Graphs with mod $(2p+1)$-orientations and mod $(2p+1)$-contractible graphs

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Flow/coloring duality for plane graphs

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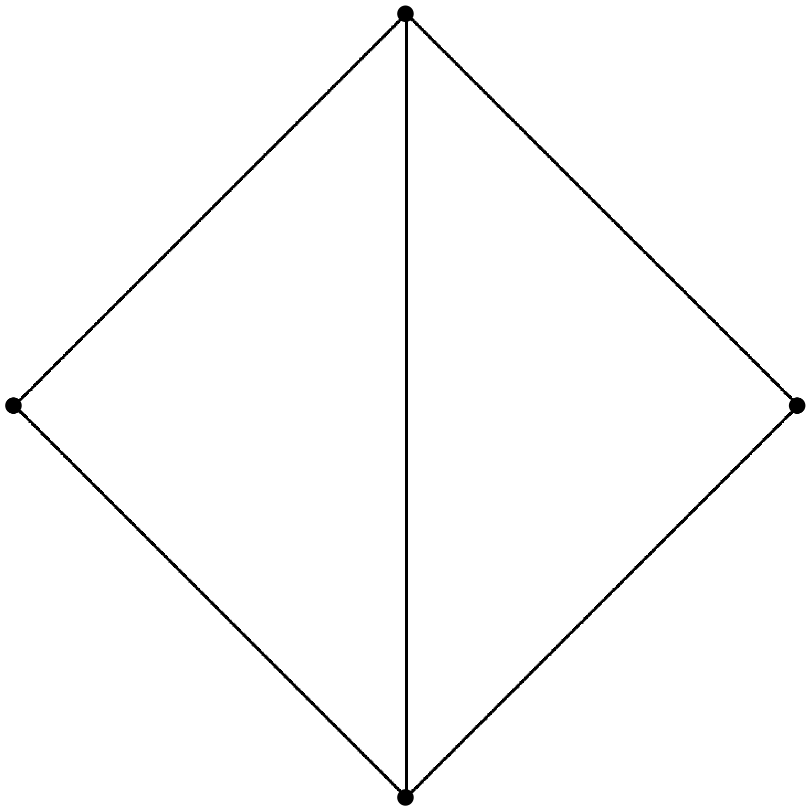
Flow/coloring duality for plane graphs

- Four Color Theorem: Given a plane graph, the faces can be colored using at most four colors so that no two adjacent faces have the same color.
- Given a plane graph G, a k-face-coloring is a map $f: \{\text{faces}\} \to \{0, 1, \cdots, k-1\}$ such that no two adjacent faces have the same color.

Flow/coloring duality for plane graphs



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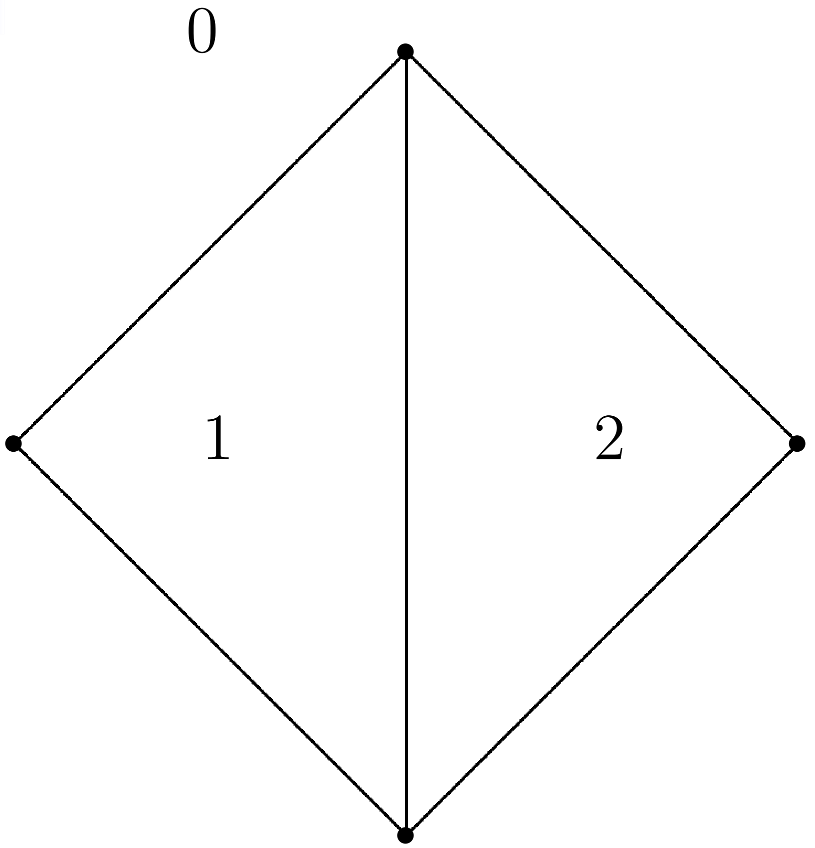

