EXAM 4 - Math 156 SOlutions

NAME:

Instruction: Circle your answers and show all your work CLEARLY. Partial credit will be given only when you present what belongs to part of a correct solution.

1. (4 % each, total 20 %) Answer each of the following. (No need to show your work for this problem).

(a) Write down the Maclaurin series of $f(x) = e^x$.

Answer: $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, Interval of convergence: $(-\infty, \infty)$.

(b) Write down the Maclaurin series of $f(x) = \frac{1}{1-x}$.

Answer: $\sum_{n=0}^{\infty} x^n$, Interval of convergence: $(-1, 1)$.

(c) What is the sum of this convergent series $\sum_{n=1}^{\infty} \frac{3^n}{n!}$?

Answer: As $\sum_{n=0}^{\infty} \frac{3^n}{n!} = e^3$, then answer is $e^3 - 1$.

Discussion: Many get the answer e^3. But $e^3 = \sum_{n=0}^{\infty} \frac{3^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{3^n}{n!}$. The problem wants to see if students know what e^3 really is in terms of Maclaurin series, and if students know how to compute the first term of the Maclaurin series.

(d) For the curve $\begin{cases} x = e^t + t + 1 \\ y = t^3 - 12t + 15 \end{cases}$, find the value(s) t at which the curve has horizontal tangent lines.

Answer: Compute $\frac{dy}{dt} = 3(t^2 - 4)$. Therefore, the value(s) t at which the curve has horizontal tangent lines are $t = -2$ and $t = 2$

(e) Find a Cartesian equation for the curve $r = 4\cos(\theta)$.

Answer: Multiplying both sides by r to get $r^2 = 4r\cos(\theta)$. This leads to $x^2 + y^2 = 4x$, or, after completing the squares, $(x-2)^2 + y^2 = 4$.

Discussion: Common errors are from those students who do not know what to compute. Some of us use $y = \sqrt{4x - x^2}$. But $y = \sqrt{4x - x^2}$ represents only part of the original curve, and so it is a wrong answer.

2. (10 %) Find the Maclaurin series for the function xe^{x+2} and determine the radius of convergence.

Solution. Since this is a Maclauring series, this means $a = 0$. Therefore, we first need to single out e^x. To do that, we use Laws of exponents to obtain the wanted Maclaurin series:

$$xe^{x+2} = xe^2 \cdot e^x = xe^2 \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{e^2}{n!} x^{n+1}.$$

To determine the radius of convergence, we compute

$$\lim_{n\to\infty} \frac{\frac{e^2}{(n+1)!}}{\frac{e^2}{n!}} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Hence the radius of convergence in ∞.

Discussion: The purposes of this exercise is two fold: one is to see if students know how to use algebra to correctly compute the Maclaurin series using the known ones (such as e^x); the other is to see if students know the difference between a Taylor series and a Maclaurin series, which is the case when $a = 0$.

The most common wrong solution is:

$$xe^{x+2} = x \sum_{n=0}^{\infty} \frac{(x+2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x(x+2)^n}{n!}.$$

This indicates that the student did not understand Maclauring series must expand the series at $a = 0$, and the series can only have x^n terms, and cannot have $(x+2)^n$ terms. Only a Taylor series at $a = -2$ can have $(x+2)^n$ terms.

3. (15 %) Find the Maclaurin series for the function $\tan^{-1}(\frac{x}{3})$, and determine the radius of convergence and the interval of convergence.

Solution 1. If we can remember that $\tan^{-1}(u) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} u^{2n+1}$, with I.C. $= [-1, 1]$, then we can directly use $u = \frac{x}{3}$ to get

$$\tan^{-1}(\frac{x}{3}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{x}{3}\right)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+1}(2n+1)} x^{2n+1}.$$

As $-1 \leq \frac{x}{3} \leq 1$, we conclude that the interval of convergence of this Maclaurin series is $[-3, 3]$.

Solution 2. Suppose that we do not remember the Maclaurin series of $\tan^{-1}(u)$. We can use

$$\frac{1}{1+u} = \sum_{n=0}^{\infty} (-1)^n u^n,$$

together with differentiation and integration to get it. Let $f(x) = \tan^{-1}(\frac{x}{3})$. Then

$$f'(x) = \frac{1}{3} \cdot \frac{1}{1 + (\frac{x}{3})^2} = \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{3}\right)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+1}} x^{2n}.$$

Then

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+1}} \int x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+1}(2n+1)} x^{2n+1} + C.$$

Set $x = 0$ to get $0 = \tan^{-1}(0) = C$, and so tht Maclauring series is

$$\tan^{-1}(\frac{x}{3}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+1}} \int x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+1}(2n+1)} x^{2n+1}.$$

To determine the interval of convergence, we use ration test. For us to conclude that the seriesis convergent, we need

$$\lim_{n\to\infty} \frac{\frac{|x^{2n+3}|}{3^{2n+3}(2n+3)}}{\frac{|x^{2n+1}|}{3^{2n+1}(2n+1)}} = \lim_{n\to\infty} \frac{|x^2|(2n+1)}{3^2(2n+3)} = \frac{|x^2|}{3^2} < 1.$$

It follows that $|x^2| < 3^2$ or $|x| < 3$. We now need to check the convergency at $x = -3$ and $x = 3$.

