Math156 Review for Exam 3

1. What will be covered in this exam: Using integration to solve differential equations with initial conditions, computing limits of sequences and determining convergence of sequences, computing sums of series and testing series for convergence, power series and their convergence radius and interval of convergence.

2. Exam Rules: This is a closed note and close-text exam. Formula sheet is not allowed. Calculators of any kind will not be allowed. Any electronic devices, including a cell phone, will **not** be allowed.

3. Expectations In this exam, students are expected to do each of the following:
(A) (**differential equations**) to apply integration to solve differential equations with initial conditions.
(B) (**sequences**) to understand the meaning of the nth term of a sequence, to determine if a sequence is convergent, and if possible, to compute the limit of a sequence when it is convergent.
(C) (**series**) to understand the relationship between the sum of a series and the sequence of the nth partial sum of the series, to apply various of convergence/divergence tests to determine if a a given series is convergent/divergent.
(D) (**power series**) to understand the what the radius of convergence and interval of convergence of a power series are, and to compute the radius and the interval of convergence for a give power series.

4. Summary Sheet (distributed in class)

5. Warming Up Exercises

By working on these and similar exercises, you will become more familiar with the related concepts and skills. It will help to warm you up and prepare well for the coming exam. Please note that there is no implication of any kind that any of these problem will be in the exam.

Differential Equations Solve the following differentia equations.

(1) $\dfrac{dy}{dx} = y\sin(x)$ and $y(0) = \frac{1}{2}$. (Answer: $y = \frac{1}{\cos(x)+1}$.)

(2) $\dfrac{ydy}{dx} = 2x$ and $y(1) = 2$. (Answer: $y^2 = 2x^2 + 2$.)

(3) $(3y^2 + 2y)\dfrac{dy}{dx} = x\cos(x)$ and $y(0) = 1$. (Answer: $y^3 + y^2 = \cos(x) + x\sin(x) + 1$.)

(4) $\dfrac{dx}{dt} = 1 - t + x - tx$ and $x(0) = 2$. (Answer: $x = -1 + 3e^{t-t^2/2}$.)

Sequences Determine if the given sequence is convergent or divergent, and find the limit if convergent.

(1) $a_n = \dfrac{n}{1-2n}$. (Ans: convergent, limit is $\frac{-1}{2}$.)

(2) $a_n = \dfrac{n^2}{2^n - 1}$. (Ans: convergent, limit is 0.)

(3) $a_n = \dfrac{(-1)^n}{n!}$. (Ans: convergent, limit is 0.)

(4) $a_n = \dfrac{n}{1-2n}$. (Ans: convergent, limit is $\frac{-1}{2}$.)

(5) $a_n = \cos(\dfrac{n\pi}{2})$. (Ans: divergent.)

(6) $a_n = \dfrac{\sin(n)}{n}$. (Ans: convergent, limit is 0.)

(7) $a_n = \dfrac{\ln(n^2)}{n}$. (Ans: convergent, limit is 0.)

(8) $a_n = \dfrac{(\ln(n))^2}{n}$. (Ans: convergent, limit is 0.)

(9) $a_n = \dfrac{n^{1000}}{e^n}$. (Ans: convergent, limit is 0.)

(10) $a_n = \dfrac{n^{0.01}}{(\ln(n))^2}$. (Ans: divergent.)
(11) $a_n = \tan^{-1}(2n)$. (Ans: convergent, limit is $\frac{\pi}{2}$.)
(12) $a_n = \ln(2n^2+1) - \ln(n_2+1)$. (Ans: convergent, limit is $\ln(2)$.)
(13) $a_n = \dfrac{2+n^3}{1+2n^3}$. (Ans: convergent, limit is $\frac{1}{2}$.)
(14) $a_n = \dfrac{9^{n+1}}{10^n}$. (Ans: convergent, limit is 0.)
(15) $a_n = \dfrac{n^3}{1+n^2}$. (Ans: divergent.)
(16) $a_n = \cos(\dfrac{n\pi}{2}$. (Ans: divergent.)
(17) $a_n = \dfrac{n\sin(n)}{1+n^2}$. (Ans: convergent, limit is 0.)
(18) $a_n = \dfrac{\ln(n)}{n}$. (Ans: convergent, limit is 0.)
(19) $a_n = (1+\dfrac{3}{n})^{4n}$. (Ans: convergent, use logarithm to find that limit is e^{12}.)
(20) $a_n = \dfrac{(-10)^n}{n!}$. (Ans: convergent, consider "the tail" when $n \geq 11$, limit is 0.)

