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Hamiltonian index of directed multigraph

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ABSTRACT

The index of a property $\mathcal{P}$ for a directed multigraph D is the smallest nonnegative integer k such that the iterated line digraph $L^k(D)$ has the property $\mathcal{P}$. Let $e(D)$ denote the eulerian index of D and $h(D)$ denote the hamiltonian index of D. Directed multigraphs families $\mathcal{F}$ and $\mathcal{H}$ are defined such that a directed multigraph D has a finite value $e(D)$ if and only if $D \in \mathcal{F}$, and D has a finite value $h(D)$ if and only if $D \in \mathcal{F} \cup \mathcal{H}$. Furthermore, the values of the hamiltonian indices for members in $\mathcal{F} \cup \mathcal{H}$ are determined. In addition, line digraph stable properties are investigated, and sufficient and necessary conditions are obtained for a subfamily of strong directed multigraphs in which being eulerian and being hamiltonian are line digraph stable.

1. The problem

A directed graph (or just a digraph) D consists of a non-empty finite set $V(D)$ of elements called vertices and a finite set $A(D)$ of ordered pairs of distinct vertices called arcs. Thus a digraph may have arcs incident with the same pair of vertices but with opposite orientations, and does not contain parallel arcs, that is, arcs with the same tail and the same head, or loops (i.e. arcs whose head and tail coincide). When parallel arcs and loops are admissible we speak of directed pseudographs; directed pseudographs without loops are directed multigraphs. We consider finite graphs and finite directed multigraphs.

Undefined terms and notation one can refer to [2] for directed multigraphs and [4] for graphs. Let $D = (V(D), A(D))$ be a directed multigraph. For subsets $X_1, X_2 \subseteq V(D)$, define

$$(X_1, X_2)_D = \{(x_1, x_2) \in A(D) : x_1 \in X_1, x_2 \in X_2\}.$$

If $X_1 = \{x_1\}$ or $X_2 = \{x_2\}$, we often use $(x_1, X_2)_D$ for $(X_1, X_2)_D$ or $(X_1, x_2)_D$ for $(X_1, X_2)_D$, respectively. Hence $(x_1, x_2)_D = (\{x_1\}, \{x_2\})_D$. For a vertex $x \in V(D)$, let $\partial_D^+(x) = (x, V(D) - \{x\})_D$ and $\partial_D^-(x) = (V(D) - \{x\}, x)_D$. Thus $d_D^+(x) = |\partial_D^+(x)|$ and $d_D^-(x) = |\partial_D^-(x)|$. As in [2], a directed multigraph D is strong if and only if for any proper non-empty subset $X \subset V(D)$, $(X, V(D) - X)_D \neq \emptyset$. An **empty graph** is one with at least one vertex such that it does not have any arcs. For convenience of our discussions, we shall use $\mathcal{D}$ to denote the family of all strong directed multigraphs.

Let $G = (V(G), E(G))$ be a graph. The **line graph** $L(G)$ of a graph G is a simple graph whose vertex set is $E(G)$, where ab is an edge of line graph $L(G)$ if and only if a and b are adjacent in graph G. **Iterated line graphs** are recursively defined: $L^0(G) = G$, and $L^n(G) = L(L^{n-1}(G))$ for $n \geq 1$. Let $\overline{\mathcal{G}}$ denote the family of all connected nontrivial graphs that are not

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isomorphic to a path, a cycle or a star $K_{1,3}$. In recent years, the index problem of iterated line graph has became a hot topic in graph theory. Chartrand and Wall [6] launched the study of the smallest integer $k \geq 0$ such that the iterated line graph $L^k(G)$ becomes hamiltonian, which called the hamiltonian index of a graph G. In [8], Clark and Wormald discussed some other hamiltonian like indices. Additional researches on hamiltonian index can be found in [7,9–12,16–21], among others. Assume that $\overline{\mathcal{P}}$ is a graphical property and $G \in \overline{\mathcal{G}}$ is a connected graph. Simultaneously, $\overline{\mathcal{P}}$ can be used to denote the family of graphs that has property $\overline{\mathcal{P}}$. Hence $G \in \overline{\mathcal{P}}$ if and only if G has property $\overline{\mathcal{P}}$. More generally, Lai and Shao introduced the general definition as follows.

Definition 1.1 (Definition 5.8 of [13])**.** The $\overline{\mathcal{P}}$-index of G, denoted by $\overline{\mathcal{P}}(G)$, has the following form:

$$\overline{\mathcal{P}}(G) = \begin{cases} min\{k : L^k(G) \in \overline{\mathcal{P}}\}, & if\ for\ a\ nonnegative\ integer\ t,\ L^t(G) \in \overline{\mathcal{P}}, \\ \infty, & otherwise. \end{cases}$$

A property $\overline{\mathcal{P}}$ is **line graph stable** if $L(G)$ has $\overline{\mathcal{P}}$ whenever G has $\overline{\mathcal{P}}$. Chartrand [5] showed that for every graph $G \in \overline{\mathcal{G}}$, the hamiltonian index exists as a finite number, and the characterization of hamiltonian line graphs by Harary and Nash-Williams implies that being hamiltonian is line graph stable. Ryjáček et al. showed that it is difficult to determine the value of hamiltonian index [15]. Clark and Wormald [8] proved that for all graphs in $\overline{\mathcal{G}}$, other hamiltonian-like indices also exist as finite numbers; and in [13], it was stated that these hamiltonian-like properties are also line graph stable.

It is natural to consider the index problem of directed multigraphs. A **line digraph** $L(D)$ of a directed multigraph D is a digraph whose vertex set is $A(D)$, where $(a, b) \in A(L(D))$ if and only if there exist vertices x_1, x_2 and x_3 in D such that $a = (x_1, x_2)$ and $b = (x_2, x_3)$ are in $A(D)$. As in the graph case, we have the following.

Definition 1.2. Let D be a directed multigraph, and $n \geq 1$ be an integer. Define $L^0(D) = D$ and $L^n(D) = L(L^{n-1}(D))$.

Aigner [1] proved the following fundamental theorem.

Theorem 1.1 ([1])**.** *Assume that D is a directed multigraph without isolated vertices. Then the line digraph $L(D)$ is strong if and only if D is strong.*

It follows from Theorem 1.1 and Definition 1.2 that for any directed multigraph D and integer $n \geq 0$, the iterated line digraph $L^n(D)$ is strong if and only if D is strong.

