Mod $(2p+1)$-orientation of Graphs

Hong-Jian Lai

Department of Mathematics, West Virginia University

Motivation: Score Sequence

- Suppose that n teams $v_1, v_2, \cdots, v_n$ are in a tournament. A non negative integer sequence $s_1, s_2, \cdots, s_n$ is a score sequence if it is a possible outcome that there is a permutation π on $\{1, 2, \cdots, n\}$ such that for each i, the team v_i will win exactly $s_{\pi(i)}$ games in the tournament.

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- Landau's score sequence problem: Given a non negative integer sequence $s_1, s_2, \cdots, s_n$, how do you know if this sequence is a score sequence?

Score Sequence

- Theorem 1 (Landau 1953) Let $n \geq 1$ be an integer. A nondecreasing sequence $(s_1, s_2, \cdots, s_n)$ of nonnegative integers is a score sequence if and only if

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2}, \forall k \text{ with } 1 \leq k \leq n,$$

where equality holds if and only if $n = k$.

Graph Formulation

- G: = a graph with vertices $\{v_1, v_2, \cdots, v_n\}$. (Representing the teams).

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- $\forall v \in V(G)$, $d_D^+(v) =$ # of edges directed from v (out-degree), $d_D^-(v) =$ # of edges directed into v (in-degree).
- Formulation: A function $c : V(G) \mapsto \mathbf{Z}$ represents a score sequence if and only if the complete graph K_n has an orientation D such that at each v, $d_D^+(v) = c(v)$.

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$D(G)$: = an orientation of G. (Team v_i beats Team v_j is represented as an oriented edge from v_i to v_j).

Problem: Suppose that G is a graph (not necessarily complete). Given an integer valued function $c : V(G) \mapsto \mathbf{Z}$, can we find an orientation $D(G)$ such that at each v, $d_D^+(v) = c(v)$?

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- Get an orientation D on $E(G)$: each edge e is oriented so that the face with greater color number is on the right side of the oriented edge $e \Leftrightarrow$ the greater $-$ the smaller $= 1$.

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- Get an orientation D on $E(G)$: each edge e is oriented so that the face with greater color number is on the right side of the oriented edge e $\Leftrightarrow$ the greater $-$ the smaller $= 1$.
- The resulted orientation D satisfied that $\forall v$, $d_D^+(v) - d_D^-(v) \equiv 0$ (mod 3), (called a mod 3-orientation of G).

Face 3-Coloring of Planar Graphs

- **Theorem** (Tutte, 1954) A plane graph G is face-3-colorable if and only if G has a mod 3-orientation.

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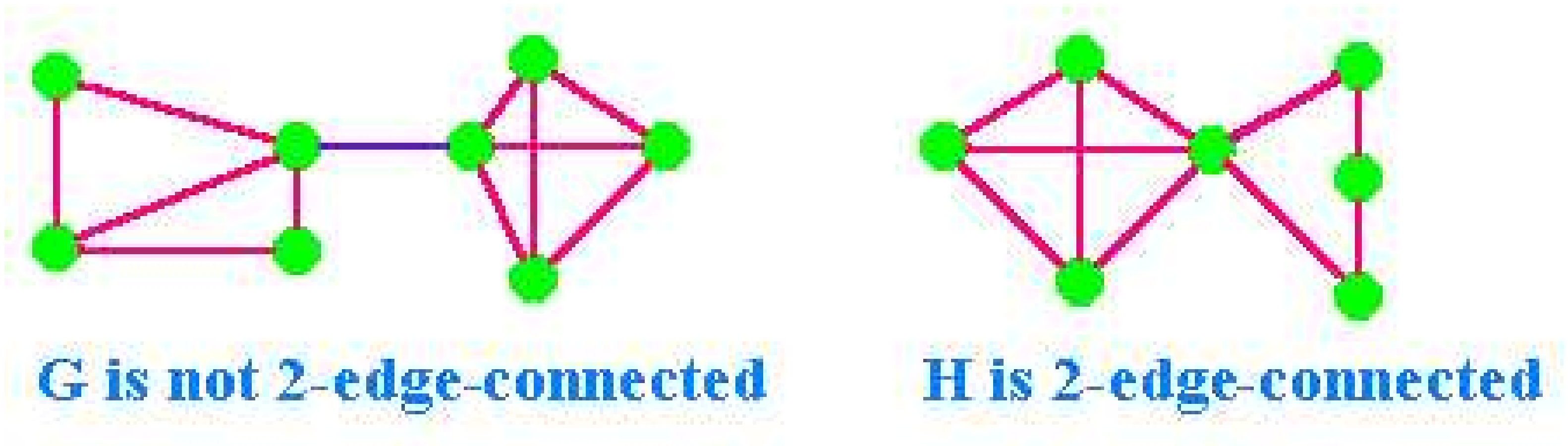
- 

