MATH 251 - QUIZ 2

NAME:
I.D.:

Instruction: Circle your answers and show all your work CLEARLY. Solutions with answer only and without supporting procedures will have little credit.

1. (Exercise 5 on Page 770) Given a curve with parametric equations $x = 3t\sin t, y = 3t\cos t$ and $z = 2t^2$, find the arc length from $t = 0$ to $t = 4/5$.

Solution Compute the derivatives to get $x'(t) = 3\sin t + 3t\cos t$, $y'(t) = 3\cos t - 3t\sin t$ and $z'(t) = 4t$. Thus the arc length S is

$$
\begin{aligned}
S &= \int_0^{4/5} \sqrt{(3\sin t + 3t\cos t)^2 + (3\cos t - 3t\sin t)^2 + (4t)^2 dt} \qquad \text{by binomial formula} \\
&= \int_0^{4/5} \sqrt{9\sin^2 t + 9t^2\cos^2 t + 9\cos^2 t + 9t^2\sin^2 t + 16t^2 dt} \qquad \text{apply } \sin^2 t + \cos^2 t = 1 \\
&= \int_0^{4/5} \sqrt{9 + 25t^2} dt \qquad \text{set } t = \frac{3}{5}\tan\theta \\
&= \frac{9}{5}\int_0^{\tan^{-1}(4/3)} sec^3\theta d\theta \qquad \text{integration by parts} \\
&= \left(\frac{9}{5}\right)\left(\frac{1}{2}\right)\left[\frac{\sqrt{9+25t^2}}{3}\frac{5t}{3} + \ln\left(\frac{\sqrt{9+25t^2}}{3} + \frac{5t}{3}\right)\right]_0^{4/5} \qquad \text{use } \sec\theta = \frac{\sqrt{9+25t^2}}{3} \text{ and } \tan\theta = \frac{5t}{3} \\
&= \frac{9}{10}\left(\frac{20}{9} + \ln 3\right) = 2 + \frac{9}{10}\ln 3.
\end{aligned}
$$

2. Given a curve $\mathbf{r}(t) = (t, t^2, t^3)$, find the unit tangent vector and unit normal vector of the curve at $(1, 1, 1)$.

Solution (Step 1) When $t = 1$, $\mathbf{r}(1) = (1, 1, 1)$. $\mathbf{v} = (1, 2t, 3t^2)$ and $v(t) = \sqrt{1 + 4t^2 + 9t^4}$.
(Step 2) At (1,1,1), the unit tangent vector is $\mathbf{T}(1) = (1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14})$.
(Step 3) Note that $\mathbf{a}(t) = (0, 2, 6t)$. At $t = 1$, $\mathbf{a}(1) = (0, 2, 6)$, and so $a_T = \frac{\mathbf{a}\cdot\mathbf{v}}{v} = \frac{0+4+18}{\sqrt{14}} = \frac{22}{\sqrt{14}}$.
(Step 4) At $t = 1$, $\mathbf{v} \times \mathbf{a} = (1, 2, 3) \times (0, 2, 6) = (12 - 6, -6, -2)$. Thus at $t = 1$,

$$a_N = \frac{|\mathbf{v} \times \mathbf{a}|}{v} = \frac{\sqrt{36 + 36 + 4}}{\sqrt{14}} = \sqrt{\frac{76}{14}}.$$

It follows by $\mathbf{N} = (\mathbf{a} - a_T\mathbf{T})/a_N$ that

$$
\begin{aligned}
\mathbf{N} &= \frac{\sqrt{14}}{\sqrt{76}}\left((0, 2, 6) - \frac{22}{\sqrt{14}}(1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14})\right) \\
&= \frac{\sqrt{14}}{\sqrt{76}}\left((0, 2, 6) - \left(\frac{11}{7}, \frac{22}{7}, \frac{33}{7}\right)\right) \\
&= \frac{\sqrt{14}}{\sqrt{76}}\left(-\frac{11}{7}, -\frac{8}{7}, \frac{9}{7}\right).
\end{aligned}
$$