Let $D = (V(D), A(D))$ be a directed multigraph with vertex set $V(D)$ and arc set $A(D)$, and let $k > 1$ be an integer. A **walk** W in D is an alternating sequence with

$$W = x_1 a_1 x_2 \cdots x_{k-1} a_{k-1} x_k a_k x_{k+1} \tag{1}$$

of vertices x_i $(1 \leq i \leq k+1)$ and arcs $a_j = (x_j, x_{j+1})$ $(1 \leq j \leq k)$ being in D. As the walk W in (1) uses k arcs, W is called a **k-walk**. For presentational simplicity, we often write the vertex sequence $x_1 x_2 \cdots x_{k-1} x_k x_{k+1}$ for the walk in (1). To emphasize the beginning and ending vertices (arcs, respectively) we often call W an (x_1, x_{k+1})-walk ((a_1, a_k)-walk, respectively). A walk presented as in (1) is a **closed walk** if $x_1 = x_{k+1}$. While a walk is defined as a sequence, it is not defined as a subdigraph. A trail is a walk in which all arcs are distinct. A trail is an **Euler (or eulerian) trail** if the trail is closed. If the vertices of a walk W are distinct, then W is called a **path**. The **length** of a path is the number of arcs in the path. A **k-path** is a path of length k with $k+1$ vertices. A closed path is a **cycle** and a **k-cycle** is a cycle with k vertices and k arcs. A cycle C is a **hamiltonian cycle** of D if $V(C) = V(D)$. We know that D is **eulerian** if D itself is an Euler trail, and D is **hamiltonian** if D contains a hamiltonian cycle. Assume that $\mathcal{P}$ is a graphical property and D is a strong directed multigraph. Simultaneously, $\mathcal{P}$ can be used to denote the family of directed multigraphs that has property $\mathcal{P}$. Hence $D \in \mathcal{P}$ if and only if D has property $\mathcal{P}$. To study the digraph index problem, we introduce the general definition as follows.

Definition 1.3. The $\mathcal{P}$-index of D, denoted by $\mathcal{P}(D)$, has the following form:

$$\mathcal{P}(D) = \begin{cases} \min\ \{k : L^k(D) \in \mathcal{P}\}, & if\ one\ such\ integer\ k\ exists, \\ \infty, & \text{otherwise}. \end{cases}$$

The goal of this research is to investigate the indices for the properties of being eulerian and being hamiltonian of directed multigraphs in $\mathcal{D}$. We use $e(D)$ to denote the **eulerian index** and $h(D)$ to denote the **hamiltonian index** of D. In the consequent sections, we shall identify a directed multigraph family $\mathcal{F}$ such that each digraph $D \in \mathcal{F}$ has a finite $e(D)$ and each digraph $D \in \mathcal{F} \cup \mathcal{H}$ has a finite $h(D)$. We also determine the values of the hamiltonian indices for members in $\mathcal{F} \cup \mathcal{H}$. A concrete example will be presented and discussed in the third section. In Section 4, we will introduce line digraph stable properties and obtain sufficient and necessary conditions for a subfamily of strong directed multigraphs in which being eulerian and being hamiltonian are both line digraph stable properties.

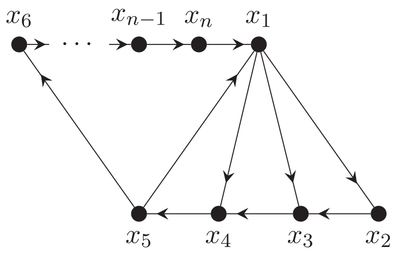

Fig. 1. Digraph $D_0(n)$.

2. Hamiltonian and eulerian indices of directed multigraphs

Throughout the rest of this paper, for a positive integer $n \geq 2$, we use $\mathbb{Z}_n = \{1, 2, \cdots, n\}$ to denote the cyclic group of order n, which is often used as vertex subscript indices in the discussions. There have been some well-known results on line digraphs with certain eulerian and hamiltonian properties, as shown in Theorems 2.1 and 2.2 below.

Theorem 2.1 ([1]). *Assume that D is a strong directed multigraph. Then*

(i) *$L(D)$ is eulerian if and only if $d^-(x_1) = d^+(x_2)$ for any $(x_1, x_2) \in A(D)$.*
(ii) *$L(D)$ is hamiltonian if and only if D is eulerian.*

Theorem 2.2 ([14]). *Assume that D is a strong directed multigraph with $|A(D)| \geq 1$. Then for any two distinct arcs a_1 and a_m of D, $L(D)$ possesses a hamiltonian (a_1, a_m)-path if and only if D itself is an (a_1, a_m)-trail.*

In this section, $D \in \mathcal{D}$, the family of all strong directed multigraphs, will be used all the time. Given a positive integer $n \geq 1$, let $W_n(D)$ denote the set of all n-walks of D. It is routine to prove by induction (as seen in Exercise 2.10 of [2]) that $L^n(D)$ is isomorphic to the digraph D', whose vertex set is $W_n(D)$, where there exists an arc from $v_0v_1 \cdots v_{n-1}v_n$ to $v_1 \cdots v_nv_{n+1}$ for every vertex $v_{n+1} \in V(D)$ with $(v_n, v_{n+1}) \in A(D)$. We shall investigate certain relationship between the out(in)-degree of strong directed multigraph and out(in)-degree of its iterated line graph $L^k(G)$ for $k \geq 1$. By the definition of line digraphs, we summarize several observations as in the following routine proposition.

Proposition 2.1. *Let $n \geq 1$ be an integer, $D \in \mathcal{D}$ be a strong directed multigraph and $L^{n-1}(D)$ be the $(n-1)$th iterated line digraph of D. Then each of the following holds.*

(i) *(Proposition 11.2.2 of [3]) $d^-_{L(D)}((z, w)) = d^-_D(z)$ and $d^+_{L(D)}((z, w)) = d^+_D(w)$ for any arc $(z, w) \in A(D)$.*
(ii) *$V(L^k(D)) = W_k(D)$.*
(iii) *If $W = x_1x_2 \cdots x_n \in W_{n-1}(D)$, then $d^-_D(x_1) = d^-_{L^{n-1}(D)}(W)$ and $d^+_D(x_n) = d^+_{L^{n-1}(D)}(W)$.*
(iv) *For any $W = x_1x_2 \cdots x_n \in W_{n-1}(D)$, $d^-_D(x_1) = d^+_D(x_n)$ if and only if $d^-_{L^{n-1}(D)}(W) = d^+_{L^{n-1}(D)}(W)$.*

In order to study the eulerian index and the hamiltonian index, we introduce the following directed multigraph families, whose properties would play useful roles in our discussions.

Definition 2.1. Let s, s' be integers with $s \geq 1$ and $s' \geq 2$.

(i) Define a family $\mathcal{M}(s') \subset \mathcal{D}$ of strong directed multigraphs such that $D \in \mathcal{M}(s')$ if and only if there exists an s'-cycle C of D such that for any m with $0 \leq m \leq s' - 1$, C contains an m-path $P = x_1^m x_2^m \cdots x_{m+1}^m$ satisfying $d^-_D(x_1^m) \neq d^+_D(x_{m+1}^m)$.
(ii) Define a family $\mathcal{F}(s) \subset \mathcal{D}$ of strong directed multigraphs such that $D \in \mathcal{F}(s)$ if and only if for any $(s-1)$-walk $W = x_1x_2 \cdots x_s$ of D satisfying $d^-_D(x_1) = d^+_D(x_s)$.
(iii) Define $\mathcal{M} = \bigcup_{s' \geq 2} \mathcal{M}(s')$ and $\mathcal{F} = \bigcup_{s \geq 1} \mathcal{F}(s)$.

In Example 2.1 below, we shall show that there is an infinite sequence of hamiltonian directed multigraphs that are in $\mathcal{M}$ but not in $\mathcal{F}$. Define

$$\mathcal{H} = \{D \in \mathcal{D} : \text{ D is hamiltonian}\}$$

be a family of all hamiltonian directed multigraphs.

