Math156 Review for Exam 1

1. What will be covered in this exam: Fundamental Theorem of Calculus, The average of a function over a closed interval, Integration skills (substitution rule, integration by parts, integrations with trigonometric functions, trigonometric substitutions, integration using partial fraction decompositions).

2. Exam Rules: This is a closed note and close-text exam. Formula sheet is not allowed. Calculators of any kind will not be allowed. Any electronic devices, including a cell phone, will not be allowed.

3. In this exam, students are expected:
(A) (**Fundamental Theorem of Calculus**) to understand what the FTC means and how to use it.
(B) (**the average of a function over a closed interval**) to know how to compute the average of a function over a closed interval.
(C) (**substitution rule**) to understand the relationship between substitution rule and the differentiation chain rule, and to become so experienced that for common integrals in Calculus 2, we can quickly determine whether the problem can be done by using substitution rule and what is an appropriate substitution to take.
(D) (**integration by parts**) to understand the relationship between substitution rule and the differentiation product rule, and to become so experienced that for common integrals in Calculus 2, we can quickly determine whether the problem can be done by using integration by parts and what is an appropriate choices for u and v.
(E) (**integrations with trigonometric functions**) to be **very** familiar with the **differentiation formulas** related to all six trigonometric functions and with the double angle/half angle formulas, and to gain sufficient experienced in using appropriate substitutions and integration by parts to evaluate such integrals.
(G) (**trigonometric substitutions**) to be **very** familiar with the trigonometric function identities related to trigonometric substitutions, and to become so experienced that for common integrals in Calculus 2, we can quickly determine whether the problem can be done by using trigonometric substitutions and what is an appropriate such substitution.
(H) (**partial fractions**) to understand how to compute the partial fraction decomposition of a reactional function.

4. Related Formulas/Procudure

4.1. Substitution If the integral has the form $\displaystyle\int f(u(x))u'(x)dx$ and if $\int f(u)du$ can be evaluated, then set $u = u(x)$.

4.2. Integration by parts

$$\int udv = uv - \int vdu.$$

4.3. Partial Fraction Decomposition
The procedure of finding partial fraction decompositions of fractional functions $\dfrac{P(x)}{Q(x)}$:

Step 1: Use long division to reduce the degree of the numerator until the degree of the numerator is less than that of the denominator $Q(x)$.

Step 2: When the degree of $P(x)$ is less than that of $Q(x)$, factor $Q(x)$ completely.

Step 3: Write down the partial fractions corresponding to the complete factoring of $Q(x)$, as follows. For any factor $(ax-b)^n$, the corresponding partial fractions are

$$\frac{A_1}{ax-b} + \frac{A_2}{(ax-b)^2} + \cdots + \frac{A_n}{(ax-b)^n}.$$

For any factor $(ax^2+bx+c)^n$ with $b^2 < 4ac$, the corresponding partial fractions are

$$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}.$$

Step 4: Use the comparison technique and/or the vanishing a factor (or variable) technique to find the constants A_i's, B_i' and the C_i's.

5. Warming Up Exercises

By working on these and similar exercises, you will become more familiar with the related concepts and skills. It will help to warm you up and prepare well for the coming exam. Please note that there is no implication of any kind that any of these problem will be in the exam (midterm or final).

Fundamental Theorem of Calculus and Its Applications For each of the function $F(x)$ below, find the derivative $F'(x)$.

(1) $F(x) = \int_0^x \tan^4(t)dt$. (Ans: $F'(x) = \tan^4(x)$).

(2) $F(x) = \int_{\sqrt{x}}^1 \cos(t^3)dt$. (Ans: $F'(x) = -\cos((\sqrt{x})^3) \cdot \dfrac{1}{2\sqrt{x}}$).

(3) $F(x) = \int_0^{x^2} e^{\sqrt{1+t^3}}dt$. (Ans: $F'(x) = e^{\sqrt{1+x^6}} \cdot (2x)$.

(4) $F(x) = \int_{\sqrt{x}}^{x^2+1} \sin(t^3)dt$. (Ans: $F'(x) = \int_{\sqrt{x}}^{x^2+1} \sin((x^2+1)^3) \cdot (2x) - \sin((\sqrt{x})^3 \cdot \dfrac{1}{2\sqrt{x}}$).

For each of the function $F(x)$ below, determine the intervals on which the function $F(x)$ is increasing, and those on which $F(x)$ is decreasing.