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- Fact: There exists a 3-edge-connected graph that does not have a mod 3-orientation. (For example, K_4).
- Conjecture (Jaeger 1988) There exists an integer $k \geq 4$ such that every k-edge-connected graph has a mod 3-orientation.

Graph Formulation

- Tutte's mod 3-orientation problem: (also known as the 3-flow Problem) Given a graph G, can we find an orientation $D(G)$ such that at v, $d_D^+(v) - d_D^-(v) \equiv 0$ (mod 3)? Or: does G have a mod 3-orientation? (NP-complete even within planar graphs).

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- Problem: Let A be an abelian (additive) group. Given an A-valued function $b : V(G) \mapsto A$, can we find an orientation $D(G)$ such that at v, $d_D^+(v) - d_D^-(v) = b(v)$ in A?

Notation

G: = a graph, with vertex set

$$V = V(G) = \{v_1, v_2, \cdots, v_n\},$$

and edge set

$$E = E(G) = \{e_1, e_2, \cdots, e_m\}.$$

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- $D = (d_{ij})_{n\times m}$: = vertex-edge incidence matrix, where

$$d_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is oriented away from } v_i \\ -1 & \text{if } e_j \text{ is oriented into } v_i \\ 0 & \text{otherwise} \end{cases} .$$

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- $\mathbf{1} = (1, 1, \cdots, 1)^T$ is a vector each of whose component equals 1.
- The ith row (component) of $D\mathbf{1}$ equals the net out degree of v_i.

Non-Homogeneous System Formulation

- A: = an abelian (additive) group with identity 0, and with $|A| \geq 3$.

Description of an Orientation

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- Suppose we reverse the orientation of e_i in D to obtain a new orientation D'. If $\exists f : E \mapsto \{1, -1\}$ such that $f^{-1}(-1) = \{e_i\}$, then $D\mathbf{1} = D'f$. Therefore, with an arbitrarily given orientation of G, any other orientation of G can be viewed as a function $\exists f : E \mapsto \{1, -1\}$.

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- For an undirected graph G, whether G has a mod $(2p+1)$-orientation or not is independent of the choice of the orientation of G.

Mod $(2p+1)$-orientation of Graphs

- A necessary Condition: If $\forall b : V(G) \mapsto \mathbf{Z}_{2p+1}$, $\exists f : E \mapsto \{1, -1\}$ such that $Df = b$ over $\mathbf{Z}_{2p+1}$, then

$$\sum_{v \in V(G)} b(v) \equiv 0 \pmod{2p+1}.$$

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$$\sum_{v \in V(G)} b(v) \equiv 0 \ (\text{mod } 2p+1).$$

- Proof: View f as an orientation D. Then

$$\sum_{v \in V(G)} b(v) = \sum_{v \in V(G)} [d_D^+(v) - d_D^-(v)].$$

Each edge is counted on the right hand side exactly twice, once positive and once negative.

