EXAM 3 - Math 156 Solutions

NAME:

Instruction: Circle your answers and show all your work CLEARLY. For each series testing problem, 4% for using the appropriate test, 4% for showing the appropriate mathematics verifying the test, and 4% or 3% for correct conclusions.

1. (4 % each, total 20 %) Determine which of the following sequence or series is convergent or divergent by circling exactly one answer. No credit if both answers are circled. Fill up the blank if appropriate.

(a) $\{(-1)^n\}_{n=1}^{\infty}$.

Answer: The sequence is: **divergent**

(b) $\left\{\dfrac{2n}{n+1}\right\}_{n=1}^{\infty}$

Answer: The sequence is: **convergent**

If the sequence is convergent, then limit is: $\underline{2}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{5n^2-4}$

Answer: The series is: **divergent**

(d) $\displaystyle\sum_{n=1}^{\infty} e^{-n}$.

Answer: The series is: **convergent**

If the series is convergent, then sum is: $\underline{\frac{e^{-1}}{1-e^{-1}} = \frac{1}{e-1}}$.

(e) $1 - \dfrac{2}{5} + \dfrac{4}{25} - \dfrac{8}{125} + \dfrac{16}{625} - \cdots$

Answer: The series is: **convergent**

If the series is convergent, then sum is: $\underline{\frac{1}{1-(-\frac{2}{5})} = \frac{5}{7}}$.

2. (10 %) Find the sum of $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$.

Solution By partial fraction, we have

$$\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}.$$

This is a telescope series, and so for any integer $m \geq 1$, write down the first m terms of the series using the partial fraction we have found. (If you cannot see the pattern easily, then write a few more terms and line them up to make yourself easier to see the pattern.)

$$\begin{array}{rl}
n=1 & a_1 = \dfrac{2}{1(1+2)} = \dfrac{1}{1} - \dfrac{1}{1+2}. \\
n=2 & a_2 = \dfrac{2}{2(2+2)} = \dfrac{1}{2} - \dfrac{1}{2+2}. \\
n=3 & a_3 = \dfrac{2}{3(3+2)} = \dfrac{1}{3} - \dfrac{1}{3+2}. \\
n=4 & a_4 = \dfrac{2}{4(4+2)} = \dfrac{1}{4} - \dfrac{1}{4+2}. \\
n=5 & a_5 = \dfrac{2}{5(n+2)} = \dfrac{1}{5} - \dfrac{1}{5+2}. \\
\ldots & \ldots \\
n=m-1 & a_{m-1} = \dfrac{2}{(m-1)((m-1)+2} = \dfrac{1}{m-1} - \dfrac{1}{(m-1)+2}. \\
n=m & a_m = \dfrac{2}{m(m+2)} = \dfrac{1}{m} - \dfrac{1}{m+2}.
\end{array}$$

Now we compute the mth partial sum by canceling the fractions whose absolute values are the same but with opposite signs (such as $-\frac{1}{1+2}$ and $+\frac{1}{3}$, $-\frac{1}{2+2}$ and $+\frac{1}{4}$, $-\frac{1}{3+2}$ and $+\frac{1}{5}$ ... $-\frac{1}{(m-3)+2}$ and $+\frac{1}{m-1}$, and $-\frac{1}{(m-2)+2}$ and $+\frac{1}{m}$):

$$\begin{array}{rcl}
S_m & = & \displaystyle\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \frac{2}{1(1+2)} + \frac{2}{2(2+2)} + \cdots \frac{2}{(m-1)((m-1)+2)} + \frac{2}{m(m+2)} \\
 & = & \dfrac{1}{1} - \dfrac{1}{1+2} \\
 & = & \dfrac{1}{2} - \dfrac{1}{2+2} \\
 & & +\dfrac{1}{3} - \dfrac{1}{3+2} \\
 & & +\dfrac{1}{4} - \dfrac{1}{4+2} \\
 & & +\dfrac{1}{5} - \dfrac{1}{5+2} \\
\ldots & & \ldots \\
 & & +\dfrac{2}{(m-1)((m-1)+2} = \dfrac{1}{m-1} - \dfrac{1}{(m-1)+2} \\
 & & +\dfrac{2}{m(m+2)} = \dfrac{1}{m} - \dfrac{1}{m+2}
\end{array}$$

$$
\begin{aligned}
&= \frac{1}{1}+\frac{1}{2}-\frac{1}{(m-1)+2}-\frac{1}{m+2} \\
&= \frac{3}{2}-\frac{1}{m+1}-\frac{1}{m+2}.
\end{aligned}
$$

Finally, we compute the sum:

$$\sum_{n=1}^{\infty}\frac{2}{n(n+2)} = \lim_{m\to\infty} S_m = \lim_{m\to\infty}\left(\frac{3}{2}-\frac{1}{m+1}-\frac{1}{m+2}\right) = \frac{3}{2}.$$

Explanation: To make it easier for students to review the solutions, we include the **full statements** of the convergence/divergence tests used in the solutions.