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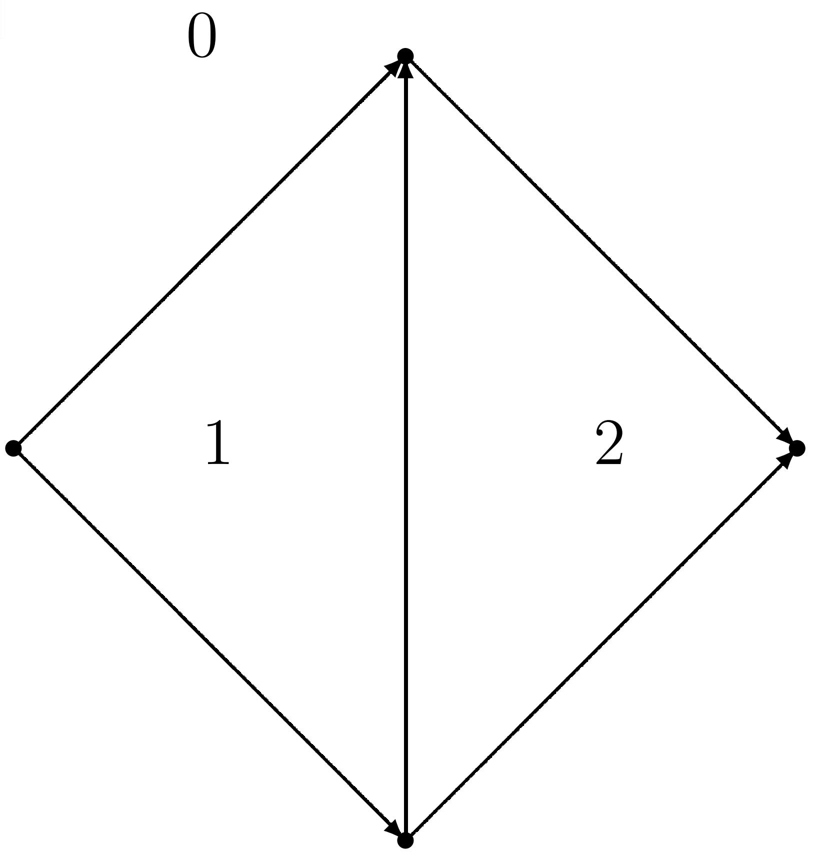

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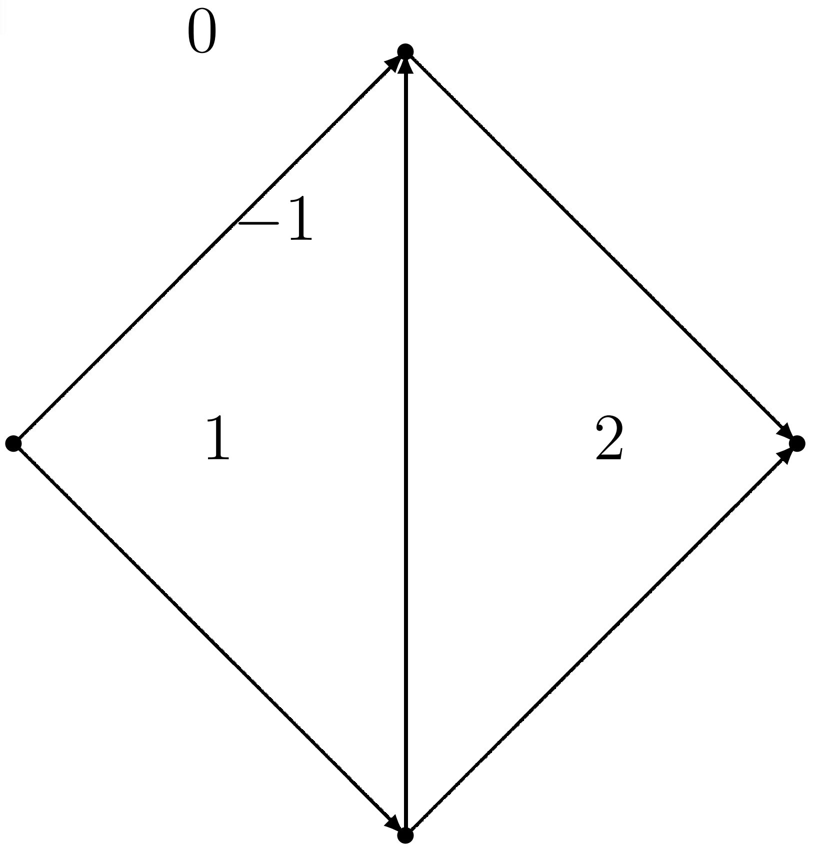