Example 2.1. Let $n \geq 6$ be an integer. Define a family $\mathcal{D}_0$ of digraphs such that a digraph $D_0(n) \in \mathcal{D}_0$ if and only if $D_0(n)$ has vertex set $V(D_0(n)) = \{x_i : i \in \mathbb{Z}_n\}$ and arc set $A(D_0(n)) = \{(x_i, x_{i+1}) : i \in \mathbb{Z}_n\} \cup \{(x_1, x_3), (x_1, x_4), (x_5, x_1)\}$. Then every digraph $D_0(n)$ in the family $\mathcal{D}_0$ is strong and lies in $\mathcal{M}$.(See Fig. 1 for an illustration.)

Proof. Since $D_0(n)$ has a hamiltonian cycle $x_1x_2x_3 \cdots x_nx_1$, we conclude that $D_0(n)$ is strong and $D_0(n) \in \mathcal{H}$. Let $C = x_1x_4x_5x_1$ denote a 3-cycle of $D_0(n)$. We are to show that for any m with $0 \leq m \leq 2$, C contains an m-path $P = x_1^m x_2^m \cdots x_{m+1}^m$ satisfying $d^-_{D_0(n)}(x_1^m) \neq d^+_{D_0(n)}(x_{m+1}^m)$. If $m = 0$, as $d^-_{D_0(n)}(x_1) = 2$ and $d^+_{D_0(n)}(x_1) = 3$, then $d^-_{D_0(n)}(x_1) \neq d^+_{D_0(n)}(x_1)$. If $m = 1$, then as

$d^-_{D_0(n)}(x_1) = 2$ and $d^+_{D_0(n)}(x_4) = 1$, we have $d^-_{D_0(n)}(x_1) \neq d^+_{D_0(n)}(x_4)$. If $m = 2$, then as $d^-_{D_0(n)}(x_4) = 2$ and $d^+_{D_0(n)}(x_1) = 3$, we have $d^-_{D_0(n)}(x_4) \neq d^+_{D_0(n)}(x_1)$. As $(x_1, x_4) \in A(C)$ and $x_4x_5x_1$ is a 2-path of C, then by Definition 2.1, $D_0(n) \in \mathcal{M}$. □

Lemma 2.1. *With the Definition 2.1 and Example 2.1, we conclude that each of the following holds.*

(a') *$\mathcal{M} \neq \emptyset$ and $\mathcal{F} \neq \emptyset$.*
(b') *For any $D \in \mathcal{M}$, we have $D \notin \mathcal{F}$.*
(c') *For any $D_0(n) \in \mathcal{D}_0$, we have $D_0(n) \in \mathcal{H} - \mathcal{F}$.*

Proof. By Example 2.1, for any $D_0(n) \in \mathcal{D}_0$ we have $D_0(n) \in \mathcal{M}$, and so $\mathcal{M} \neq \emptyset$. Since for any eulerian digraph D', we have $D' \in \mathcal{F}$, hence $\mathcal{F} \neq \emptyset$. Thus (a') holds.

Since $D \in \mathcal{M}$, there exists an integer s' with $s' \geq 2$ such that $D \in \mathcal{M}(s')$. Thus there exists an s'-cycle C of D such that for any m with $0 \leq m \leq s' - 1$,

$$C \text{ contains an } m\text{-path } P = x_1^m x_2^m \cdots x_{m+1}^m \text{ satisfying } d_D^-(x_1^m) \neq d_D^+(x_{m+1}^m). \tag{2}$$

For any $s \geq 1$, then $s = s'k + t$ with $k \geq 0$ and $1 \leq t \leq s'$. By (2), D contains a $(t-1)$-path $P = x_1^{t-1}x_2^{t-1}\cdots x_t^{t-1}$ satisfying $d_D^-(x_1^{t-1}) \neq d_D^+(x_t^{t-1})$. Since C is a cycle, hence let $C = y_1^{t-1}y_2^{t-1}\cdots y_{s'}^{t-1}y_1^{t-1}$ such that $y_1^{t-1}y_2^{t-1}\cdots y_t^{t-1} = x_1^{t-1}x_2^{t-1}\cdots x_t^{t-1}$. Thus D has an $(s-1)$-walk $y_1^{t-1}y_2^{t-1}\cdots y_{s'}^{t-1}\cdots y_1^{t-1}y_2^{t-1}\cdots y_{s'}^{t-1}x_1^{t-1}x_2^{t-1}\cdots x_t^{t-1}$ with $d_D^-(x_1^{t-1}) \neq d_D^+(x_t^{t-1})$. Hence $D \notin \mathcal{F}$. Thus (b') holds.

Since $D_0(n) \in \mathcal{M}$, by (b'), $D_0(n) \notin \mathcal{F}$. As $D_0(n) \in \mathcal{H}$, so $D_0(n) \in \mathcal{H} - \mathcal{F}$. Thus (c') holds. □

Theorem 2.3. *Assume that D is a strong directed multigraph. Then*

(i) *$e(D)$ exists as a finite number if and only if $D \in \mathcal{F}$.*
(ii) *$h(D)$ exists as a finite number if and only if $D \in \mathcal{F} \cup \mathcal{H}$.*

Proof. Assume that $e(D)$ exists as a finite number. Then let t be an integer with $t \geq 0$ such that $L^t(D)$ is eulerian. Thus randomly pick a vertex $W \in V(L^t(D))$, we have $d^-_{L^t(D)}(W) = d^+_{L^t(D)}(W)$. By Proposition 2.1 (ii), we have $W \in W_t(D)$. Let $W = x_1x_2\cdots x_{t+1}$. By Proposition 2.1 (iv) and $d^-_{L^t(D)}(W) = d^+_{L^t(D)}(W)$, we have $d_D^-(x_1) = d_D^+(x_{t+1})$. Therefore $D \in \mathcal{F}$.

We now assume that $D \in \mathcal{F}$. Then D is strong and there exists an integer s with $s \geq 1$ such that for any $(s-1)$-walk $W = x_1x_2\cdots x_s$ of D, we obtain that $d_D^-(x_1) = d_D^+(x_s)$. As D is strong, every iterated line digraph of D is also strong. By Proposition 2.1, $W \in V(L^{s-1}(D))$ and $d^-_{L^{s-1}(D)}(W) = d^+_{L^{s-1}(D)}(W)$. Hence $L^{s-1}(D)$ is eulerian. This proves (i).

If $h(D)$ exists as a finite number, then let t be an integer with $t \geq 0$ satisfying $L^t(D)$ is hamiltonian. If $t = 0$, then directed multigraph D is hamiltonian, and so $D \in \mathcal{H}$. If $t \geq 1$, then by Theorem 2.1 (ii), $L^{t-1}(D)$ is eulerian. Hence for any vertex $W' \in V(L^{t-1}(D))$, $d^-_{L^{t-1}(D)}(W') = d^+_{L^{t-1}(D)}(W')$. By Proposition 2.1 (ii), $W' \in W_{t-1}(D)$. Let $W' = x_1x_2\cdots x_t$. By Proposition 2.1 (iv) and $d^-_{L^{t-1}(D)}(W') = d^+_{L^{t-1}(D)}(W')$, we have $d_D^-(x_1) = d_D^+(x_t)$. Hence $D \in \mathcal{F}$.