At $x = 3$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+1}(2n+1)} 3^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

By Alternating series test, this is convergent.

At $x = -3$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+1}(2n+1)} (-3)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}.$$

By Alternating series test, this is convergent. Therefore, the interval of convergence is $[-3, 3]$.

4. (15 %) Find the Taylor series of $\frac{1}{3-x}$ at $a = 2$, and determine the radius of convergence and the interval of convergence.

Solution. Write

$$\frac{1}{3-x} = \frac{1}{1-(x-2)} = \sum_{n=0}^{\infty} (x-2)^n.$$

Using Ratio test, the radius of convergence is 1, and so the interval of convergence contains $(2-1, 2+1) = (1, 3)$. At $x = 1$, the series becomes $\sum_{n=0}^{\infty}(1-2)^n = \sum_{n=0}^{\infty}(-1)^n$. By the Divergence Test, the series is divergent. At $x = 3$, the series becomes $\sum_{n=0}^{\infty}(3-2)^n = \sum_{n=0}^{\infty} 1^n$. By the Divergence Test, the series is divergent. Thus the interval of convergence is $(1, 3)$.

Discussion: There are only two Taylor series problems in the test. This is one of them. As we are seeking Taylor series at $a = 2$, the answer must have the form

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n.$$

Some of us, mostly those not attending the classes, chose to directly compute $f^{(n)}(2)$ and end up nowhere, due to the complexity of the higher order of differentiations. The uniqueness of Taylor is the way we recommended in class.

One common error is the following:

$$\frac{1}{3-x} = \frac{1}{3(1-\frac{x}{3})} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n.$$

Those who answered the problem this way might be trying to remember what they did in the past, without understanding what we are now computing. That was for the Maclaurin series (that is, Taylor series at $a = 0$). But we are now computing Taylor series at $a = 2$, a totally different problem.

5. (10 %) Find a function $f(x)$ whose Maclaurin series is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{3n}}{n!}$.

Solution. Compare it with the e^x Maclaurin series, we note that

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{3n}}{n!} = \sum_{n=0}^{\infty}(-1)^n \frac{(-x^3)^n}{n!} = e^{-x^3}.$$

Thus the answer is $f(x) = e^{-x^3}$.

6. (10 %) Find the first 4 terms of the Taylor series of $f(x) = \sqrt[3]{1+x}$ at $a = 1$.

Solution. Use the formula $a_n = \dfrac{f^{(n)}(a)}{n!}$ and compute

$f(x) = (1+x)^{\frac{1}{3}}$	$f(1) = 2^{\frac{1}{3}}$	$a_0 = 2^{\frac{1}{3}}$
$f'(x) = \frac{1}{3}(1+x)^{\frac{-2}{3}}$	$f'(1) = \frac{2^{\frac{-2}{3}}}{3}$	$a_1 = \frac{2^{\frac{-2}{3}}}{3}$
$f''(x) = \frac{-2}{9}(1+x)^{\frac{-5}{3}}$	$f''(1) = \frac{(-2)2^{\frac{-5}{3}}}{9}$	$a_2 = \frac{-2^{\frac{-5}{3}}}{9}$
$f'''(x) = \frac{10}{27}(1+x)^{\frac{-8}{3}}$	$f''(1) = \frac{10\cdot 2^{\frac{-8}{3}}}{27}$	$a_3 = \frac{5\cdot 2^{\frac{-8}{3}}}{81}$

Thus,

$$f(x) = \sqrt[3]{1+x} = 2^{\frac{1}{3}} + \frac{2^{\frac{-2}{3}}}{3}(x-1) + \frac{-2^{\frac{-5}{3}}}{9}(x-1)^2 + \frac{5\cdot 2^{\frac{-8}{3}}}{81}(x-1)^3 + \ldots$$

Discussion: Many chose to apply the formula:

$$(1+x)^k = \sum_{n=0}^{\infty} \begin{pmatrix} k \\ n \end{pmatrix} x^n = 1 + \sum_{n=1}^{\infty} \frac{k(k-1)(k-2)...(k-n+1)}{n!} x^n, \quad r = 1.$$

