2}^{4} dx \\ &= \int_{-2}^{2} \left(8 - 4x^2 + \frac{x^4}{2} \right) dx = \left[8x - \frac{4x^3}{3} + \frac{x^5}{10} \right]_{-2}^{2} = 64 \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{10} \right) = \frac{256}{15}. \end{aligned}$$