Conversely, if $D \in \mathcal{H}$, then D is hamiltonian. If $D \in \mathcal{F} - \mathcal{H}$, then there is a positive integer s with $s \geq 1$ such that for any $(s-1)$-walk $W = x_1x_2\cdots x_s$ of D satisfying $d_D^-(x_1) = d_D^+(x_s)$. By Proposition 2.1, $W \in V(L^{s-1}(D))$ and $d^-_{L^{s-1}(D)}(W) = d^+_{L^{s-1}(D)}(W)$. Hence $L^{s-1}(D)$ is eulerian. By Theorem 2.1, $L^s(D)$ is hamiltonian. □

Theorem 2.3 characterizes digraph D for which $e(D)$ or $h(D)$ exists. Thus by Theorem 2.3, in the consequent discussion of this section, we shall suppose $D \in \mathcal{F} \cup \mathcal{H}$. We are to investigate the exact values of the eulerian index and hamiltonian index for each digraph $D \in \mathcal{F} \cup \mathcal{H}$. Aiming at this goal, we present a relationship of $e(D)$ and $h(D)$ as follows.

Proposition 2.2. *Let D be a strong directed multigraph with $D \in \mathcal{F} \cup \mathcal{H}$. Then*

$$h(D) = \begin{cases} 0, & \text{if } D \in \mathcal{H}, \\ e(D) + 1, & \text{if } D \in \mathcal{F} - \mathcal{H}. \end{cases}$$

Proof. If $D \in \mathcal{H}$, then D is hamiltonian, and so $h(D) = 0$. If $D \in \mathcal{F} - \mathcal{H}$, then by Theorem 2.3 (ii), there is an integer $s \geq 0$ satisfying $h(D) = s$. We may claim that $s \geq 1$. Otherwise, if $s = 0$, then D is hamiltonian, thus $D \in \mathcal{H}$, contrary to $D \in \mathcal{F} - \mathcal{H}$. Hence $s \geq 1$. If $s = 1$, then $L(D)$ is hamiltonian. By Theorem 2.1 (ii), D is eulerian and $e(D) = 0$. Thus $h(D) = e(D) + 1 = 1$. Assume now that $s \geq 2$. Hence $L^s(D)$ is hamiltonian. By Theorem 2.1 (ii), $L^{s-1}(D)$ is eulerian, and so $e(D) \leq s - 1$. If there exists an integer s' with $1 \leq s' < s - 1$ such that $L^{s'}(D)$ is eulerian, then by Theorem 2.1 (ii), $L^{s'+1}(D)$ is hamiltonian, contrary to $h(D) = s$. Hence $e(D) = s - 1$. Thus if $D \in \mathcal{F} - \mathcal{H}$, we have $h(D) = s = e(D) + 1$. □

By Proposition 2.2 and definition of the hamiltonian index, we have

$$h(D) = 0 \text{ if and only if } D \in \mathcal{H}.$$

We shall consider $D \in \mathcal{F} - \mathcal{H}$ in the following results.

Theorem 2.4. *Let $D \in \mathcal{F} - \mathcal{H}$ be a strong directed multigraph. Then for an integer s with $s \geq 1$, the following statements are equivalent.*

(i) $h(D) = s$.
(ii) $D \in \mathcal{F}(s)$ *and for any* $1 \le t \le s-1$, $D \notin \mathcal{F}(t)$.

Proof. $(i) \Rightarrow (ii)$. If $h(D) = s$, then $L^s(D)$ is hamiltonian. By Theorem 2.1 (ii), $L^{s-1}(D)$ is eulerian. Hence for any vertex $W \in V(L^{s-1}(D))$, we have $d^+_{L^{s-1}(D)}(W) = d^-_{L^{s-1}(D)}(W)$. Thus by Proposition 2.1 (ii), $W \in W_{s-1}(D)$. Let $W = x_1x_2\cdots x_s$. Since $d^+_{L^{s-1}(D)}(W) = d^-_{L^{s-1}(D)}(W)$, hence by Proposition 2.1 (iv), $d^-_D(x_1) = d^+_D(x_s)$, and so $D \in \mathcal{F}(s)$. Since $h(D) = s$, it follows that for any t with $1 \le t \le s-1$, $L^t(D)$ is not hamiltonian. Thus by Theorem 2.1 (ii), $L^{t-1}(D)$ is not eulerian, and so $L^{t-1}(D)$ has a vertex W' satisfying $d^+_{L^{t-1}(D)}(W') \ne d^-_{L^{t-1}(D)}(W')$. By Proposition 2.1 (ii), $W' \in W_{t-1}(D)$. Let $W' = y_1y_2\cdots y_t$. Since $d^+_{L^{t-1}(D)}(W') \ne d^-_{L^{t-1}(D)}(W')$, by Proposition 2.1 (iv), it follows that $d^-_D(y_1) \ne d^+_D(y_t)$, and so $D \notin \mathcal{F}(t)$.

$(ii) \Rightarrow (i)$. Since $D \in \mathcal{F}(s)$, D is strong and for any $(s-1)$-walk $W = x_1x_2\cdots x_s$, we have $d^-_D(x_1) = d^+_D(x_s)$. Hence each iterated line digraph of D is also strong and by Proposition 2.1, $W \in V(L^{s-1}(D))$ and $d^-_{L^{s-1}(D)}(W) = d^+_{L^{s-1}(D)}(W)$. It follows that $L^{s-1}(D)$ is eulerian. By Theorem 2.1 (ii), $L^s(D)$ is hamiltonian, and so $h(D) \le s$. Since for any t with $1 \le t \le s-1$, $D \notin \mathcal{F}(t)$, hence there exists a $(t-1)$-walk $W' = y_1y_2\cdots y_t$ with $d^-_D(y_1) \ne d^+_D(y_t)$, and so by Proposition 2.1, $W' \in V(L^{t-1}(D))$ and $d^+_{L^{t-1}(D)}(W') \ne d^-_{L^{t-1}(D)}(W')$. Thus $L^{t-1}(D)$ is not eulerian. By Theorem 2.1 (ii), $L^t(D)$ is not hamiltonian. Since $D \in \mathcal{F} - \mathcal{H}$, hence D is not hamiltonian. It follows that $h(D) > t$. Since $h(D) \le s$, hence $h(D) = s$. □

3. An example for directed multigraphs with prescribed hamiltonian index

Let $s \ge 0$ be a given integer. While Theorem 2.4 characterizes directed multigraphs with hamiltonian index s, the existence of directed multigraphs D with $h(D) = s$ is warranted in theory by the arguments in the proof. Thus it would be interesting to have a concrete example for such directed multigraphs. We shall present the constructions of such examples to illustrate the result.

