EXAM 3 - Math 251

1. (20 %) Let $f(x, y, z) = 3x^2 + y^2 + 4z^2$ and let $P(1, 5, -2)$ be a point.
(1A) Compute the gradient vector ∇f at P.
(1B) Compute the directional derivative of $f(x, y, z)$ at P along the direction $\mathbf{v} = (1, 1, 2)$.
(1C) Find an equation of the plane tangent to the surface $3x^2 + y^2 + 4z^2 = 0$ at the point P.

Solution Compute $\nabla f = (6x, 2y, 8z)$. (1A) $\nabla f(P) = (6, 10, -16)$.
(1B) Compute $|\mathbf{v}| = \sqrt{1 + 1 + 4} = \sqrt{6}$. Thus

$$D_{\mathbf{v}} f(P) = \frac{(6, 8, -16) \cdot (1, 1, 2)}{\sqrt{6}} = \frac{6 + 10 - 32}{\sqrt{6}} = \frac{-16}{\sqrt{6}}.$$

(1C) Use $\nabla f(P) = (6, 10, -16)$ as a normal vector. The answer is $6(x-1)+10(y-5)-16(z+2) = 0$.

2. (10 %) Find and classify the critical points of the function $f(x, y) = x^3 + 6xy + 3y^2$.

Solution $f_x = 3x^2 + 6y$ and $f_y = 6x + 6y$. From $f_y = 0$, we have $y = -x$. Substitute $y = -x$ into $f_x = 0$ to get $x^2 - 2x = 0$, and so $x = 0$ or $x = 2$. Accordingly, $(0, 0)$ and $(2, -2)$ are the only two critical points.

$f_{xx} = 6x$, $f_{yy} = 6$ and $f_{xy} = 6$. At $(0, 0, 0)$, $\Delta < 0$ and so this is a saddle point. At $(2, -2, -4)$, $\Delta = 72 - 36 > 0$ and $f_{xx} = 12 > 0$, and so this is a local minimum.

3. (10 %) Compute the double integral

$$\int_0^1 \int_y^1 (x + y) dx dy.$$

Solution

$$\int_0^1 \int_y^1 (x + y) dx dy = \int_0^1 \left[\frac{x^2}{2} + yx \right]_y^1 dy = \int_0^1 \left(\frac{1}{2} + y - \frac{3}{2} y^2 \right) dy = \left[\frac{y}{2} + \frac{y^2}{2} - \frac{y^3}{2} \right]_0^1 = \frac{1}{2}.$$

4. (6 % for setting up the integral with correct bounds and 6 % for accuracy) Compute the double integral of the function $f(x, y) = x^2$ over the region bounded by the parabola $y = 2 - x^2$ and the line $y = -7$.

Solution Solve $y = 2 - x^2$ and $y = 2 - x^2$ for x to get $x^3 = 9$, and so the (vertically simple) x-bounds are $a = -3$ and $b = 3$. Accordingly, $-7 \le y \le 2 - x^2$, and so the integral is

$$\int_{-3}^{3} \int_{-7}^{2-x^2} x^2 dy dx = \int_{-3}^{3} [(2x^2 - x^4) - (-7x^2)] dx = \int_{-3}^{3} (9x^2 - x^4) dx = \left[3x^3 - \frac{x^5}{5} \right]_{-3}^{3}$$

$$= 81\left(1-\frac{3}{5}\right) - 81\left(-1-\frac{-3}{5}\right) = \frac{324}{5}.$$

5. (6 % for setting up the integral with correct bounds and 7 % for accuracy) Find the volume of the solid that lies below $z = 3x + 2y$ and above the region R on the $z = 0$ plane, where R is bounded by $x = 0$, $y = 0$ and $x + 2y = 4$.

Solution View the region as a horizontally simple one. Then $0 \le y \le 2$ and $0 \le x \le 4 - 2y$. The volume is

$$\begin{aligned}
\int_0^2 \int_0^{4-2y} (3x+2y)dxdy &= \int_0^2 \left[\frac{3x^2}{2} + 2xy\right]_0^{4-2y} dy = \int_0^2 \left(\frac{3(16-16y+4y^2)}{2} + 2y(4-2y)\right) dy \\
&= \left[24y - 8y^2 + \frac{2y^3}{3}\right]_0^2 = 48 - 32 + \frac{16}{3} = \frac{64}{3}.
\end{aligned}$$

For the vertically simple view, we have

$$\begin{aligned}
\int_0^4 \int_0^{2-\frac{x}{2}} (3x+2y)dydx &= \int_0^4 \left[3x\left(2-\frac{x}{2}\right) + \left(2-\frac{x}{2}\right)^2\right] dx = \int_0^4 \left[6x - \frac{3x^2}{2} + 4 - 2x + \frac{x^2}{4}\right] dx \\
&= \int_0^4 \left[4 + 4x - \frac{5x^2}{4}\right] dx = \left[4x + 2x^2 - \frac{5x^3}{12}\right]_0^4 = \frac{256}{12} = \frac{64}{3}.
\end{aligned}$$

6. (6 % for setting up the integral with correct bounds and 6 % for accuracy) Find the volume of the solid that is under $z = xy$ and above the triangle R on the $z = 0$ plane, with vertices $(2, 1)$, $(4, 1)$ and $(3, 0)$.

Solution Compute to find that an equation of the line connecting $(2, 1)$ and $(3, 0)$ is $x = 3 - y$ and an equation of the line connecting $(4, 1)$ and $(3, 0)$ is $x = 3 + y$. With the horizontal simple view, the volume is

$$\int_0^1 \int_{3-y}^{3+y} xydxdy = \int_0^1 \left[\frac{yx^2}{2}\right]_{3-y}^{3+y} dy = \int_0^1 6y^2 dy = 2.$$

Another way of computing this is

$$\begin{aligned}
&\int_2^3 \int_{3-x}^1 xydydx + \int_3^4 \int_{x-3}^1 xydydx \\
=& \int_2^3 \left[\frac{x}{2}(1-(3-x)^2)\right] dx + \int_3^4 \left[\frac{x}{2}(1-(x-3)^2)\right] dx \\
=& \int_2^4 \frac{x}{2}(6x - 8 - x^2)dx = \left[x^3 - 2x^2 - \frac{x^4}{8}\right]_2^4 \\
=& 64 - 32 - 32 - (8 - 8 - 2) = 2.
\end{aligned}$$