(5) $F(x) = \int_0^x (t-1)(t+2)dt$. (Ans: $F'(x) = (x-1)(x+2)$. Thus $F'(x) > 0$ on $(-\infty, -2) \cup (1, \infty)$ and $F'(x) < 0$ on $(-2, 1)$.)

(6) $F(x) = \int_0^{x^3-3x} \sqrt{1+t^4}dt$. (Ans: $F'(x) = \sqrt{1+(x^3-3x)^4} \cdot 3(x-1)(x+1)$. Thus $F'(x) > 0$ on $(-\infty, -1) \cup (1, \infty)$ and $F'(x) < 0$ on $(-1, 1)$.)

Average of a function Find the average of the following function on the given interval.

(1) $f(x) = x^2$ on $[-1, 1]$. (Ans: $\dfrac{1}{1-(-1)} \int_{-1}^1 x^2dx = \dfrac{1}{3}$).

(2) $f(x) = \sec(x)\tan(x)$ on $[0, \frac{\pi}{4}]$. (Ans: $\dfrac{1}{\frac{\pi}{4} - 0} \int_0^{\frac{\pi}{4}} \sec(x)\tan(x)dx = \dfrac{4(\sqrt{2}-1)}{\pi}$).

Substitution Rule Evaluate the integrals

(1) $\int x(x^2-1)^4dx$. (Ans: $u = x^2 - 1$, $\frac{(x^2-1)^5}{10} + C$.)

(2) $\int x(x^2+1)dx$. (Ans: $u = x^2 + 1$, $\frac{(x^2+1)^3}{6} + C$.)

(3) $\int \sqrt{2x-1}dx$. (Ans: $u = 2x - 1$, $\frac{(2x-1)^{3/2}}{3} + C$.)

(4) $\int x\sqrt{2x-1}dx$. (Ans: $u = 2x - 1$, $\frac{(2x-1)^{5/2}}{10} + \frac{(2x-1)^{3/2}}{6} + C$.)

(5) $\int \sin^2(3x)\cos(3x)dx$. (Ans: $u = \cos(3x)$, $\frac{\sin^3(3x)}{9} + C$.)

(6) $\int x^2(x^3-1)^4dx$. (Ans: $u = x^3 - 1$, $\frac{(x^3-1)^5}{15} + C$.)

(7) $\int x\sin(x^2)dx$. (Ans: $u = x^2$, $-\frac{\cos(x^2)}{2} + C$.)

(8) $\int \tan^4(x)\sec^2(x)dx$. (Ans: $u = \tan(x)$, $\frac{\tan^5(x)}{5} + C$.)

(9) $\int \frac{\sin(x)}{\cos^2(x)} dx$. (Ans: $\sec(x) + C$.)

(10) $\int x^2\sqrt{1-x}dx$. (Ans: $-\frac{2}{105}(1-x)^{3/2}(15x^2+12x+8)+C$.)

(11) $\int_0^1 x(1+x^2)^3 dx$. (Ans: $u = x^2+1$, $\frac{15}{8}$.)

(12) $\int_1^5 \frac{x}{\sqrt{2x-1}} dx$. (Ans: $u = \sqrt{2x-1}$, $\frac{16}{3}$.)

(13) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$. (Ans: $u = \sqrt{x}$, $2e^{\sqrt{x}} + C$.)

(14) $\int \frac{(\ln(x))^2}{x} dx$. (Ans: $u = \ln(x)$, $\frac{(\ln(x))^3}{3} + C$.)

(15) $\int \frac{1+x}{1+x^2} dx$. (Ans: $\tan^{-1}(x) + \frac{\ln(1+x^2)}{2} + C$.)

(16) $\int \sec^3(x)\tan(x)dx$. (Ans: $u = \sec(x)$, $\frac{\sec^3(x)}{3} + C$.)

(17) $\int \frac{e^x}{e^x+1} dx$. (Ans: $u = e^x+1$, $\ln(e^x+1)+C$.)

(18) $\int \sec^2(x)e^{\tan(x)}dx$. (Ans: $u = \tan(x)$, $e^{\tan(x)} + C$.)

(19) $\int \frac{2}{x\ln(x^2)} dx$. (Ans: $u = \ln(x)$, $\ln|\ln|x|| + C$.)

(20) $\int \frac{\cos(x)}{1+\sin(x)} dx$. (Ans: $u = 1+\sin(x)$, $\ln|1+\sin(x)| + C$.)

Integration By Parts Evaluate the integrals

(1) $\int x^2\ln(x^2)dx$. (Ans: $u = 2\ln(x), dv = x^2dx$, $\frac{x^3\ln(x)}{3} - \frac{x^3}{9} + C$.)

