Hamiltonian Line Graphs and Related Problems

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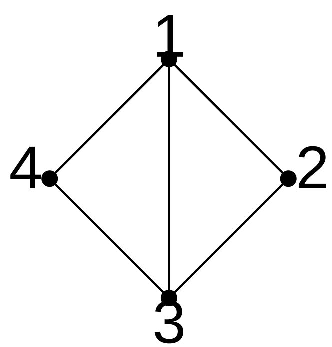
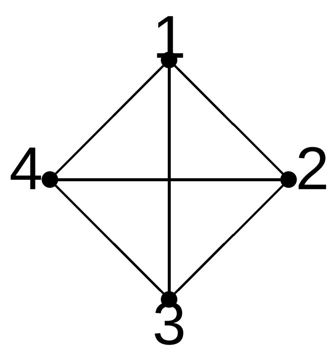
- Examples



Hamiltonian, not
Hamiltonian Connected



Hamiltonian Connected

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- $L(G)$: the line graph of a graph G, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are linked by k edges if and only if the corresponding edges in G share exactly k vertices in common.

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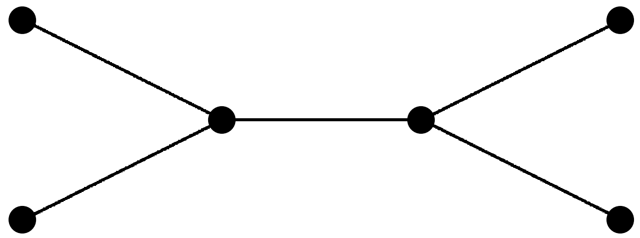

G

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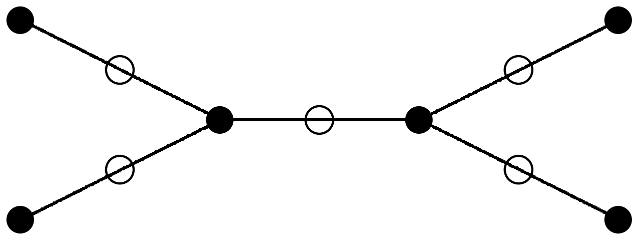

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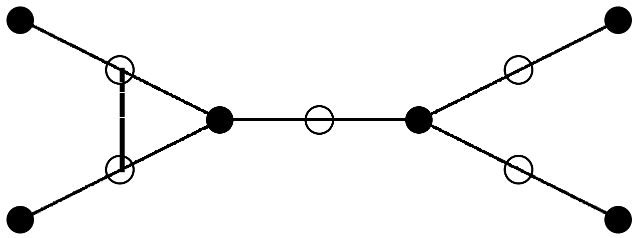

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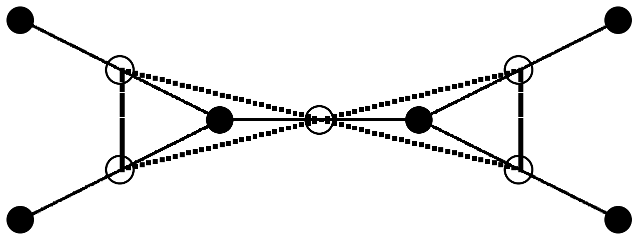

G: solid lines and closed circles

$L(G)$: dash lines and open circles

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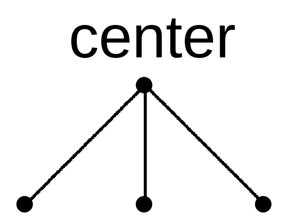
- Every essential edge cut in G corresponds to a vertex cut in $L(G)$; and vice versa when $L(G)$ is not complete.
- If $L(G)$ is k-connected, then G is essentially k-edge-connected. Moreover, when $L(G)$ is not complete, G is essentially k-edge-connected if and only if $L(G)$ is k-connected.

Claw-free Graphs

- a claw: an induced $K_{1,3}$



a claw

Figure 1.3

Claw-free Graphs

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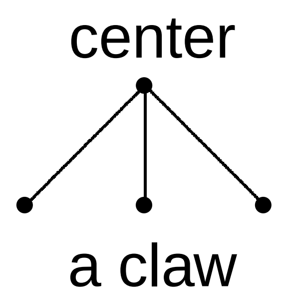
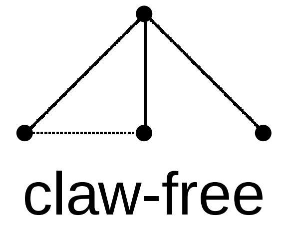


Figure 1.3

- **claw free** graph G: G does not contain an induced $K_{1,3}$



Questions and Conjectures

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- Conjecture (Matthews and Sumner) Every 4-connected claw-free graph is hamiltonian.
- Theorem (Ryjáček) These two conjectures are equivalent.

Known Results

- Theorem (Zhan) Every 7-connected line graph is hamiltonian-connected.