A collection $\mathscr{C} = \{C_1, C_2, \ldots, C_k\}$ of cycles is a **cycle cover** of D, if members in $\mathscr{C}$ are cycles of D satisfying $A(D) = \cup_{C\in\mathscr{C}} A(C)$ and $C_s \ne C_t$ whenever $1 \le s < t \le k$. For any two distinct cycles $C_i, C_j \in \mathscr{C}$, since C_i and C_j are cycles, hence $C_i \cap C_j$ is a union of some vertex-disjoint paths. Set $C_i \cap C_j = P_{ij_1} \cup P_{ij_2} \cup \cdots \cup P_{ij_{\ell_{ij}}}$, where $P_{ij_1}, P_{ij_2}, \ldots, P_{ij_{\ell_{ij}}}$ are ℓ_{ij} vertex-disjoint paths.

Example 3.1. Let s be an integer with $s \ge 0$. Define a family $\mathcal{D}(s)$ of strong directed multigraphs such that $D \in \mathcal{D}(s)$ if and only if there exists a cycle cover $\mathscr{C} = \{C_1, C_2, \ldots, C_k\}$ in D satisfying for any two distinct cycles $C_i, C_j \in \mathscr{C}$, $C_i \cap C_j = P_{ij_1} \cup P_{ij_2} \cup \cdots \cup P_{ij_{\ell_{ij}}}$ is a union of some vertex-disjoint paths and for any $P \in \bigcup_{i=1}^{k}\bigcup_{j=i+1}^{k}\{P_{ij_1}, P_{ij_2}, \ldots, P_{ij_{\ell_{ij}}}\}$, $|V(P)| = s$.

For any $D \in \mathcal{D}(0)$, by Example 3.1 and D is strong, thus D is a cycle. For any $D \in \mathcal{D}(1)$, by Example 3.1, D has a cycle cover $\mathscr{C} = \{C_1, C_2, \ldots, C_k\}$ satisfying for any two distinct cycles $C_i, C_j \in \mathscr{C}$, $C_i \cap C_j$ is an empty graph. Thus, we obtain that D is eulerian. Let s be an integer with $s \ge 2$ and $D \in \mathcal{D}(s)$ be a strong directed multigraph defined as in Example 3.1, then there is a cycle cover $\mathscr{C} = \{C_1, C_2, \ldots, C_k\}$ of D, such that

$$\begin{aligned} &\forall\ C_i, C_j \in \mathscr{C} \text{ with } i \ne j,\ C_i \cap C_j = P_{ij_1} \cup P_{ij_2} \cup \cdots \cup P_{ij_{\ell_{ij}}} \text{ is a union of some vertex-disjoint paths} \\ &\qquad\qquad \text{and for any } P \in \textstyle\bigcup_{i=1}^{k}\bigcup_{j=i+1}^{k}\{P_{ij_1}, P_{ij_2}, \ldots, P_{ij_{\ell_{ij}}}\},\ |V(P)| = s. \end{aligned} \tag{3}$$

Proposition 3.1. *Let s be a positive integer with $s \ge 2$ and $D \in \mathcal{D}(s)$ be a strong directed multigraph defined as in Example 3.1. Suppose that there is a cycle cover $\mathscr{C} = \{C_1, C_2, \ldots, C_k\}$ in D satisfying* (3). *For any two distinct cycles $C_i, C_j \in \mathscr{C}$, let $P = v_1v_2\cdots v_s$ be a maximal subpath of $C_i \cap C_j$. We can get the following results.*

(i) *For any $C_{j'} \in \mathscr{C} - \{C_i, C_j\}$, if $C_{j'} \cap P \ne \emptyset$, then P is a subpath of $C_{j'}$.*
(ii) *For any arc $a \in \partial^-_D(v_1) \cup \partial^+_D(v_s)$, there exists only one $C \in \mathscr{C}$ such that $a \in A(C)$.*

Proof. Since P is a maximal subpath of $C_i \cap C_j$. By contradiction, assume that there is $C_{j'} \in \mathscr{C} - \{C_i, C_j\}$ with $C_{j'} \cap P \ne \emptyset$ and P is not subpath of $C_{j'}$. If $v_1, v_s \notin V(C_{j'})$, then $C_i \cap C_{j'}$ has a maximal subpath P' such that $A(P') \subset A(P)$ and $v_1, v_s \notin V(P')$. Thus $|V(P')| \le s-2$, contrary to (3). Hence assume that $v_1 \in V(C_{j'})$ or $v_s \in V(C_{j'})$. If $v_1 \in V(C_{j'})$, then there exists an integer t with $2 \le t \le s$ such that $v_t \notin V(C_{j'})$ and $v_1, v_2, \ldots, v_{t-1} \in V(C_{j'})$. Hence $v_1v_2\cdots v_{t-1}$ is a subpath of both $C_i \cap C_{j'}$ and $C_j \cap C_{j'}$. Let $x \in N^-_{C_i}(v_1)$ and $y \in N^-_{C_j}(v_1)$. Since P is a maximal subpath of $C_i \cap C_j$, $x \ne y$. By the assumption of Proposition 3.1 and as $v_t \notin V(C_{j'})$, we conclude that either $x \in V(C_{j'})$ or $y \in V(C_{j'})$. Since $C_{j'}$ is a cycle, if $x \in V(C_{j'})$, then $y \notin V(C_{j'})$. Thus $v_1v_2\cdots v_{t-1}$ is a maximal subpath of $C_j \cap C_{j'}$, contrary to (3). If $y \in V(C_{j'})$, then $x \notin V(C_{j'})$. Thus $v_1v_2\cdots v_{t-1}$ is a maximal subpath of $C_i \cap C_{j'}$, contrary to (3). Likewise, if $v_s \in V(C_{j'})$, then a contradiction will be obtained similarly. Thus (i) holds.

For any arc $a \in \partial^-_D(v_1) \cup \partial^+_D(v_s)$, as $\mathscr{C}$ is a cycle cover of D, hence there is a cycle $C \in \mathscr{C}$ such that $a \in A(C)$. If D has another cycle $C' \in \mathscr{C} - \{C\}$ with $a \in A(C')$, then as $a \in \partial^-_D(v_1) \cup \partial^+_D(v_s)$ and P is a maximal subpath of $C_i \cap C_j$, hence $\{C', C\} \ne \{C_i, C_j\}$. We assume that $C \notin \{C_i, C_j\}$. By (i), path P is a subpath of cycle C. If $C' \in \{C_i, C_j\}$, then P is a subpath of cycle C'; if $C' \notin \{C_i, C_j\}$, then by (i), P is a subpath of cycle C'. Hence P is a subpath of $C \cap C'$. Likewise, if $C' \notin \{C_i, C_j\}$, then the result that P is a subpath of $C \cap C'$ is obtained similarly. Since $a \in A(C \cap C')$, hence $P \cup \{a\}$ is a subpath of $C \cap C'$, contrary to (3). This proves the proposition. □

Lemma 3.1. *Let s be a positive integer with $s \geq 2$, and $D \in \mathcal{D}(s)$ be a strong directed multigraph defined as in Example 3.1. Then $L(D) \in \mathcal{D}(s-1)$.*

Proof. Since $D \in \mathcal{D}(s)$ with $s \geq 2$, by (3), D has a cycle cover $\mathscr{C} = \{C_1, C_2, \ldots, C_k\}$ such that for any two distinct cycles $C_i, C_j \in \mathscr{C}$, $C_i \cap C_j = P_{ij_1} \cup P_{ij_2} \cup \cdots \cup P_{ij_{\ell_{ij}}}$ is a union of some vertex-disjoint paths and for any $P \in \bigcup_{i=1}^{k} \bigcup_{j=i+1}^{k} \{P_{ij_1}, P_{ij_2}, \ldots, P_{ij_{\ell_{ij}}}\}$, $|V(P)| = s$.