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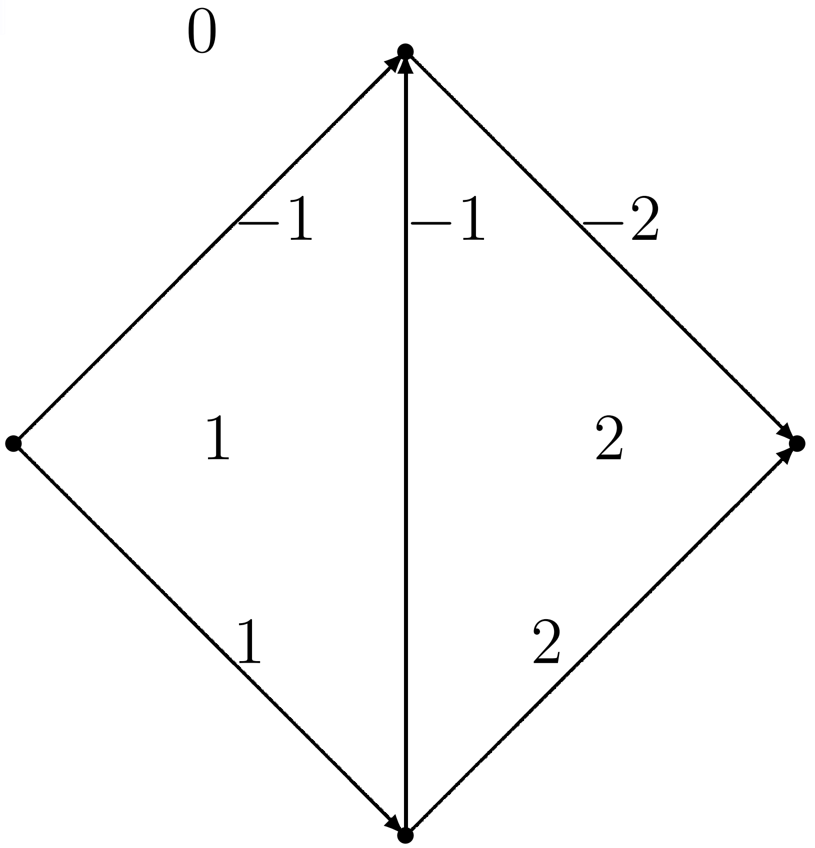

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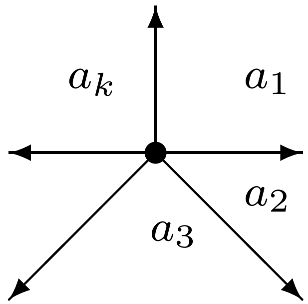

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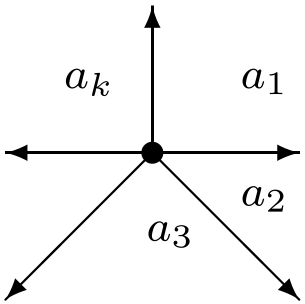
- 

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$$\sum_{e\in\delta^+(v)} \Phi(e) - \sum_{e\in\delta^-(v)} \Phi(e) = \sum_{e\in\delta^+(v)} \Phi(e)$$

$$= (a_1 - a_2) + (a_2 - a_3) + \cdots + (a_k - a_1) = 0$$

.