3. (12 %) Test for the convergence of $\displaystyle\sum_{n=2}^{\infty}\frac{3}{n^2+5}$.

Solution (i) **Which test will be applied?** We use **Comparison Test**: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
(a) If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is convergent.
(b) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is divergent.

(ii) **How the test is applied?** Let $b_n = \frac{3}{n^2}$. Then for each $n \ge 1$,

$$a_n = \frac{3}{n^2+5} \le \frac{3}{n^2} = b_n.$$

Since $\sum_{n=1}^{\infty}\frac{3}{n^2}$ is a p-series with $p = 2 > 1$, it is convergent.
(iii) **Conclusion.** By the comparison test, $\displaystyle\sum_{n=2}^{\infty}\frac{3}{n^2+5}$ is convergent.

4. (12 %) Test for the convergence of $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\sqrt{n^3-2}}$.

Solution (i) **Which test will be applied?** We use **Limiting Comparison Test**. If $a_n \ge 0$ and $b_n \ge 0$ and if

$$\lim_{n\to\infty}\frac{a_n}{b_n} = c,$$

where $c > 0$ is a finite number, then either both $\sum a_n$ and $\sum b_n$ converge or both $\sum a_n$ and $\sum b_n$ diverge.

(ii) **How the test is applied?** Let $b_n = \frac{1}{n^{\frac{3}{2}}}$. Then

$$
\begin{aligned}
\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{1}{\sqrt{n^3-2}} \div \frac{1}{n^{\frac{3}{2}}} \\
&= \lim_{n\to\infty}\frac{n^{\frac{3}{2}}}{\sqrt{n^3-2}} = \lim_{n\to\infty}\frac{1}{\sqrt{1-\frac{2}{n^3}}} = 1.
\end{aligned}
$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ is a p-series with $p = \frac{3}{2} > 1$, it is convergent.

(iii) **Conclusion.** By the comparison test, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3 - 2}}$ is convergent.

5. (12 %) Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ converges absolutely, converges conditionally, or diverges.

Solution (i) **Which test will be applied?** This is not a positive termed series. We shall **first test for absolute convergency**. If it is not absolute convergent, **then we test for conditional convergency**.

(ii) **Is it absolutely convergent?** Let $a_n = |\frac{(-1)^n}{\sqrt[3]{n}}| = \frac{1}{n^{\frac{1}{3}}}$. As a p-series with $p = \frac{1}{3} < 1$ is divergent, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ is NOT absolutely convergent.

(iii) **Is it conditionally convergent?** This is an alternating series, we use **Alternating Series Test:** If the alternating series $\sum(-1)^n b_n$ satisfies

(i) $b_{n+1} \geq b_n \geq 0$ for all n, and

(ii) $\lim_{n\to\infty} b_n = 0$, then $\sum(-1)^n b_n$ converges.

How the test is applied? We compute

$$\lim_{n\to\infty} a_n == \lim_{n\to\infty} \frac{1}{n^{\frac{1}{3}}} = 0,$$

and verify

$$a_n = \frac{1}{n^{\frac{1}{3}}} > \frac{1}{(n+1)^{\frac{1}{3}}} = a_{n+1}.$$

(iv) **Conclusion.** By the Alternating Series Test, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ is convergent. Since it is not absolutely convergent, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ is conditionally convergent.

6. (10 %) Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n\pi/3)}{n^2}$ converges absolutely, converges conditionally, or diverges.

Solution (i) **Which test will be applied?** This is not a positive termed series. We shall **first test for absolute convergency**. If it is not absolute convergent, **then we test for conditional convergency**.

(ii) **Is it absolutely convergent?** Let $a_n = |\frac{\sin(n\pi/3)}{n^2}|$. Since $|sin(x)| \leq 1$ for any x we have $a_n \leq \frac{1}{n^2}$. We are to apply **Comparison Test:** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(a) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is convergent.

(b) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is divergent.

As $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$, it is convergent. It follows by **Comparison Test** that $\displaystyle\sum_{n=1}^{\infty} \frac{|\sin(n\pi/3)|}{n^2}$ is convergent.

(iii) **Conclusion.** $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n\pi/3)}{n^2}$ is absolutely convergent, and so it is convergent. (Therefore, there is no need to test for conditionally convergency.)