(2) $\int x^2\sin(x)dx$. (Ans: $u = x^2, dv = \sin(x)dx$, $-x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$.)

(3) $\int \sin^{-1}(x)dx$. (Ans: $x\sin^{-1}(x) + \sqrt{1-x^2} + C$.)

(4) $\int x\sec^2(x)dx$. (Ans: $u = x$, $x\tan(x) - \ln|\cos(x)| + C$.)

(5) $\int (x^2-1)e^x dx$. (Ans: $u = e^x$, $(x-1)^2e^x + C$.)

(6) $\int x\ln(x+1)dx$. (Ans: $u = \ln(x+1)$, $\frac{1}{4}[2(x^2-1)\ln|x+1| - x^2 + 2x] + C$.)

(7) $\int_1^2 (\ln(x))^2 dx$. (Ans: $2(\ln 2)^2 - 4\ln(2) + 2$.)

(8) $\int e^x\cos(2x)dx$. (Ans: $\frac{e^x}{5}(\cos(2x) + 2\sin(2x)) + C$.)

(9) $\int \frac{xe^x}{(x+1)^2} dx$. (Ans: $u = xe^x, dv =$
$fracdx(x+1)^2$, $\frac{e^x}{x+1} + C$.)

(10) $\int \sin(\ln(x))dx$.

Trigonometric Integrals Evaluate the integrals

(1) $\int \sin^3(x)\cos^4(x)dx$. (Ans: $u = \cos(x)$, $\frac{-\cos^5(x)}{5} + \frac{\cos^7(x)}{7} + C$.)

(2) $\int \sin^2(x)\cos^5(x)dx$. (Ans: $u = \sin(x)$, $\frac{\sin^3(x)}{3} - \frac{2\sin^5(x)}{5} + \frac{\sin^7(x)}{7} + C$.)

(3) $\int_0^{\pi/3} \frac{\sin^3(x)}{\sqrt{\cos(x)}} dx$. (Ans: $u = \cos(x)$, $\left[\frac{2\cos^{5/2}(x)}{5} - 2\cos^{1/2}(x)\right]_0^{\pi/3} = \frac{32-19\sqrt{2}}{20}$.)

(4) $\int_0^{\pi/2} \cos^4(x)dx$. (Ans: $\left[\frac{3x}{8} + \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32}\right]_0^{\pi/2} = \frac{3\pi}{16}$.)

(5) $\displaystyle\int \frac{\tan^3(x)}{\sqrt{\sec(x)}}dx$. (Ans: $u=\sec(x)$, $(\frac{2}{3}(\sec(x))^{3/2}+2(\sec(x))^{-1/2}+C.$)

(6) $\displaystyle\int \tan^3(3x)\sec^4(3x)dx$. (Ans: $u=\tan(3x)$, $\frac{\tan^4(3x)}{12}+\frac{\tan^6(3x)}{18}+C.$)

(7) $\displaystyle\int \tan^4(x)dx$. (Ans: $u=\tan(x)$, $\frac{\tan^3(x)}{3}-\tan(x)+x+C.$)

(8) $\displaystyle\int \csc^4(x)\cot^4(x)dx$. (Ans: $u=\cot(x)$, $-\frac{\cot^5(x)}{5}-\frac{\cot^7(x)}{7}+C.$)

(9) $\displaystyle\int \frac{\sec(x)}{\tan^2(x)}dx$. (Ans: convert to $\sin(x)$ and $\cos(x)$, $\frac{-1}{\sin(x)}+C.$)

Trigonometric Substitutions Evaluate the integrals

(1) $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}}$. (Ans: $x=3\sin(\theta)$, $\frac{-\sqrt{9-x^2}}{9x}+C.$)

(2) $\displaystyle\int \frac{dx}{\sqrt{4x^2+1}}$. (Ans: $2x=\tan(\theta)$, $\frac{1}{2}\ln|\sqrt{4x^2+1}+2x|+C.$)

(3) $\displaystyle\int \frac{dx}{(x^2+1)^{3/2}}$. (Ans: $x=\tan(\theta)$, $\frac{x}{\sqrt{x^2+1}}+C.$)

(4) $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}dx$. (Ans: $x=\sqrt{3}\sec(\theta)$, $\sqrt{3}\int_0^{\pi/6}\left[\sec^2(\theta)-1\right]d\theta=1-\frac{\sqrt{3}\pi}{6}$.)

