Math 156 Quiz 7 (Extended Quiz)

Name:

Instruction. (This is counted as two quizzes. For problems of testing series for convergency, we must indicate (A) which test is applied, and (B) how the test is applied to claim your conclusion. Failing to do so would result in point deductions.

1: Determine if the series $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$ is absolutely convergent, conditionally convergent or divergent.

Solution (i) **Which test will be applied?** We use **Ratio Test**: For a series $\sum a_n$, let $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$.
(a) If $L < 1$, then $\sum a_n$ converges absolutely, (therefore, it is convergent).
(b) If $L > 1$, then $\sum a_n$ diverges.
(c) If $L = 1$, then the Root test is inconclusive.

(ii) **How the test is applied?** Let $a_n = \frac{3^n n^2}{n!}$. Compute

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{3^{n+1}(n+1)^2}{(n+1)!} \frac{n!}{3^n n^2} = \lim_{n\to\infty} \frac{3(n+1)^2}{n^2(n+1)} = 0.$$

(iii) **Conclusion.** By the Ratio Test, and as $L < 1$, $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$is absolutely convergent, and so it is convergent. (Therefore, there is no need to test for conditionally convergency.)

2: Determine if the series $\sum_{n=1}^{\infty} \frac{(-2)^{n-1}3^{n+1}}{n^n}$ is absolutely convergent, conditionally convergent or divergent.

Solution (i) **Which test will be applied?** We use **Root Test**: For a series $\sum a_n$, let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.
(a) If $L < 1$, then $\sum a_n$ converges absolutely, (therefore, it is convergent).
(b) If $L > 1$, then $\sum a_n$ diverges.
(c) If $L = 1$, then the Root test is inconclusive.

(ii) **How the test is applied?** Let $a_n = \frac{(-2)^{n-1}3^{n+1}}{n^n}$. Compute

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{2^{n-1}3^{n+1}}{n^n}} = \lim_{n\to\infty} \frac{2^{1-\frac{1}{n}}3^{1+\frac{1}{n}}}{n} = 0.$$

(iii) **Conclusion.** By the Root Test, and as $L < 1$, $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$is absolutely convergent, and so it is convergent. (Therefore, there is no need to test for conditionally convergency.)

3: Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(-2)^n}{n} x^n$.

Solution (i) **Find the radius of convergence.** Here $a = 0$, and $c_n = \frac{(-2)^n}{n}$. Compute

$$L = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n\to\infty} \frac{2^{n+1}}{n+1} \cdot \frac{n}{2^n} = \lim_{n\to\infty} = \frac{2n}{n+1} = 2.$$

Therefore, the radius of convergence is $RC = \frac{1}{L} = \frac{1}{2}$.

(ii) **Find the interval of convergence.** The interval of convergence contains $(0 - \frac{1}{2}, 0 + \frac{1}{2})$. We will examine the convergency at the end points.

At $x = -\frac{1}{2}$, the power series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n}\left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n}$. This is a p-series with $p = 1$, and so it is divergent.

At $x = \frac{1}{2}$, the power series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. As shown above, this is not absolutely convergent. We note that it is an alternating series and so we use **Alternating Series Test:** If the alternating series $\sum(-1)^n b_n$ satisfies

(i) $b_{n+1} \geq b_n \geq 0$ for all n, and

(ii) $\lim_{n\to\infty} b_n = 0$, then $\sum(-1)^n b_n$ converges.

Compute $\frac{n+1}{<}\frac{1}{n}$ and $\lim_{n\to\infty} \frac{1}{n} = 0$. By Alternating Series Test, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges. Therefore, the interval of convergence is $(\frac{-1}{2}, \frac{1}{2}]$.

4: Find the radius of convergence and interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2}$.

Solution (i) **Find the radius of convergence.** Here $a = 2$, and $c_n = \frac{(1}{n^2}$. Compute

$$L = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n\to\infty} \frac{1}{(n+1)^2} \cdot \frac{n^2}{1} = \lim_{n\to\infty} = \frac{n^2}{(n+1)^2} = 1.$$

Therefore, the radius of convergence is $RC = \frac{1}{L} = 1$.

(ii) **Find the interval of convergence.** The interval of convergence contains $(2-1, 2+1) = (1, 3)$. We will examine the convergency at the end points.

At $x = 3$, the power series is $\displaystyle\sum_{n=1}^{\infty} \frac{(3-2)^n}{n^2} == \sum_{n=1}^{\infty} \frac{1}{n^2}$. This is a p-series with $p = 2$, and so it is convergent.