Find the nth term a_n of each of the following sequence:
(1) $\dfrac{2}{1}, \dfrac{4}{3}, \dfrac{8}{5}, \dfrac{16}{7}, \dfrac{32}{9}, \cdots$ (Ans: $a_n = \dfrac{2^n}{2n-1}$.)
(2) $1+\dfrac{1}{2}, 1+\dfrac{3}{4}, 1+\dfrac{7}{8}, 1+\dfrac{15}{16}, 1+\dfrac{31}{32}, \cdots$ (Ans: $a_n = 1+\dfrac{2^n-1}{2^n}$.)
(3) $\dfrac{1}{6}, \dfrac{2}{12}, \dfrac{3}{20}, \dfrac{4}{30}, \dfrac{5}{42}, \cdots$ (Ans: Note that $6 = 2\cdot 3$, $12 = 3\cdot 4$, $20 = 4\cdot 5$ and $30 = 5\cdot 6$, $42 = 6\cdot 7$, ...$a_n = \dfrac{n}{(n+1)(n+2)}$.)

Series: Convergence Testing/Sum Computing Determine if each of the following series is convergent or divergent, and find the sum if convergent.
(1) $\displaystyle\sum_{n=1}^{\infty} \frac{(-6)^{n-1}}{5^{n-1}}$. (Ans: divergent.)
(2) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(\sqrt{2})^n}$. (Ans: convergent, sum is $2+\sqrt{2}$.)
(3) $\displaystyle\sum_{n=0}^{\infty} \frac{3}{2^n}$. (Ans: convergent, sum is 6.)
(4) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{4n^2-1}$. (Ans: convergent, sum is 1.)
(5) $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1}$. (Ans: convergent, sum is $\frac{3}{2}$.)
(6) $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2+4n+3}$. (Ans: convergent, sum is $\frac{6}{5}$.)
(7) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5n}$. (Ans: divergent.)
(8) $\displaystyle\sum_{n=2}^{\infty} \frac{1+2^n}{3^n}$. (Ans: convergent, sum is $\frac{3}{2}$.)
(9) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{5n+1}$. (Ans: divergent.)

Express the number as a ratio of integers.
(1) $0.\overline{73} = 0.73737373...$ (Ans: $0.\overline{73} = 73\sum_{n=1}^{\infty}(\frac{1}{10^2})^n = \frac{73}{99}$.)
(2) $6.2\overline{54} = 6.2545454...$ (Ans: $6.2\overline{54} = 6.2 + \frac{54}{10}\sum_{n=1}^{\infty}(\frac{1}{10^2})^n = \frac{62}{10} + \frac{54}{990} = \frac{6192}{990} = \frac{344}{55}$.)

Convergence Testing

(1) $\sum_{n=1}^{\infty}\left(\frac{5}{n^4}+\frac{4}{n\sqrt{n}}\right)$ (Use any of: integral test, comparison test, limiting comparison test, Ans: convergent)

(2) $\sum_{n=1}^{\infty}\frac{n^2}{n^3+1}$ (Use any of: integral test, Ans: divergent)

(3) $\sum_{n=1}^{\infty}\frac{n^2-1}{5n^4+1}$ (Use any of: comparison test, limiting comparison test, Ans: convergent)

(4) $\sum_{n=1}^{\infty}\frac{4+3^n}{2^n}$. (Use any of: general term not going to zero, or comparison test, Ans: divergent)

(5) $\sum_{n=1}^{\infty}\frac{n+5}{\sqrt[3]{n^7+n^2}}$ (Use any of: comparison test, limiting comparison test, Ans: convergent)

(6) $\sum_{n=1}^{\infty}\frac{n+5}{\sqrt[3]{n^6+n^2}}$. (Use any of: limiting comparison test, Ans: divergent)

(7) $\sum_{n=1}^{\infty}\frac{n+5}{n4^n}$. (Use any of: comparison test, limiting comparison test, Ans: convergent)

(8) $\sum_{n=1}^{\infty}\frac{5}{\sqrt{n^2+2}}$. (Use any of: limiting comparison test, Ans: divergent)

(9) $\sum_{n=1}^{\infty}\frac{\cos^2(n)}{n^2+1}$ (Use any of: comparison test, limiting comparison test, Ans: convergent)

(10) $\sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right)$ (Use any of: limiting comparison test, Ans: divergent)

(11) $\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+1}$ (Use: alternating series test, Ans: convergent, not absolute convergent)

(12) $\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt{n}}{\sqrt{n}+1}$ (Use: general term not going to zero, Ans: divergent)

(13) $\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^3+1}$ (Use: alternating series test, Ans: convergent, absolute convergent)

(14) $\sum_{n=1}^{\infty}\frac{(-1)^n(3n-1)}{2n+3}$ (Use: general term not going to zero, Ans: divergent)