For any $C_i \in \mathscr{C}$, since C_i is a cycle, hence by Theorem 2.1 (*ii*), $L(C_i)$ is a cycle of $L(D)$. Let $C'_i = L(C_i)$ for any $1 \leq i \leq k$ and let $\mathscr{C}' = \{C'_1, C'_2, \ldots, C'_k\}$. Firstly, we are to prove that $\mathscr{C}'$ is a cycle cover of $L(D)$. Since $A(\bigcup_{C \in \mathscr{C}} C) = A(D) = V(L(D))$, hence $V(\bigcup_{C' \in \mathscr{C}'} C') = V(L(D))$, and so $\bigcup_{C' \in \mathscr{C}'} C'$ is a spanning subdigraph of $L(D)$. If $A(L(D)) \neq A(\bigcup_{C' \in \mathscr{C}'} C')$, then there exists an arc $(a, b) \in A(L(D)) - A(\bigcup_{C' \in \mathscr{C}'} C')$, where $a = (u_1, v_1)$ and $b = (v_1, v_2)$. It follows that either $u_1 \neq v_2$ and $u_1 v_1 v_2$ is a 2-path of D, or $u_1 = v_2$ and $u_1 v_1 v_2$ is a 2-cycle of D. Since $\mathscr{C}$ is a cycle cover of D, there exist two cycles C_1 and C_2 (say) such that $a = (u_1, v_1) \in A(C_1)$ and $b = (v_1, v_2) \in A(C_2)$. If $C_1 \neq C_2$, since $(a, b) \in A(L(D)) - A(\bigcup_{C' \in \mathscr{C}'} C')$, hence $u_1 v_1 v_2$ is not a subpath of both C_1 and C_2. Thus $u_1 \in V(C_1) - V(C_2)$ and $v_2 \in V(C_2) - V(C_1)$. Hence v_1 is a maximal path of $C_1 \cap C_2$, a contradiction to (3) with $s \geq 2$. If $C_1 = C_2$, then $u_1 v_1 v_2$ is a subpath of C_1, or C_1 is 2-cycle, thus $(a, b) \in A(L(C_1))$, a contradiction to the assumption that $(a, b) \in A(L(D)) - A(\bigcup_{C' \in \mathscr{C}'} C')$. Hence $A(L(D)) = A(\bigcup_{C' \in \mathscr{C}'} C')$, and so $\mathscr{C}'$ is a cycle cover of $L(D)$.

Now, we are to show that $L(D) \in \mathcal{D}(s-1)$. For any two distinct cycles $C'_i, C'_j \in \mathscr{C}'$, let $C'_i \cap C'_j = P'_{ij_1} \cup P'_{ij_2} \cup \cdots \cup P'_{ij_{\ell'_{ij}}}$ be a union of some vertex-disjoint paths. Since $C'_i = L(C_i)$, $C'_j = L(C_j)$ and $P_{ij_1}, P_{ij_2}, \ldots, P_{ij_{\ell_{ij}}} \in C_i \cap C_j$, hence $L(P_{ij_1}), L(P_{ij_2}), \ldots, L(P_{ij_{\ell_{ij}}}) \in L(C_i) \cap L(C_j)$, and so $L(P_{ij_1}), L(P_{ij_2}), \ldots, L(P_{ij_{\ell_{ij}}}) \in C'_i \cap C'_j$. Next, we are to prove that $C'_i \cap C'_j = L(P_{ij_1}) \cup L(P_{ij_2}) \cup \cdots \cup L(P_{ij_{\ell_{ij}}})$. Suppose, by contradiction, that $\{P'_{ij_1}, P'_{ij_2}, \cdots, P'_{ij_{\ell'_{ij}}}\} - \{L(P_{ij_1}), L(P_{ij_2}), \ldots, L(P_{ij_{\ell_{ij}}})\} \neq \emptyset$. Let $P' \in \{P'_{ij_1}, P'_{ij_2}, \cdots, P'_{ij_{\ell'_{ij}}}\} - \{L(P_{ij_1}), L(P_{ij_2}), \ldots, L(P_{ij_{\ell_{ij}}})\}$. Since P' is a maximal subpath of $C'_i \cap C'_j$, hence, applying Theorem 2.2, D has a trail T with $A(T) = V(P')$. Since $A(T) = V(P')$, $V(P') \subseteq V(C'_i \cap C'_j)$ and $V(C'_i \cap C'_j) = A(C_i \cap C_j)$, hence $A(T) \subseteq A(C_i \cap C_j)$. Since $C_i \cap C_j$ is a union of some vertex-disjoint paths, hence T is a path of $C_i \cap C_j$. Next, we can claim that T is a maximal subpath of $C_i \cap C_j$. Otherwise, if $C_i \cap C_j$ has a path T' with $A(T) \subset A(T')$, then by Theorem 2.2, $C'_i \cap C'_j$ has a path P'' with $V(P'') = A(T')$. Since $A(T) \subset A(T')$, hence $|V(P'')| > |V(P')|$, a contradiction to assumption that P' is a maximal path of $C'_i \cap C'_j$. Hence $C'_i \cap C'_j = P'_{ij_1} \cup P'_{ij_2} \cup \cdots \cup P'_{ij_{\ell'_{ij}}} = L(P_{ij_1}) \cup L(P_{ij_2}) \cup \cdots \cup L(P_{ij_{\ell_{ij}}})$. Since for any $P \in \bigcup_{i=1}^{k} \bigcup_{j=i+1}^{k} \{P_{ij_1}, P_{ij_2}, \ldots, P_{ij_{\ell_{ij}}}\}$, we have $|V(P)| = s$, and so for any $P' \in \bigcup_{i=1}^{k} \bigcup_{j=i+1}^{k} \{P'_{ij_1}, P'_{ij_2}, \ldots, P'_{ij_{\ell'_{ij}}}\}$, $|V(P')| = s-1$ with $s \geq 2$. Hence $L(D) \in \mathcal{D}(s-1)$. This proves the lemma. □

By Lemma 3.1, if $D \in \mathcal{D}(s)$ with $s \geq 2$, then $L(D) \in \mathcal{D}(s-1)$, and so $L^2(D) \in \mathcal{D}(s-2)$. Applying Lemma 3.1 repeatedly, we conclude that

$$\text{if } D \in \mathcal{D}(s) \text{ with } s \geq 2, \text{ then for any } t \text{ with } 1 \leq t \leq s-1,\ L^t(D) \in \mathcal{D}(s-t). \tag{4}$$

Theorem 3.1. *Let s be an integer with $s \geq 1$, and $D \in \mathcal{D}(s)$ be a strong directed multigraph defined as in Example 3.1. Then*

$$h(D) = \begin{cases} 0, & \text{if } D \in \mathcal{D}(0) \text{ or } D \in \mathcal{D}(s) \cap \mathcal{H}, \\ s, & \text{if } D \in \mathcal{D}(s) - \mathcal{H}. \end{cases}$$

Proof. For any $D \in \mathcal{D}(0)$, by Example 3.1 and D is strong, we have D is a cycle. Thus $D \in \mathcal{H}$, and so $h(D) = 0$. If $D \in \mathcal{D}(s) \cap \mathcal{H}$, then D is hamiltonian, we have $h(D) = 0$.