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- Theorem (Ryjáček) Every 7-connected claw-free graph is hamiltonian.

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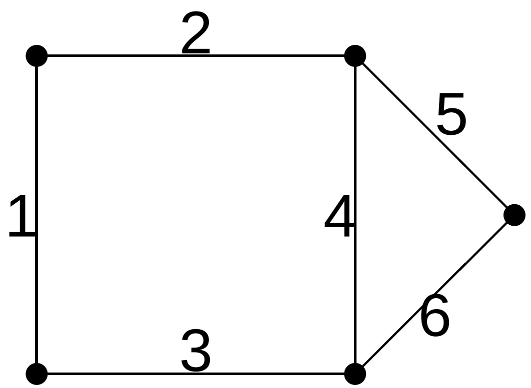
- A locally connected graph is N_2-locally connected. Not vice versa.



A Settled Conjecture

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- Theorem (Lai, Shao and Zhan, 2005) This conjecture is a theorem.

Main Ideas in the Proof

- Step 1 Apply Ryjáček's closure to convert the problem into a line graph problem: It suffices to show that every 3-connected, N_2-locally connected line graph $L(G)$ is hamiltonian.

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- Step 4 Apply the Harary and Nash-Williams' Theorem to show that $L(G)$ is hamiltonian.

Hamiltonian Line Graphs with High Essential Connectivity

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- Corollary Every 3-connected, essentially 11-connected claw-free graph is hamiltonian.

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- Theorem (Lai, Shao, Wu, and Zhou, 2006) Every 3-connected, essentially 11-connected line graph is hamiltonian.
- Corollary Every 3-connected, essentially 11-connected claw-free graph is hamiltonian.
- Problem What is the smallest positive integer k such that every 3-connected, essentially k-connected line graph (or claw-free graph) is hamiltonian?

Hamiltonian Cycles in 3-connected Claw-free Graphs

- Conjecture (Kuipers and Veldman 1998) Every 3-connected claw-free simple graph G with order n and minimum degree $\delta(G) \geq \frac{n+6}{10}$ is Hamiltonian for sufficiently large n.

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- **Theorem** (Favaron and Fraisse, 2001, JCT(B)) If G is a 3-connected claw-free simple graph with order n, and if $\delta(G) \geq \frac{n+37}{10}$, then G is hamiltonian.

Hamiltonian Cycles in 3-connected Claw-free Graphs

- Theorem (Lai, Shao and Zhan, 2006) if G is a 3-connected claw-free simple graph with sufficiently large order n, and if $\delta(G) \geq \frac{n+5}{10}$, then either G is hamiltonian, or $\delta(G) = \frac{n+5}{10}$ and the Ryjáček's closure $cl(G)$ of G is the line graph of a graph obtained from the Petersen graph P_{10} by adding $\frac{n-15}{10}$ pendant edges at each vertex of P_{10}.

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- Idea of Proof Use Ryjáček's closure, to find a hamiltonian cycle of G, it suffices to find a dominating eulerian subgraph in H, where $L(H) = cl(G)$.

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- Theorem (Lai, Xiong and Yan 2006) Every 3-connected $\{K_{1,3}, Z_8\}$-free graph is hamiltonian. Moreover, there exists a non hamiltonian 3-connected $\{K_{1,3}, Z_9\}$-free graph.

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- Step 4 Discuss the cases when the circumference is 9, 10 and 11.

Hamiltonian Cycles in 3-connected Claw-free Graphs

- The same steps also prove the following: if G is a connected simple graph without subgraphs isomorphic to P_{12}, and if $\kappa(L(G)) \geq 3$, then $L(G)$ is hamiltonian.

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  This gives another proof of the following theorem.

- Theorem (Luczak and Pfender, 2004 JGT) Every 3-connected $\{K_{1,3}, P_{11}\}$-free graph is hamiltonian.

s-hamiltonian Line Graphs

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- **Fact** If G is s-hamiltonian, then G must be (s+2)-connected.
- **Fact** There exist arbitrarily high connected non hamiltonian graphs ($K_{n,n+1}$ for example).

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- Let G be a k-triangular graph. Then $L(G)$, the line graph of G, is s-hamiltonian if and only $L(G)$ is $(s+2)$-connected.

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- **Theorem** (Broersma and Veldman, 1987 JGT) Let $k \geq s \geq 0$ be integers and let G be a k-triangular simple graph. Then $L(G)$ is s-hamiltonian if and only $L(G)$ is $(s+2)$-connected.

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- Problem (Broersma and Veldman, 1987 JGT) Is the statement above remains valid if $s = 2k$?
- Theorem (Chen, Lai, Li, and Siu) Let k and s be positive integers such that $0 \leq s \leq \max\{2k, 6k-16\}$, and let G be a k-triangular simple graph. Then $L(G)$ is s-hamiltonian if and only $L(G)$ is $(s+2)$-connected.

Thank You