(5) $\displaystyle\int \frac{dx}{(x^2+1)^2}$. (Ans: $x=\tan(\theta)$, $\frac{1}{2}\left[\tan^{-1}(x)+\frac{x}{x^2+1}\right]+C.$)

Partial Fractions Write the form of the partial fraction decomposition of the following function, Do not determine the numerical value of the coefficients.

(1) $\displaystyle\frac{5}{x^2-10x}$. (Ans: $\frac{A}{x}+\frac{B}{x-10}$).

(2) $\displaystyle\frac{4x^2+3}{(x-5)^3}$. (Ans: $\frac{A_1}{x-5}+\frac{A_2}{(x-5)^2}+\frac{A_3}{(x-5)^3}$).

(3) $\displaystyle\frac{x^3+12x-3}{x^3+10x}$. (Ans: $1+\frac{A}{x}+\frac{Bx+C}{x^2+10}$).

(4) $\displaystyle\frac{4x^2+3}{(x-5)^3}$. (Ans: $\displaystyle\frac{Bx+C}{4x^2+3}+\frac{A_1}{x-5}+\frac{A_2}{(x-5)^2}+\frac{A_3}{(x-5)^3}$.)

(5) $\displaystyle\frac{16x}{x^3-10x^2}$. (Ans: $\displaystyle\frac{B}{x-10}+\frac{A_1}{x}+\frac{A_2}{x^2}$.)

Evaluate the integrals

(1) $\displaystyle\int \frac{5x^2+20x+6}{x^3+2x^2+x}dx$. (Ans: $\displaystyle\int\left(\frac{6}{x}-\frac{1}{x+1}+\frac{9}{(x+1)^2}\right)dx=\ln\left|\frac{x^6}{x+1}\right|-\frac{9}{x+1}+C.$)

(2) $\displaystyle\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}dx$. (Ans: $\displaystyle\int\left(\frac{2}{x}-\frac{2}{x-1}+\frac{2x+4}{x^2+4}\right)dx=\ln\left(\frac{x^2(x^2+4)}{(x-1)^2}\right)+\tan^{-1}\frac{x}{2}+C.$)

(3) $\displaystyle\int \frac{8x^2+13x}{(x^2+2)^2}dx$. (Ans: $\displaystyle\int\left(\frac{8x}{x^2+2}-\frac{3x}{(x^2+2)^2}\right)dx=4\ln(x^2+2)+\frac{3}{2(x^2+2)}+C.$)

(4) $\displaystyle\int \frac{x^2+2x-1}{x^3-x}dx$. (Ans: $\displaystyle\int\frac{x^2+2x-1}{x^3-x}dx=\int\left[\frac{1}{x}-\frac{1}{x+1}+\frac{1}{x-1}\right]dx=\ln\left|\frac{x(x-1)}{x+1}\right|+C.$)

(5) $\displaystyle\int \frac{x^3}{(x-3)(x+2)^2}dx$. (Ans: $\displaystyle\int\frac{x^3}{(x-3)(x+2)^2}dx=\int\left[\frac{9/25}{x-3}+\frac{16/25}{x+2}-\frac{4/5}{(x+2)^2}\right]dx=\frac{9}{25}\ln|x-3|+\frac{16}{25}\ln|x+2|+\frac{4}{5(x+2)}+C.$)

(6) $\displaystyle\int \frac{x^2-x+6}{x^3+3x}dx$. (Ans: $\displaystyle\int\frac{x^2-x+6}{x^3+3x}dx=\int\left[\frac{2}{x}-\frac{x}{x^2+3}-\frac{1}{x^2+3}\right]dx=2\ln|x|-\frac{\ln(x^2+3)}{2}-\frac{1}{\sqrt{3}}\cdot\tan^{-1}\frac{x}{\sqrt{3}}+C.$)

(7) $\displaystyle\int \frac{x^3}{x^3+1}dx$. (Ans: $\displaystyle\int\frac{x^3}{x^3+1}dx=\int\left[1-\frac{1/3}{x+1}+\frac{x-2}{3(x^2-x+1)}\right]dx=x-\frac{\ln|x+1|}{3}-\frac{\ln|x^2-x+1|}{6}-\frac{1}{\sqrt{3}}\cdot\tan^{-1}\frac{2x-1}{\sqrt{3}}+C.$)

(8) $\int \frac{x^4+1}{x(x^2+1)^2}dx$. (Ans: $\int \frac{x^4+1}{x(x^2+1)^2}dx = \int \left[\frac{1}{x} - \frac{2x}{(x^2+1)^2}\right] dx = \ln|x| + \frac{1}{x^2+1} + C$.)