Assume now that $D \in \mathcal{D}(s) - \mathcal{H}$ with $s \geq 1$.

If $s = 1$, then $D \in \mathcal{D}(1)$, and so D is eulerian. Applying Theorem 2.1 (*ii*), $L(D)$ is hamiltonian. Hence as $D \in \mathcal{D}(s) - \mathcal{H}$, we have $h(D) = 1 = s$.

If $s \geq 2$, by (4), then $L^{s-1}(D) \in \mathcal{D}(1)$, and so $L^{s-1}(D)$ is eulerian. By Theorem 2.1 (*ii*), $L^s(D)$ is hamiltonian and $h(D) \leq s$. Next, we are to prove that $h(D) = s$ with $s \geq 2$. By Theorem 2.4, it suffices to prove that $D \in \mathcal{F} - \mathcal{H}$, $D \in \mathcal{F}(s)$ and $D \notin \mathcal{F}(t)$ for any t with $1 \leq t \leq s-1$.

Firstly, since $L^{s-1}(D)$ is eulerian, hence for any vertex $W \in V(L^{s-1}(D))$, we have $d^-_{L^{s-1}(D)}(W) = d^+_{L^{s-1}(D)}(W)$. By Proposition 2.1 (*ii*), $W \in W_{s-1}(D)$. Let $W = x_1 x_2 \cdots x_s$. Since $d^+_{L^{s-1}(D)}(W) = d^-_{L^{s-1}(D)}(W)$, applying Proposition 2.1 (*iv*), thus $d^-_D(x_1) = d^+_D(x_s)$, and so $D \in \mathcal{F}(s)$ with $s \geq 2$. Since $\mathcal{F}(s) \subset \mathcal{F}$ and $D \in \mathcal{D}(s) - \mathcal{H}$, hence $D \in \mathcal{F} - \mathcal{H}$.

Secondly, we are to show that for any t with $1 \leq t \leq s-1$, $D \notin \mathcal{F}(t)$. Since D is non-hamiltonian, and $\mathscr{C}$ is a cycle cover of strong directed multigraph D, hence there exist two distinct cycles $C_i, C_j \in \mathscr{C}$ such that $\{P_{ij_1}, P_{ij_2}, \cdots, P_{ij_{\ell_{ij}}}\} \neq \emptyset$. Since $D \in \mathcal{D}(s)$ with $s \geq 2$, hence let $P = v_1 v_2 \cdots v_s \in \{P_{ij_1}, P_{ij_2}, \cdots, P_{ij_{\ell_{ij}}}\}$. Then $d^-_D(v_1) \geq 2$. For any t with $1 \leq t \leq s-1$, since $s \geq 2$, hence $d^+_D(v_t) = 1$, and so $d^-_D(v_1) \neq d^+_D(v_t)$. Thus $v_1 v_2 \cdots v_t$ is a $(t-1)$-path of D with $d^-_D(v_1) \neq d^+_D(v_t)$, and so $D \notin \mathcal{F}(t)$. Hence by Theorem 2.4, $h(D) = s$. □

4. Line digraph stable properties

Let $\mathcal{P}$ be a graphical property and $\mathcal{D}' \subset \mathcal{D}$ denote a subfamily of strong directed multigraphs. A property $\mathcal{P}$ is **line digraph stable** in $\mathcal{D}'$ if for any $D \in \mathcal{D}'$, and for any positive integer $k \geq 1$, every $L^k(D)$ has property $\mathcal{P}$, whenever D has property $\mathcal{P}$.

Lemma 4.1. *Let D be a strong directed multigraph. Then $L(D)$ is regular if and only if D is regular.*

Proof. If D is regular, then there is a positive integer t with $t \geq 1$ such that for each vertex $x \in V(D)$, $d_D^-(x) = d_D^+(x) = t$. Hence for any arc $a' = (u, v) \in A(D)$, we have $d_D^-(u) = d_D^+(v) = t$. By Proposition 2.1, $a' \in V(L(D))$ and $d_{L(D)}^-(a') = d_{L(D)}^+(a') = t$, and so $L(D)$ is regular.

Conversely, if $L(D)$ is regular, then there exists an integer t' with $t' \geq 1$ satisfying

$$\text{for each vertex } a = (u, v) \in V(L(D)), d_{L(D)}^-(a) = d_{L(D)}^+(a) = t'. \tag{5}$$

Next, we are to prove that D is regular. Assume by contradiction that D is not regular. Then there exist two distinct vertices $x, y \in V(D)$ such that $d_D^-(x) = t_1, d_D^+(y) = t_2$ and $t_1 \neq t_2$. Since D is a strong directed multigraph, hence there exist two arcs $b_1 \in \partial_D^+(x)$ and $b_2 \in \partial_D^-(y)$. Thus $b_1, b_2 \in V(L(D))$. By Proposition 2.1 (iii), $d_{L(D)}^+(b_2) = d_D^+(y) = t_2$ and $d_{L(D)}^-(b_1) = d_D^-(x) = t_1$. Since $t_1 \neq t_2$, hence $d_{L(D)}^-(b_1) \neq d_{L(D)}^+(b_2)$, contrary to (5). This proves the lemma. □

Lemma 4.2. *A strong directed multigraph D is eulerian and $d_D^-(u) = d_D^+(v)$ for any arc $a = (u, v) \in A(D)$ if and only if D is regular.*

Proof. Assume first that

$$D \text{ is eulerian and } d_D^-(u) = d_D^+(v) \text{ for any arc } a = (u, v) \in A(D). \tag{6}$$

Next, we are to prove that D is regular. Assume by contradiction that D is not regular. As D is eulerian, hence

$$\text{for any vertex } x \in V(D),\ d_D^-(x) = d_D^+(x). \tag{7}$$

Since D is not regular, hence there are two distinct vertices $x_1, x_t \in V(D)$ and two distinct positive integers t_1 and t_2 with $d_D^+(x_1) = d_D^-(x_1) = t_1$ and $d_D^+(x_t) = d_D^-(x_t) = t_2$. As D is a strong directed multigraph, hence there is a path P from x_1 to x_t of D. Let $P = x_1x_2 \cdots x_t$. By (6) and (7), $d_D^+(x_t) = d_D^-(x_t) = d_D^-(x_2) = d_D^+(x_2) = \cdots = d_D^-(x_{t-1}) = d_D^+(x_{t-1}) = t_1$. Since $t_1 \neq t_2$, hence $d_D^-(x_{t-1}) \neq d_D^+(x_t)$, contrary to (6).