Why this is **NOT** applicable here? Because the formula above is for the Maclaurin series (that is, Taylor series at $a = 0$), and in this problem, we are computing the Taylor series at $a = 1$. As discussed in class, we recommended that in this case, we should use

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \text{ with } f(x) = (1+x)^{\frac{1}{3}} \text{ and } a = 1.$$

Can we use the Maclaurin series in this problem? Yes, we can. But we must formulate the problem correctly. Here is how. Using algebra, firstly we rewrite the function as $f(x) = (1+x)^{\frac{1}{3}} = (2+(x-1))^{\frac{1}{3}} = 2^{\frac{1}{3}} \cdot (1 + \frac{(x-1)}{2})^{\frac{1}{3}}$. Then we can apply the formula as follows:

$$\begin{aligned} f(x) &= 2^{\frac{1}{3}} \cdot \left(1 + \frac{(x-1)}{2}\right)^{\frac{1}{3}} \\ &= 2^{\frac{1}{3}} \left(1 + \sum_{n=1}^{\infty} \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)...(\frac{1}{3}-n+1)}{n!} \left(\frac{x-1}{2}\right)^n\right). \\ &= 2^{\frac{1}{3}} + \sum_{n=1}^{\infty} \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)...(\frac{1}{3}-n+1)2^{\frac{1}{3}}}{n!} \left(\frac{x-1}{2}\right)^n. \end{aligned}$$

Then, we compute: (it takes a bit high school algebra to see the two solutions yield the same answers).

$$\begin{aligned} n = 0, \quad a_0 &= 2^{\frac{1}{3}} \\ n = 1, \quad a_1 &= \frac{2^{\frac{1}{3}}}{3} \cdot \frac{1}{2} = \frac{2^{\frac{1}{3}}}{6} \\ n = 2, \quad a_2 &= \frac{2^{\frac{1}{3}}}{3} \cdot \frac{-2}{3} \cdot \frac{1}{2!2^2} = \frac{-2^{\frac{1}{3}}}{3^3 \cdot 2^2} \\ n = 3, \quad a_3 &= \frac{2^{\frac{1}{3}}}{3} \cdot \frac{-2}{3} \cdot \frac{-5}{3} \cdot \frac{1}{3!2^3} = \frac{5 \cdot 2^{\frac{1}{3}}}{3^4 \cdot 2^3} \end{aligned}$$

7. (5 % each, 20% total) Given a parametric curve $x = 4 + t^2$ and $y = t^2 + t^3$,

(i) Find $\frac{dy}{dx}$.

(ii) Find $\frac{d^2y}{dx^2}$.

(iii) Find the values of t at which the curve has horizontal tangent line and at which the curve has vertical tangent line.

(iv) Write down the integral that computes the length of the curve with $1 \le t \le 2$. (No need to simplify your answer. Do **NOT** evaluate the integral.)

Solution. Compute $\frac{dy}{dt} = 2t + 3t^2$ and $\frac{dx}{dt} = 2t$.

(i) Thus $y' = \frac{dy}{dx} = \frac{2t+3t^2}{2t} = (2+3t)/2$.

(ii) Compute $\frac{dy'}{dt} = \frac{3}{2}$. Hence $\dfrac{d^2y}{dx^2} = \dfrac{3}{2} \cdot \dfrac{1}{2t} = \dfrac{3}{4t}$.

(iii) Set $y' = 0$ to get $t = \frac{-2}{3}$. Thus at $t = \frac{-2}{3}$, we have a horizontal tangent line. As $\lim_{t\to 0} \frac{dy}{dx} = 1 \neq \infty$, the curve has no vertical tangent line. (Review Calculus I for vertical tangent lines).

(iv) The integral computing the Arc length is

$$\int_1^2 \sqrt{(2t+3t^2)^2 + (2t)^2}dt.$$

Grade Distribution of Exam 4:

Meaning of the scores: The highest score of exam 4 is 94/100.
at least 90 = Very good, familiar with the related materials and skillful, with minimal computational errors. Keep on!
80-89 = good, familiar with most of the related materials, with a few computations errors. Make an effort to do better.
70-79 = OK, not so familiar with the related materials, with relatively more computational errors. We have room to improve. (For this quiz, not familiar with differentiation).
60-69 = Passing, We are ont the borderline of failing. It indicates that we are less familiar with the related materials and more computational errors and algebraic errors. We have lots of room to improve.
at most 59 = We failed. We need to catch it up. It should definitely be the time for us to see the instructor and get assistance to understand the materials and to practice MORE.

Scores	90-99	80-89	70-79	60-69	≤ 59
Frequency	4	10	10	5	2
Percentage	12.9	32.3	32.3	16.14	6.4