Flow/coloring duality for plane graphs

- A map $f : E(G) \mapsto \{-(k-1), \cdots, k-1\} - \{0\}$ is a **nowhere-zero k-flow** (k-NZF) if under orientation D,

$$\partial f(v) = \sum_{e \in \delta^+(v)} f(e) - \sum_{e \in \delta^-(v)} f(e) = 0$$

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- An undirected G has an NZF or not is independent of the choice of the orientation of G.
- Theorem (Tutte): Let G and G^* are plane dual graphs. Then G^* is k-colorable if and only if G has a k-NZF.

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- $\mathbf{Z}_{2p+1}$: = the additive group of order $2p+1$.

Mod 3-orientation

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- Observation: A graph G, under orientation D, has $\mathbf{Z}_3$-NZF $f : E(G) \mapsto \{1, -1\}$. Can adjust the orientation so that $f \equiv 1$.
- Observation: If $f \equiv 1$ in $\mathbf{Z}_3$, then at $\forall v$, $d^+_D(v) - d^-_D(v) \equiv 0$ (mod 3).

Mod $(2p+1)$-orientation

- Incidence matrix $D = (a_{ij})$ of G with orientation:

$$a_{ij} = \begin{cases} 1 & \text{if edge } e_j \text{ is directed from } v_i \\ -1 & \text{if edge } e_j \text{ is directed into } v_i \\ 0 & \text{if edge } e_j \text{ is not incident to } v_i \end{cases}$$

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- An undirected G has a mod $(2p+1)$-orientation or not is independent of the choice of the orientation of G.

Mod $(2p+1)$-orientation

- Let M_{2p+1} denote the set of all graphs which have mod $(2p+1)$-orientations.

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- Conjecture (Jaeger, Selected Topics in Graph Theory, III, Beineke and Willson): Every (4p)-edge-connected graph is in M_{2p+1}.

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- Theorem (Zhang & HJL, 1992 in DM) Every $4\log_2(|V(G)|)$-edge-connected graph is in M_3.

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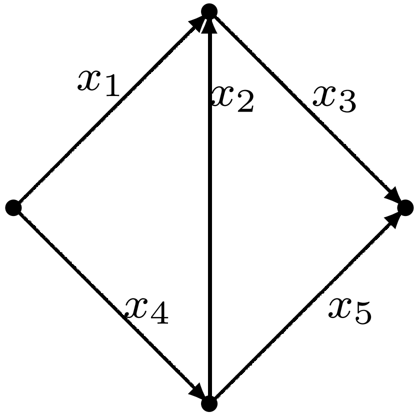

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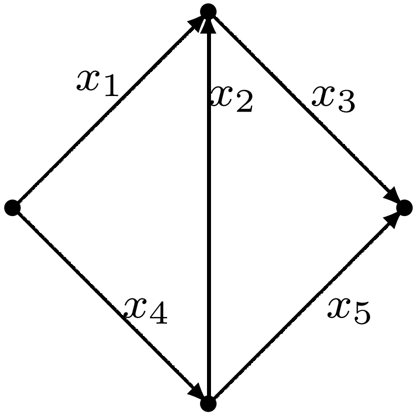

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$$\left\{\begin{array}{llllllll} -x_1 & + & (-x_2) & + & x_3 & \equiv & 0 & \\ x_1 & + & x_4 & \equiv & 0 & & & x_i \in \{\pm 1\} \\ -x_3 & + & (-x_5) & \equiv & 0 & & & \\ -x_4 & + & x_2 & + & x_5 & \equiv & 0 & \end{array}\right.$$

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- **Question:** When does $Df \equiv \vec{b}$ (mod $2p+1$) have a ± 1 solution f in $\mathbf{Z}_{2p+1}$?

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- A necessary Condition: If $\forall b : V(G) \mapsto \mathbf{Z}_{2p+1}$, $\exists f : E \mapsto \{1, -1\}$ with $Df = b$ in $\mathbf{Z}_{2p+1}$, then $\sum_{v \in V(G)} b(v) \equiv 0 \pmod{2p+1}$.

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- Proof: View f as an orientation D.

$$\sum_{v \in V(G)} b(v) = \sum_{v \in V(G)} [d_D^+(v) - d_D^-(v)].$$

Each edge occurs as positive and negative exactly once.

Non-Homogeneous System Formulation

- Define M^o_{2p+1} to be the collection of graphs such that $G \in M^o_{2p+1}$ if and only if for any zero sum function b of G in $\mathbf{Z}_{2p+1}$, there exists an orientation D such that $d^+_D(v) - d^-_D(v) \equiv b(v)$ (mod $2p+1$) at every vertex $v \in V(G)$.