Conversely, assume that D is regular. Thus there is a positive integer k such that for each vertex $x \in V(D)$, $d_D^-(x) = d_D^+(x) = k$. Hence D is eulerian and $d_D^-(u) = d_D^+(v)$ for any arc $a = (u, v) \in A(D)$. □

Theorem 4.1. *Let D be a strong directed multigraph. Then for each $k \geq 1$, $L^k(D)$ is both eulerian and hamiltonian if and only if D is regular.*

Proof. If D is regular, then by Lemma 4.1, for each $k \geq 1$, $L^k(D)$ is regular. Thus, for any $k' \geq 0$, $L^{k'}(D)$ is regular. By Lemma 4.2, $L^{k'}(D)$ is eulerian and $d_{L^{k'}(D)}^-(u) = d_{L^{k'}(D)}^+(v)$ for any arc $a = (u, v) \in A(L^{k'}(D))$. Thus by Theorem 2.1, for each $k \geq 1$, $L^k(D)$ is eulerian and hamiltonian.

Conversely, for each $k \geq 1$, if $L^k(D)$ is eulerian and hamiltonian, then by Theorem 2.1, for each $k' \geq 0$, $L^{k'}(D)$ is eulerian, and for any arc $a = (u, v) \in A(L^{k'}(D))$, we get that $d_{L^{k'}(D)}^-(u) = d_{L^{k'}(D)}^+(v)$. Thus by Lemma 4.2, $L^{k'}(D)$ is regular, and so D is regular as well. □

Define $\mathcal{R} = \{D \in \mathcal{D}\colon D \text{ is regular}\}$. By Definition 2.1 (ii), we have $\mathcal{R} \subset \mathcal{F}(1)$, and so $\mathcal{R} \subset \mathcal{F}$ as well.

Theorem 4.2. *Let $\mathcal{H}' \subseteq \mathcal{D}$ be a subfamily of strong directed multigraphs. Each of the following holds.*

(i) *If $\mathcal{H}' \subseteq \mathcal{R}$, then being eulerian and being hamiltonian are line digraph stable in $\mathcal{H}'$.*
(ii) *If $\mathcal{H}' \subseteq \mathcal{F}$, then being eulerian is line digraph stable in $\mathcal{H}'$ if and only if $\mathcal{H}' \subseteq \mathcal{R}$.*
(iii) *If $\mathcal{H}' \subseteq \mathcal{F} \cup \mathcal{H}$, then being hamiltonian is line digraph stable in $\mathcal{H}'$ if and only if $\mathcal{H}' \subseteq \mathcal{R}$.*

Proof. For any $D \in \mathcal{H}'$, if $\mathcal{H}' \subseteq \mathcal{R}$, then D is regular. By Theorem 4.1, for any $k \geq 1$, $L^k(D)$ is eulerian and hamiltonian. Thus, being eulerian and being hamiltonian are line digraph stable in $\mathcal{H}'$. Hence (i) holds.

If $\mathcal{H}' \subseteq \mathcal{F}$ and being eulerian is line digraph stable in $\mathcal{H}'$, then for any $D \in \mathcal{H}'$ and any k with $k \geq 1$, $L^k(D)$ is eulerian. Applying Theorem 2.1 (ii), for any $k' \geq 2$, we conclude that $L^{k'}(D)$ is eulerian and hamiltonian. Thus by Theorem 4.1, $L(D)$ is regular, and so by Lemma 4.1, D is regular. Hence $\mathcal{H}' \subseteq \mathcal{R}$. Conversely, if $\mathcal{H}' \subseteq \mathcal{R}$, then by (i), being eulerian is line digraph stable in $\mathcal{H}'$. Thus (ii) holds.

If $\mathcal{H}' \subseteq \mathcal{F} \cup \mathcal{H}$ and being hamiltonian is line digraph stable in $\mathcal{H}'$, then for any $D \in \mathcal{H}'$ and any $k \geq 1$, $L^k(D)$ is hamiltonian. Applying Theorem 2.1 (ii), we conclude that $L^k(D)$ is eulerian and hamiltonian. By Theorem 4.1, D is regular, and so $\mathcal{H}' \subseteq \mathcal{R}$. Conversely, if $\mathcal{H}' \subseteq \mathcal{R}$, then by (i), being hamiltonian is line digraph stable in $\mathcal{H}'$. Thus (iii) holds. This proves the theorem. □

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References

[1] M. Aigner, On the line graph of a directed graph, Math. Z. 102 (1) (1967) 56–61.
[2] J. Bang-Jensen, G. Gutin, Digraphs: Theory, Algorithms and Applications, 2nd, Springer-Verlag, London, 2009.
[3] J. Bang-Jensen, G. Gutin, Classes of Directed graphs, Springer-Verlag, London, 2018.
[4] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, New York, 2008.
[5] G. Chartrand, On hamiltonian line-graphs, Trans Am Math Soc 134 (3) (1968) 559–566.
[6] G. Chartrand, C.E. Wall, On the hamiltonian index of a graph, Studia Scientiarum Mathematicarum Hungarica 8 (1973) 43–48.
[7] Z.-H. Chen, H.-J. Lai, L.M. Xiong, H.Y. Yan, M.Q. Zhan, Hamilton-connected indices of graphs, Discrete Math 309 (14) (2009) 4819–4827.
[8] L.K. Clark, N.C. Wormald, Hamiltonian like indices of graphs, Ars Combinatoria 15 (1983) 131–148.
[9] R.J. Gould, On line graphs and the hamiltonian index, Discrete Math 34 (1981) 111–117.
[10] L.S. Han, H.-J. Lai, L.M. Xiong, H.Y. Yan, The chvátal-erdös condition for supereulerian graphs and the hamiltonian index, Discrete Math 310 (15–16) (2010) 2082–2090.
[11] Y. Hong, J.-L. Lin, Z.-S. Tao, Z.H. Chen, The hamiltonian index of graphs, Discrete Math 309 (1) (2009) 288–292.
[12] H.J. Lai, On the hamiltonian index, Discrete Math 69 (1) (1988) 43–53.
[13] H.-J. Lai, Y.H. Shao, Some problems related to hamiltonian line graphs, AMS/IP Studies in Advance Mathematics 39 (2007) 149–159.
[14] J. Liu, H. Yang, H.-J. Lai, X.D. Zhang, Hamiltonian-connected line digraphs, Advances in Mathematics(China)Accepted.
[15] Z. Ryjáček, G.J. Woeginger, L.M. Xiong, Hamiltonian index is NP-complete, Discrete Appl. Math. 159 (4) (2011) 246–250.
[16] M.L. Saražin, On the hamiltonian index of a graph, Discrete Math 122 (1–3) (1993) 373–376.
[17] M.L. Saražin, A simple upper bond for the hamiltonian index of a graph, Discrete Math 134 (1–3) (1994) 85–91.
[18] L.M. Xiong, H.J. Broersma, X.L. Li, M.C. Li, The hamiltonian index of a graph and its branch-bonds, Discrete Math 285 (1–3) (2004) 279–288.
[19] L.M. Xiong, Z.H. Liu, Hamiltonian iterated line graphs, Discrete Math 256 (1–2) (2002) 407–422.
[20] L.M. Xiong, Q.X. Wu, The hamiltonian index of a 2-connected graph, Discrete Math 308 (24) (2008) 6373–6382.
[21] L.L. Zhang, E. Eschen, H.-J. Lai, Y.H. Shao, The s-hamiltonian index, Discrete Math 308 (20) (2008) 4779–4785.