Graphs That Are Mod ($2p+1$)-Contractible

- If $\forall b : V(G) \mapsto \mathbf{Z}_{2p+1}$ with $\sum_{v \in V(G)} b(v) \equiv 0 \pmod{2p+1}$, $\exists f : E \mapsto \{1, -1\}$ such that $Df = b$ over $\mathbf{Z}_{2p+1}$, then we say that G is mod ($2p+1$)-contractible.

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- Every mod $(2p+1)$-contractible graph has a mod $(2p+1)$-orientation.

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- Every mod $(2p+1)$-contractible graph has a mod $(2p+1)$-orientation.
- For an undirected graph G, whether G is mod $(2p+1)$-contractible or not is independent of the choice of the orientation of G.
- Examples: $2K_2$ and K_5 are mod 3-contractible.

Graphs That Are Mod $(2p+1)$-Contractible

- Proposition: If H is a subgraph of G, and if H is mod $(2p+1)$-contractible, then the following are equivalent:
  (i) G has mod $(2p+1)$-orientation,
  (ii) the contraction G/H has mod $(2p+1)$-orientation.

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  (i) G has mod $(2p+1)$-orientation,
  (ii) the contraction G/H has mod $(2p+1)$-orientation.
- **Proof:** Note that mod $(2p+1)$-contractible implies mod $(2p+1)$-orientation. This follows from Theorem 2 of next page.

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- Let $H \subseteq G$ (a connected subgraph of G). The contraction G/H is obtained by identifying all vertices of H into a single vertex and removing all edges of H.

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- Theorem 1 (Z.-H. Chen, H.-J. Lai, H. Y. Lai 2001) Suppose that $H \subseteq G$ and that H is mod 3-contractible. Then G is mod 3-contractible if and only if G/H is mod 3-contractible.

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- Theorem 2 (Lai, Shao, Wu and Zhou 2006) Suppose that $H \subseteq G$ and that H is mod $(2p+1)$-contractible. Then G is mod $(2p+1)$-contractible if and only if G/H is mod $(2p+1)$-contractible.

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- Theorem 7 (Steinburg and D. H. Younger 1989, Thomassen 1994) Every 4-edge-connected projective planar graph has a mod 3-orientation.
- Theorem 8 (Lai and Zhang 1992) Every $4\log_2(|V(G)|)$-edge-connected graph has a mod 3-orientation.

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- Let G be an undirected graph, D be an orientation of G. Let $S \subseteq V(G)$ be a vertex subset.

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Recent Progress

- Necessity: Let $c : V(G) \mapsto \mathbf{Z}$. If G has an orientation D such that $d_D^+(v) = c(v), \forall v \in V(G)$, then $\forall S \subseteq V(G)$

$$|E(S)| \leq \sum_{v \in S} c(v) \leq |E(S)| + |\partial_G(S)|.$$

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- A function $c : V(G) \mapsto \mathbf{Z}$ satisfying inequality above will be called a feasible function of G.

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- $\forall S \subseteq V(G)$, $\sum_{v \in S} c(v) = \sum_{v \in S} d_D^+(v)$.
- $|E(S)| = \sum_{v \in S} d_D^+(v) - |\delta_D^+(S)| \leq \sum_{v \in S} d_D^+(v)$
  $\leq \sum_{v \in S} d_D^+(v) + |\delta_D^-(S)| = |E(S)| + |\partial_G(S)|$.

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- (ii) c is a feasible function of G. That is, $\forall S \subseteq V(G)$

$$|E(S)| \leq \sum_{v \in S} c(v) \leq |E(S)| + |\partial_G(S)|.$$

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- Corollary (Landau 1953) Let $n \geq 1$ be an integer. A nondecreasing sequence $(s_1, s_2, \cdots, s_n)$ of nonnegative integers is a score sequence if and only if

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2}, \forall k \text{ with } 1 \leq k \leq n,$$

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- **Proof** Directly verify that the function s is feasible.

