Math 156 Fall 2015 Quiz 3 Solutions

1: (Similar to Section 6.6, Exercise 25) Evaluate the integral $\int_0^3 \frac{dx}{\sqrt{x}}$.

Solution. The domain of $\frac{1}{\sqrt{x}}$ is $(0, \infty)$, and the integration interval is $[0, 3]$. Therefore, this is an improper integral with a discontinuity at $x = 0$. Choose $\delta > 0$ and compute

$$\int_{0+\delta}^3 \frac{dx}{\sqrt{x}} = \left.\frac{\sqrt{x}}{\frac{1}{2}}\right|_\delta^3 = 2\sqrt{3} - 2\sqrt{\delta}.$$

Then find the value of the improper integral

$$\int_0^3 \frac{dx}{\sqrt{x}} = \lim_{\delta \to 0^+} 2\sqrt{3} - 2\sqrt{\delta} = 2\sqrt{3}.$$

2: (Section 6.6, Exercise 12) Evaluate the integral $\int_0^\infty xe^{-x^2} dx$.

Solution. The domain of xe^{-x^2} is $(-\infty, \infty)$, and the integration interval is $[0, +\infty]$. Therefore, this is an improper integral over an infinite interval. Choose $A > 0$ and compute (use $u = -x^2$ and so $\frac{-1}{2} du = xdx$)

$$\int_0^A xe^{-x^2} dx = -\frac{1}{2} \int_0^{-A^2} e^u du = \left.\frac{-e^u}{2}\right|_0^{-A^2} = -\frac{e^{-A^2}}{2} - \frac{-1}{2} = \frac{1}{2} - \frac{1}{2e^{A^2}}.$$

Then find the value of the improper integral

$$\int_0^\infty xe^{-x^2} dx = \lim_{A \to \infty} \left(\frac{1}{2} - \frac{1}{2e^{A^2}} \right) = \frac{1}{2} - \lim_{A \to \infty} \frac{1}{2e^{A^2}} = \frac{1}{2}.$$

3: (Similar to Section 7.1, Exercises 9 and 10) Find the area of the the region bounded by $y = \frac{x^2}{4}$, $y = 5 - x^2$.

Solution. Sketching the graphs is recommended.
Bounds: As $y = 5 - x^2$ is the top curve and $y = \frac{x^2}{4}$ is the bottom curve, we should have x-bounds. Solve the system $y = \frac{x^2}{4}$ and $y = 5 - x^2$ for x, we have $\frac{x^2}{4} = 5 - x^2$ or $x^2 = 20 - 4x^2$. Thus $x^2 = 4$ or $(x - 2)(x + 2) = 0$. This gives $a = -2$ and $b = 2$.
Integral: (For those students who known the integration property of even functions, we can use the property. Otherwise, we can compute the area of half of theregion and multiply the result by 2, as seen below). The area is

$$\int_{-2}^2 \left((5 - x^2) - \frac{x^2}{4} \right) dx = 2 \int_0^2 \left(5 - \frac{5x^2}{4} \right) dx = 10 \int_0^2 \left(1 - \frac{x^2}{4} \right) dx.$$

Evaluation of integral: The area is

$$10 \int_0^2 \left(1 - \frac{x^2}{4} \right) dx = 10 \left[x - \frac{x^3}{12} \right]_0^2 = 10 \left(2 - \frac{8}{12} \right) = 10 \left(2 - \frac{2}{3} \right) = 10 \cdot \frac{4}{3} = \frac{40}{3}.$$

4: (Section 7.1, Exercise 12) Set up an integral to find the area of the the region bounded by $y = x$, $4x + y^2 = 12$. (**Do not evaluate the integral.**)

Solution. Sketching the graphs is recommended. For this purpose, write $4x + y^2 = 12$ as $x = (12 - y^2)/4 = 3 - \frac{y^2}{4}$.

Bounds: As $y = x$ is on the left of $x = 3 - \frac{y^2}{4}$, we should have y-bounds. Solve the system $y = x$ and $4x + y^2 = 12$ for y, we have $4y + y^2 = 12$ or $y^2 + 4y - 12 = 0$. Thus $(y - 2)(4 + 6) = 0$. This gives $c = -6$ and $d = 2$.

Integral: The area is $\displaystyle\int_{-6}^{2} \left(3 - \frac{y^2}{4} - y\right) dy$.

Evaluation of integral: (Not required.) The area is

$$\begin{aligned}
\int_{-6}^{2} \left(3 - \frac{y^2}{4} - y\right) dy &= \left[3y - \frac{y^3}{12} - \frac{y^2}{2}\right]_{-6}^{2} \\
&= \left(6 - \frac{8}{12} - \frac{4}{2}\right) - \left(-18 - \frac{(-6)^2 \cdot (-6)}{12} - \frac{36}{2}\right) \\
&= 6 - \frac{2}{3} - 2 + 18 - 18 + 18 = 22 - \frac{2}{3}.
\end{aligned}$$

Grade Distribution of this quiz:

Meaning of the scores:

9, 10 = Very good, familiar with the related materials and skillful, with minimal computational errors. Keep on!

8 = good, familiar with most of the related materials, with a few computations errors. Make an effort to do better.

7 = Passing, lack of familiarity of the related materials and more computational errors and algebraic errors. Set our goal as "familiar with the skills" instead of just knowing it.

6 = Borderline. We need to catch it up. If you have trouble doing your homework, it might be time to visit your instructor to get help. Do not wait to let the trouble accumulate.

At most 5 = This might be a dangerous warning signal. **We are failing!** It should definitely be the time for us to see the instructor and get assistance to understand the materials and to practice MORE.

Scores	10	9	8	7	6	≤ 5
Frequency	5	6	7	9	4	4
Percentage	14.3	17.2	20.0	25.7	11.4	11.4

Discussions and Comments

(1) **Problem:** Differentiation Errors occurred most common in solution of Problem 2. Many tried to use by-parts and set $v = e^{-x^2}$. Many of us have a variety of expression of dv.

The correct dv should be $-2xe^{-x^2}$. If we correctly computed this derivative, we could have changed our mind and directly compute the integral $\displaystyle\int xe^{-x^2}dx = \frac{-e^{-x^2}}{2} + C$.

(2) **Problem:** Various common integration errors have been observed as listed below.

Errors	Correct Way
$\int xe^{-x^2}dx = \frac{x^2}{2} - \frac{e^{-x^2}}{2x}$	Use $u = e^{-x^2}$ and $du = -2xe^{-x^2}dx$ (This would work!)
$\int \frac{dx}{\sqrt{x}} = \ln\|\sqrt{x}\|$	$\int \frac{dx}{\sqrt{x}} = \int x^{-1/2}dx = \frac{x^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} + C = 2\sqrt{x} + C$

(3) **Problem:** Not using parentheses when they are needed. Some of the algebraic errors are caused because of this. Point deduction will be applied in the future, and in all the exams.