At $x = 1$, the power series is $\displaystyle\sum_{n=1}^{\infty} \frac{(1-2)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. As shown above, this is absolutely convergent. and so it converges. Therefore, the interval of convergence is $[1, 3]$.

Take Home Part

Instruction: For full credit, any time when you are using a convergence/divergence test, write down the correct statement of the test, do the appropriate computation and explain how the test is applied to obtain your conclusion. Not following the instruction will have at most half the credit.

5: Determine if $\sum_{n=1}^{\infty}(-1)^n\left(\frac{2n+1}{3n-4}\right)^n$ is absolutely convergent, conditionally convergent or divergent.

Solution (i) **Which test will be applied?** We use **Root Test**: For a series $\sum a_n$, let $L = \lim_{n\to\infty}\sqrt[n]{|a_n|}$.
(a) If $L < 1$, then $\sum a_n$ converges absolutely, (therefore, it is convergent).
(b) If $L > 1$, then $\sum a_n$ diverges.
(c) If $L = 1$, then the Root test is inconclusive.

(ii) **How the test is applied?** Let $a_n = (-1)^n\left(\frac{2n+1}{3n-4}\right)^n$. Compute

$$L = \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{(-1)^n\left(\frac{2n+1}{3n-4}\right)^n} = \lim_{n\to\infty}\frac{2n+1}{3n-4} = \frac{2}{3}.$$

(iii) **Conclusion.** By the Root Test, and as $L < 1$, $\sum_{n=1}^{\infty}\frac{3^n n^2}{n!}$ is absolutely convergent, and so it is convergent. (Therefore, there is no need to test for conditionally convergency.)

6: If $c_n \geq 0$ and $\sum c_n 6^n$ is convergent, is $\sum c_n(-4)^n$ absolutely convergent? conditionally convergent? or divergent?

Solution (Thinking before doing: two series are given, one is convergent and the other is to be tested. We can use comparison test to test for convergence.) Let $a_n = |c_n(-4)^n| = c_n 4^n$.

(i) **Which test will be applied?** We use **Comparison Test**: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
(a) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is convergent.
(b) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is divergent.

(ii) **How the test is applied?** Let $b_n = c_n 6^n$. Then as, $4 < 6$, for each $n \geq 1$,

$$a_n = c_n 4^n \leq c_n 6^n = b_n.$$

It is known that $\sum c_n 6^n$ is convergent.
(iii) **Conclusion.** By the comparison test, $\sum c_n(-4)^n$ is absolutely convergent, and so it is convergent.

7: Determine if $\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{5^n n!}$ is absolutely convergent, conditionally convergent or divergent.

Solution (i) **Which test will be applied?** We use **Ratio Test**: For a series $\sum a_n$, let $L = \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}$.

(a) If $L < 1$, then $\sum a_n$ converges absolutely, (therefore, it is convergent).
(b) If $L > 1$, then $\sum a_n$ diverges.
(c) If $L = 1$, then the Root test is inconclusive.

(ii) **How the test is applied?** Let $a_n = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{5^n n!}$. Compute

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)\cdot(2(n+1)-1)}{5^{n+1}(n+1)!} \cdot \frac{5^n n!}{1\cdot 3\cdot 5\cdots(2n-1)} \\ &= \lim_{n\to\infty} \frac{2n+1}{5(n+1)} = \frac{2}{5}. \end{aligned}$$

(iii) **Conclusion.** By the Ratio Test, and as $L < 1$, $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$ is absolutely convergent, and so it is convergent. (Therefore, there is no need to test for conditionally convergency.)

8: Determine if $\sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n}$ is absolutely convergent, conditionally convergent or divergent.

Solution (i) **Which test will be applied?** This is not a positive termed series. We shall **first test for absolute convergency**. If it is not absolute convergent, **then we test for conditional convergency**.
(ii) **Is it absolutely convergent?** Let $a_n = |\frac{(-2)^n}{n2^n}| = \frac{1}{n}$. As a p-series with $p = 1$ is divergent, $\sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n}$ is NOT absolutely convergent.
(iii) **Is it conditionally convergent?** This is an alternating series, we use **Alternating Series Test:** If the alternating series $\sum(-1)^n b_n$ satisfies
(i) $b_{n+1} \geq b_n \geq 0$ for all n, and
(ii) $\lim_{n\to\infty} b_n = 0$, then $\sum(-1)^n b_n$ converges.