(15) $\sum_{n=1}^{\infty}\frac{(-2)^n}{n^3}$ (Use: ratio test, Ans: absolute convergent)

(16) $\sum_{n=2}^{\infty}\frac{(-1)^n}{n(\ln n)^2}$ (Use: integral test, Ans: absolute convergent)

(17) $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[4]{n}}$ (Use: alternating series test, Ans: convergent, not absolute convergent)

(18) $\sum_{n=1}^{\infty}(-1)^n\frac{4^n}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)}$ (Use: ratio test, Ans: divergent)

(19) $\sum_{n=1}^{\infty}(-1)^n\frac{(2^n)}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)}$ (Use: ratio test, Ans: convergent)

(20) $\sum_{n=1}^{\infty}(-1)^n\left(\frac{n^2+1}{2n^2+3}\right)^n$ (Use: root test, Ans: absolute convergent)

(21) $\sum_{n=2}^{\infty}\frac{(-10)^n}{(n+1)4^{2n+2}}$ (Use: ratio test, Ans: absolute convergent)

(22) $\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{(2n-1)!}$ (Use: ratio test, Ans: absolute convergent)

(23) $\sum_{n=1}^{\infty}\frac{n^3}{5^n}$ (Use: ratio test, And: absolute convergent)

(24) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ (Use: alternating series test, Ans: convergent, not absolute convergent)

(25) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{(\ln(n))^3}}$ (Use: integral test, And: convergent)

(26) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{3n+1}\right)$ (Use: Divergency test, Ans: divergent)

(27) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{n+1}$ (Use: alternating series test, And: convergent, not absolute convergent)

(28) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$ (Use: limiting comparison test with a 3/2-series, Ans: convergent)

(29) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{(-1)^n\sqrt{n}}{\ln(3n+1)}\right)$ (Use: Divergency test, Ans: divergent)

(30) $\displaystyle\sum_{n=2}^{\infty}(-1)^n n^{-3}$ (Use: comparison test, Ans: absolutely convergent)

Ratio Test:
(1) Which of the following series is the Ratio Test inconclusive?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$.

(b)) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{\sqrt{n}}$.

(d)) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$.

(2) What can you say about the series $\sum a_n$ in each of the following cases?

(a) $\displaystyle\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = 0.8$.

(b) $\displaystyle\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = 8$.

(c) $\displaystyle\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = 1$.

Power Series Find the radius of convergence and interval of convergence of each of the following series

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$ (Ans: $r = 1$ and $I = [-1, 1)$)

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n+3}$ (Ans: $r = 1$ and $I = (-1, 1]$)

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n^3}$ (Ans: $r = 1$ and $I = [-1, 1]$)

(4) $\displaystyle\sum_{n=1}^{\infty} \sqrt{n}x^n$ (Ans: $r = 1$ and $I = (-1, 1)$)

(5) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!}$ (Ans: $r = \infty$ and $I = (-\infty, \infty)$)

(6) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$ (Ans: $r = 3$ and $I = [-3, 3)$)

(7) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{5^n n^5}$ (Ans: $r = 5$ and $I = [-5, 5]$)

(8) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{\sqrt{n}}$ (Ans: $r = 1/2$ and $I = (-1/2, 1/2]$)

(9) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{3^n \ln(n)}$ (Ans: $r = 3$ and $I = (-3, 3]$)

(10) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ (Ans: $r = \infty$ and $I = (-\infty, \infty)$)

(11) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n2^n}$ (Ans: $r = 2$ and $I = (-4, 0]$)

(12) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n (x+3)^n}{\sqrt{n}}$ (Ans: $r = 1/2$ and $I = (-7/2, -5/2]$)

(13) $\displaystyle\sum_{n=1}^{\infty} \frac{n(x-4)^n}{n^3+1}$ (Ans: $r = 1$ and $I = [3, 5]$)

(14) $\displaystyle\sum_{n=1}^{\infty} n!(2x-1)^n$ (Ans: $r = 0$ and $I = \{1/2\}$)

(15) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 x^n}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}$ (Ans: $r = \infty$ and $I = (-\infty, \infty)$)

(16) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2 5^n}$ (Ans: $r = 5$ and $I = [-5, 5]$)

(17) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{n4^n}$ (Ans: $r = 4$ and $I = [-6, 2)$)

(18) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n (x-2)^n}{(n+2)!}$ (Ans: $r = \infty$ and $I = (-\infty, \infty)$)

(19) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n (x-3)^n}{\sqrt{n+3}}$ (Ans: $r = 1/2$ and $I = [5/2, 7/2)$)