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- $M^o_{2p+1} \subset M_{2p+1}$ (take b = 0).
- Graphs in M^o_{2p+1} will be referred as mod $(2p+1)$-contractible graphs, for a postponed reason.

Contractible graphs with respect to mod $(2p+1)$-orientations

- For an undirected graph G, whether G is in M^{o}_{2p+1} or not is independent of the choice of the orientation of G.

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- Examples: $2K_2$ and K_5 are mod 3-contractible.
- **Conjecture** (Lai, 2007 in SIAM JDM, Shao, Wu, Zhou & HJL, 2009 in JCTB): Every $(4p+1)$-edge-connected graph is in M^o_{2p+1}.

Orientation with given out degrees

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Orientation with given out degrees

- Necessity: Let $c : V(G) \mapsto \mathbf{Z}$. If G has an orientation D such that $d_D^+(v) = c(v), \forall v \in V(G)$, then $\forall S \subseteq V(G)$

$$|E(S)| \leq \sum_{v \in S} c(v) \leq |E(S)| + |\partial_G(S)|.$$

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- A function $c : V(G) \mapsto \mathbf{Z}$ satisfying inequality above will be called a feasible function of G.

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- $|E(S)| = \sum_{v \in S} d_D^+(v) - |\delta_D^+(S)| \leq \sum_{v \in S} d_D^+(v)$
  $\leq \sum_{v \in S} d_D^+(v) + |\delta_D^-(S)| = |E(S)| + |\partial_G(S)|$.

Orientation with given out degrees

- Theorem (Hakimi, 1965, Shao, Wu, Zhou & HJL, 2009 in JCTB) Let G be a graph, and let $c : V(G) \mapsto \mathbf{Z}$ be a function. The following are equivalent.

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- (i) G has an orientation D such that $d_D^+(v) = c(v), \forall v \in V(G)$.
- (ii) c is a feasible function of G. That is, $\forall S \subseteq V(G)$

$$|E(S)| \leq \sum_{v \in S} c(v) \leq |E(S)| + |\partial_G(S)|.$$

Application to score sequence

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- Corollary 10 (Landau 1953) $0 \le s_1 \le s_2 \le \cdots \le s_n$ of nonnegative integers is a score sequence if and only if

$$\sum_{i=1}^{k} s_i \ge \binom{k}{2}, \forall k \text{ with } 1 \le k \le n,$$

where equality holds if and only if $n = k$.

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where equality holds if and only if $n = k$.
- Proof Directly verify that the function s is feasible.

Application to orientation with given net out degrees

- **Theorem** (Shao, Wu, Zhou & HJL, 2009 in JCTB) Let G be a graph and $b : V(G) \mapsto \mathbf{Z}$ be a function such that $\sum_{v \in V(G)} b(v) = 0$ and $b(v) \equiv d_G(v)$ (mod 2), $\forall v \in V(G)$.
  Then G has an orientation D such that $d_D^+(v) - d_D^-(v) = b(v)$, $\forall v \in V(G)$ if and only if for any $\forall S \subseteq V(G)$

$$|\sum_{v \in S} b(v)| \leq |\partial_G(S)|.$$

Application to M^o_{2p+1}

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Application to M^o_{2p+1}

- Theorem (Shao, Wu, Zhou & HJL, 2009 in JCTB) Let G be a graph. The following are equivalent.
- (i) $G \in M^o_{2p+1}$.
- (ii) $\forall b : V(G) \mapsto \mathbf{Z}$ satisfying both

$$\sum_{v \in V(G)} b(v) \equiv 0 \pmod{2p+1}$$

and

$$b(v) \equiv d_G(v) \pmod{2}, \forall v \in V(G),$$

G has an orientation D such that $d^+_D(v) - d^-_D(v) \equiv b(v) \pmod{2p+1}$, $\forall v \in V(G)$.

Application to M_{2p+1}

- Fact Using the vertex splitting method, it is known that to prove $\kappa'(G) \geq 4p \Longrightarrow G \in M_{2p+1}$, it suffices to show that this holds for $(4p+1)$-regular graphs.