How the test is applied? We compute

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0,$$

and verify

$$a_n = \frac{1}{n} > \frac{1}{n+1} = a_{n+1}.$$

(iv) **Conclusion.** By the Alternating Series Test, $\sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n}$ is convergent. Since it is not absolutely convergent, $\sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n}$ is conditionally convergent.

9: Suppose that $a_n \geq 0$. If $\sum_{n=1}^{\infty} a_n(5^n)$ converges, then $\sum_{n=1}^{\infty} a_n(-3)^n$

(circle only one) (i) is always convergent (ii) may be divergent.

10: To test the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ for convergency/divergency, we can use (circle all appropriate ones)

Answer: (i) Ratio Test, (ii) Root Test, (iii) Neither Ratio Test nor Root Test.

Solution (i) Can we use Ratio Test? We compute

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1}{(n+1)^2} \cdots \frac{n^2}{1} = 1.$$

As $L = 1$, using Ratio Test is inconclusive.

(ii) Can we use Root Test? We compute

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{n^{\frac{2}{n}}} = \lim_{n\to\infty} n^{-\frac{2}{n}}.$$

We try to compute $\lim_{n\to\infty} x^{-\frac{2}{x}}$ in Calculus I. (See Section 3.7 of your text book). Let $y = x^{\frac{-2}{x}}$. Then

$$\ln(y) = \frac{-2}{x} \ln\left(\frac{1}{x}\right) = \frac{-2}{x} (\ln(1) - \ln(x)) = \frac{2\ln(x)}{x}.$$

As $x \to \infty$ means $\ln(x) \to \infty$, by L'Hospital's Rule, we have

$$\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} \frac{2\ln(x)}{x} = \lim_{x\to\infty} \frac{2}{x} = 0.$$

Hence $\lim_{x\to\infty} x^{\frac{-2}{x}} = e^{\lim_{x\to\infty} \ln(y)} = e^0 = 1$, and so

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{n^{2/n}} = 1.$$

As $L = 1$, using Ratio Test is inconclusive.

(iii) **Conclusion:** Neither Ratio Test nor Root Test.

Extra Credit Problem (the extra credit will be added to your score of Text 3. Example: If your Test 3 score is 100, and if you earn 3 points from this problem, then your Test 3 score will be 103/100).

Motivation of this exercise: We run into the problem of testing convergence for $\sum_{n=2}^{\infty} \frac{1}{\ln n}$. We tried integral test but we, including the professor, do not know how to compute $\int \frac{dx}{\ln(x)}$. Therefore, comparison test was suggested in class. How can we compare $\frac{1}{\ln n}$ with a p-series? This leads to this exercise.

Hint: We only need to use our knowledge of Calculus I. Remember, to see if a differentiable function $f(x)$ is increasing, we can compute to see if $f'(x)$ is positive.

Extra Credit Problem. (4 points total, 1 point for each question). Let $f(x) = x - \ln(x)$. Do each of the following.
(i) Find the domain of $f(x)$ and compute $f(1)$.
(ii) Compute $f'(x)$ and explain why $f'(x) > 0$ for all $x > 1$.
(iii) Explain why you can use (ii) to conclude that $f(x) > 0$ for all $x \geq 1$. (Hint: If $f(x)$ is increasing on $[1, \infty)$, then for any $x > 1$, we have $f(x) \geq f(1)$.)
(iv) Explain how you can use the above the show that for $n \geq 2$, $\frac{1}{n} \leq \frac{1}{\ln n}$, for all $n \geq 2$.

Solution (i) Since the domain of $\ln(x)$ is $(0, \infty)$, and the domain of x is $(-\infty, \infty)$, the domain of $f(x)$ is the intersection of the two: $(0, \infty)$.
(ii) Compute to get $f'(x) = 1 - \dfrac{1}{x}$. When $x > 1$, $\dfrac{1}{x} < 1$, and so $f'(x) > 0$ when $x > 1$.
(iii) From Calculus I, $f(x)$ is an increasing function in$[1, \infty)$, and so for any $x > 1$, $f(x) \geq f(1) = 1 > 0$.
(iv) From (iii), we know that if $x > 1$, then $f(x) = x - \ln(x) > 0$, or $x > \ln(x)$. Thus for $n \geq 2$, we have $n > \ln(n)$. Hence $\dfrac{1}{n} > \dfrac{1}{\ln n}$.