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- (i) G is mod $(2p+1)$-contractible.
- (ii) $\forall b : V(G) \mapsto \mathbf{Z}$ satisfying both

$$\sum_{v \in V(G)} b(v) \equiv 0 \ (\text{mod } 2p+1)$$

and

$$b(v) \equiv d_G(v) \ (\text{mod } 2), \forall v \in V(G),$$

G has an orientation D such that $d_D^+(v) - d_D^-(v) \equiv b(v)$ $(\text{mod } 2p+1)$, $\forall v \in V(G)$.

Recent Progress

- Corollary (Lai, Shao, Wu and Zhou, 2006) Let G be a $(4p+1)$-regular graph. Then G has a mod $(2p+1)$-orientation if and only if $V(G)$ has a partition (V^+, V^-) such that $\forall U \subseteq V(G)$,

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- Proof of the "only if" part Suppose G has a mod 3-orientation D. Since G is 5-regular, $\forall v \in V(G)$, either $d_D^+(v) = 4p$ or $d_D^+(v) = 1$. Define $V^+ = \{v \in V(D) : d_D^+(v) = 4p\}$ and $V^- = V(D) - V^+$. Apply Theorem 10.

Recent Progress

- Proof of the "if" part Define a map $b : V(G) \mapsto \mathbf{Z}$ satisfying $b(V^+) = \{2p+1\}$ and $b(V^-) = \{-2p-1\}$. Since G is $(4p+1)$-regular, $\forall v \in V(G)$, $b(v) \equiv d_G(v)$ (mod 2).

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- Corollary (Da Silva and Dahad, 2005) Let G be a 5-regular graph. Then G has a mod 3-orientation if and only if $V(G)$ has a partition (V^+, V^-) such that $\forall U \subseteq V(G)$,

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- Proof Let $p = 1$.

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- (i) $V(G)$ is a disjoint union $V(G) = V_1 \dot{\bigcup} V_2$ with $|V_1| = k$, $|V_2| = n - k$, and

$$\left\lceil \frac{|E(V_1, V_2)| + 1}{k} \right\rceil + \left\lceil \frac{|E(V_1, V_2)| + 1}{n - k} \right\rceil \leq 4p + 2.$$

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- (ii) $V(G)$ is a disjoint union $V(G) = V_1 \dot{\bigcup} V_2$ with $|V_1| = k$, $|V_2| = n - k$, and

$$|E(V_1, V_2)| \leq \frac{(4p+2)k(n-k)}{n}.$$

Recent Progress

- Example: For any positive $p \in \mathbf{Z}$, K_{4p+1} is mod $(2p+1)$-contractible.

Recent Progress

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- Proof: $n = 4p+1$. By Theorem 11, $V(K_n)$ can be partitioned into two subsets V_1 and V_2 with $|V_1| = k$ and $|V_2| = n-k$ satisfying inequality Theorem 11(ii). Since $|E(V_1,V_2)| = k(n-k)$, we have

$$
\begin{aligned}
& \left\lceil \frac{|E(V_1,V_2)|+1}{k} \right\rceil + \left\lceil \frac{|E(V_1,V_2)|+1}{4p-k} \right\rceil \\
= & \left\lceil \frac{k(n-k)+1}{k} \right\rceil + \left\lceil \frac{|k(n-k)+1}{n-k} \right\rceil \\
= & (n-k+1)+(k+1) = n+2 > 4p+2.
\end{aligned}
$$

Recent Progress

- **Theorem 12** (Lai, Shao, Wu and Zhou 2006) Let n, p be positive integers, and let $f(n) = \frac{(2p+1)n\log_2(n)}{2}$ be a function. If G is a graph with n vertices and if $|E(G)| \geq f(n)$, then G has a subgraph H with $E(H) \neq \emptyset$ which is mod $(2p+1)$-contractible.

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- **Theorem 13** (Lai, Shao, Wu and Zhou 2006) Let G be a graph with n vertices. If G is $(2p+1)\log_2(n)$-edge-connected, then G is mod $(2p+1)$-contractible.
- **Proof:** Use connectivity to count the number of edges and use Theorem 12 to find a contractible subgraph.