Application to M_{2p+1}

- **Fact** Using the vertex splitting method, it is known that to prove $\kappa'(G) \geq 4p \Longrightarrow G \in M_{2p+1}$, it suffices to show that this holds for $(4p+1)$-regular graphs.
- **Theorem** (Shao, Wu, Zhou & HJL, 2009 in JCTB) Let G be a $(4p+1)$-regular graph. Then $G \in M_{2p+1}$ if and only if $V(G)$ has a partition (V^+, V^-) such that $\forall U \subseteq V(G)$,

$$|\partial_G(U)| \geq (2p+1)||U \cap V^+| - |U \cap V^-||.$$

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- A graph H is mod $(2p+1)$**-contractible** if for any graph G that contains H as a subgraph,

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- (Barat and Thomassen, 2006 in JGT) If for any map $w : V(G) \mapsto \mathbf{Z}_{2p+1}$ with $\sum_{v \in V(G)} w(v) \equiv |E(G)|$ (mod $2p+1$), G has an orientation D such that $\forall v \in V(G)$, $d^+(v) \equiv w(v)$ (mod $2p+1$), then we say that G admits **all** $(2p+1)$**-orientations.**

Contractible graphs with respect to mod $(2p+1)$-orientations

- Theorem: Let $p \geq 1$ be an integer. The followings are equivalent for a connected graph H.
  (i) $H \in M^o_{2p+1}$,
  (ii) H is mod $(2p+1)$-contractible.
  (iii) $\forall G$ such that H is a subgraph of G, $G/H \in M^o_{2p+1}$ if and only if $G \in M^o_{2p+1}$.
  (iv) H admits all $(2p+1)$-orientations.

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- Theorem (Shao, Wu, Zhou & HJL, 2009 in JCTB) If $n = |V(G)|$ and G is not mod $(2p+1)$-contractible, then:

Contractible graphs with respect to mod $(2p+1)$-orientations

- Theorem (Shao, Wu, Zhou & HJL, 2009 in JCTB) If $n = |V(G)|$ and G is not mod $(2p+1)$-contractible, then:
- (i) $V(G)$ is a disjoint union $V(G) = V_1 \dot{\bigcup} V_2$ with $|V_1| = k$, $|V_2| = n - k$, and

$$\left\lceil \frac{|E(V_1, V_2)| + 1}{k} \right\rceil + \left\lceil \frac{|E(V_1, V_2)| + 1}{n-k} \right\rceil \le 4p + 2.$$

Contractible graphs with respect to mod $(2p+1)$-orientations

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- (ii) $V(G)$ is a disjoint union $V(G) = V_1 \dot{\bigcup} V_2$ with $|V_1| = k$, $|V_2| = n - k$, and

$$|E(V_1, V_2)| \leq \frac{(4p+2)k(n-k)}{n}.$$

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$$\begin{aligned} & \left\lceil \frac{|E(V_1,V_2)|+1}{k} \right\rceil + \left\lceil \frac{|E(V_1,V_2)|+1}{4p-k} \right\rceil \\ = & \left\lceil \frac{k(n-k)+1}{k} \right\rceil + \left\lceil \frac{|k(n-k)+1}{n-k} \right\rceil \\ = & (n-k+1)+(k+1) = n+2 > 4p+2. \end{aligned}$$

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\end{aligned}$$

- **Proposition** $K_m \in M^o_{2p+1}$ if and only if $m = 1$ or $m \geq 4p+1$.

Recent Progress

- Theorem (Shao, Wu, Zhou & HJL, 2009 in JCTB) Let n, p be positive integers, and let $f(n) = \frac{(2p+1)n\log_2(n)}{2}$ be a function. If G is a graph with n vertices and if $|E(G)| \geq f(n)$, then G has a subgraph H with $E(H) \neq \emptyset$ which is mod $(2p+1)$-contractible.

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- **Proof:** Use connectivity to count the number of edges and use the top Theorem to find a contractible

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- Corollary Let G be a graph with n vertices. If G is $3\log_2(n)$-edge-connected, then G is mod 3-contractible.

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- Corollary Let G be a graph with n vertices. If G is $3\log_2(n)$-edge-connected, then G is mod 3-contractible.
- Theorem (Zhang & HJL, 1992 in DM) Let G be a graph with n vertices. If G is $4\log_2(n)$-edge-connected, then G has a mod 3-orientation.

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- **Theorem** Every simple $(4p)$-connected chordal graph is in M_{2p+1}^o.
- **Theorem** Every $(4p-1)$-edge-connected graph without a K_4-minor is in M_{2p+1}^o.
- **Example** Let $m = 2p - 1$ and let $G = mC_{2p+1}$. Then G is a $(4p-2)$-edge-connected graph without K_4-minor but $G \notin M_{2p+1}^o$.

Thank you!