7. (10 %) Determine whether $\displaystyle\sum_{n=9}^{\infty} \left(\frac{-1}{\ln n}\right)^n$ converges absolutely, converges conditionally, or diverges. (Hint: When $n \geq 9$, $\ln n \geq 2$.)

Solution 1 (Without using root test).
(i) **Which test will be applied?** This is not a positive termed series. We shall **first test for absolute convergency**. If it is not absolute convergent, **then we test for conditional convergency**.
(ii) **Is it absolutely convergent?** Let $a_n = |\left(\frac{-1}{\ln n}\right)^n| = \frac{1}{(\ln n)^n}$. Seeing that when $n \geq 9$, $\ln n \geq 2$, we use **Comparison Test** (see statement above) to test if it is absolutely convergent.
How the test is applied? Let $b_n = \left(\frac{1}{2}\right)^n$. Then as when $n \geq 9$, $\ln n \geq 2$, we have (when $n \geq 9$),

$$a_n = \frac{1}{(\ln n)^n} \leq \frac{1}{2^n} = \left(\frac{1}{2}\right)^n = b_n.$$

Since $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a geometric series with $r = \frac{1}{2} < 1$, it is convergent. By the comparison test, $\displaystyle\sum_{n=9}^{\infty} \frac{1}{(\ln n)^n}$ is convergent.

(iii) **Conclusion.** $\displaystyle\sum_{n=9}^{\infty} \left(\frac{-1}{\ln n}\right)^n$ is absolutely convergent, and so it is convergent. (Therefore, there is no need to test for conditionally convergency.)

Solution 2 (Using root test).
(i) **Which test will be applied?** We will use **Root Test:** Let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.
(a) If $L < 1$, then $\sum a_n$ converges absolutely, (therefore, it is convergent).
(b) If $L > 1$, then $\sum a_n$ diverges.
(c) If $L = 1$, then the Root test is inconclusive.

(ii) **How the test is applied?** Compute

$$\lim_{n\to\infty} \sqrt[n]{|\left(\frac{-1}{\ln n}\right)^n|} = \lim_{n\to\infty} \frac{1}{\ln(n)} = 0 < 1.$$

(iii) **Conclusion.** By the Root Test, $\displaystyle\sum_{n=9}^{\infty} \left(\frac{-1}{\ln n}\right)^n$ is absolutely convergent, and so it is convergent. (Therefore, there is no need to test for conditionally convergency.)

8.

(i) (8 %) Find the values of x for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n}$ converges.

(ii) (6 %) Find the sum of the series for those values of x.

Solution As $\displaystyle\sum_{n=1}^{\infty} \left(\frac{x-1}{2}\right)^n$, this is a geometric series $\displaystyle\sum_{n=1}^{\infty} r^n$ with $r = \dfrac{x-1}{2}$.

(i) This geometric series is convergent if and only if $|r| < 1$, or $-1 < \frac{x-1}{2} < 1$. This means $-2 < x - 1 < 2$, or x is in the interval $(-1, 3)$.

Conclusion: $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n}$ converges exactly when $-1 < x < 3$ or in the interval $(-1, 3)$.

(ii) As $r = \dfrac{x-1}{2}$, we can use the formula for computing the sum of a geometric series to get the answer. Observe that the series starts with $n = 1$. We cannot directly apply the formula.

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n} = \sum_{n=1}^{\infty} \left(\frac{x-1}{2}\right)^n = \frac{x-1}{2} \sum_{n=0}^{\infty} \left(\frac{x-1}{2}\right)^n = \frac{x-1}{2} \cdot \frac{1}{1 - \frac{x-1}{2}} = \frac{x-1}{3-x}.$$

Grade Distribution of Exam 3:

Meaning of the scores: The highest score of exam 3 is 100/100.
at least 90 = Very good, familiar with the related materials and skillful, with minimal computational errors. Keep on!
80-89 = good, familiar with most of the related materials, with a few computations errors. Make an effort to do better.
70-79 = OK, not so familiar with the related materials, with relatively more computational errors. We have room to improve. (For this quiz, not familiar with differentiation).
60-69 = Passing, We are ont the borderline of failing. It indicates that we are less familiar with the related materials and more computational errors and algebraic errors. We have lots of room to improve.
at most 59 = We failed. We need to catch it up. It should definitely be the time for us to see the instructor and get assistance to understand the materials and to practice MORE.

Scores	90-99	80-89	70-79	60-69	≤ 59
Frequency	2	7	7	4	8
Percentage	6.5	22.6	22.6	35.4	12.9