Recent Progress

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- Corollary Let G be a graph with n vertices. If G is $3\log_2(n)$-edge-connected, then G is mod 3-contractible.

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- Corollary Let G be a graph with n vertices. If G is $3\log_2(n)$-edge-connected, then G is mod 3-contractible.
- Theorem 8 (Lai and Zhang 1992) Let G be a graph with n vertices. If G is $4\log_2(n)$-edge-connected, then G has a mod 3-orientation.

$K_{1,3}$-Decomposition

- A claw is an induced $K_{1,3}$

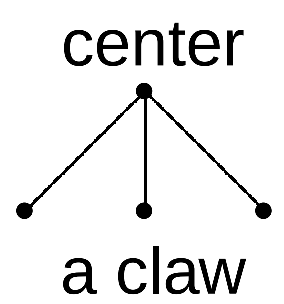

Figure 1.3

$K_{1,3}$-Decomposition

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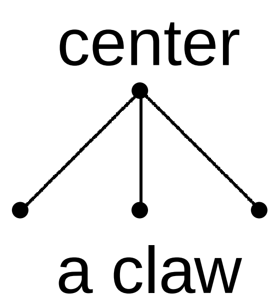

Figure 1.3

- A connected loopless graph with 3 edges and a vertex of degree 3 is called a generalized claw.

$K_{1,3}$-Decomposition

- A graph G with $|E(G)| \equiv 0$ (mod 3) has a claw-decomposition if $E(G)$ can be partitioned into disjoint unions $E(G) = X_1 \cup X_2 \cup \cdots \cup X_k$ such that for each i with $1 \leq i \leq k$, $G[X_i]$ is a generalized claw.

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- Theorem 14 (Barat and Thomassen 2004) If there exists an integer k such that every k-edge-connected graph G with $|E(G)| \equiv 0 \pmod 3$ has a claw-decomposition, then every k-edge-connected graph G has a mod 3-orientation.

$K_{1,3}$-Decomposition

- Conjecture 15 (Barat and Thomassen 2004) Every 4-edge-connected simple planar graph G with $|E(G)| \equiv 0 \pmod 3$ has a claw-decomposition.

$K_{1,3}$-Decomposition

- Conjecture 15 (Barat and Thomassen 2004) Every 4-edge-connected simple planar graph G with $|E(G)| \equiv 0 \pmod 3$ has a claw-decomposition.
- Question in our minds: How do we approach this conjecture?

$K_{1,2p+1}$-Decomposition

- A connected loopless graph with $2p + 1$ edges and a vertex of degree $2p + 1$ is called a generalized $K_{1,2p+1}$.

$K_{1,2p+1}$-Decomposition

- A connected loopless graph with $2p+1$ edges and a vertex of degree $2p+1$ is called a generalized $K_{1,2p+1}$.
- A graph G with $|E(G)| \equiv 0$ (mod $2p+1$) has a $K_{1,2p+1}$-decomposition if $E(G)$ can be partitioned into disjoint unions $E(G) = X_1 \cup X_2 \cup \cdots \cup X_k$ such that for each i with $1 \leq i \leq k$, $G[X_i]$ is a generalized $K_{1,2p+1}$.

$K_{1,2p+1}$-Decomposition

- Theorem 16 (Lai, Shao, Wu and Zhou 2006) Fix $k > 0$. The every k-edge-connected (planar) graph G is mod $(2p+1)$-contractible if and only if every k-edge-connected (planar) graph G with $|E(G)| \equiv 0$ (mod $2p+1$) has a $K_{1,2p+1}$-decomposition.

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- **Proof** Apply Theorems 16 and 17 when $p = 1$.

$K_{1,2p+1}$-Decomposition

- **Question 18** Is there an integer k such that every k-edge-connected planar graph G with $|E(G)| \equiv 0 \pmod 3$ has a $K_{1,3}$-decomposition?

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- **Corollary 20** Every 5-edge-connected planar graph with $|E(G)| \equiv 0 \pmod 3$ has a claw-decomposition.

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Thank You!