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ORIGINAL PAPER

The Extremal Sizes of Arc-Maximal (*k*, *l*)-Digraphs

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Abstract

Boesch and McHugh in [J. Combinatorial Theory Ser. B 38 (1985), 1-7] introduced the edge-maximal (k,ℓ)-graphs to study of network subcohesion, and obtained best possible upper size bounds for all edge-maximal (k,ℓ)-graphs. The best possible lower bounds are obtained in [J. Graph Theory 18 (1994), 227-240]. Let $k,\ell > 0$ be integers. A strict digraph D is a (k,ℓ)-digraph if for any subdigraph H of D, that $|V(H)| \geq \ell$ implies $\lambda(H) \leq k-1$. An arc-maximal (k,ℓ)-digraph D is one such that for any $e \in A(D^c)$, $D+e$ is not a (k,ℓ)-digraph. We show that there is a close relationship between the extremal edge-maximal (k,ℓ)-graphs and the extremal arc-maximal (k,ℓ)-digraphs. This is applied to determine the optimal upper and lower bounds of the sizes of an arc-maximal (k,ℓ)-digraphs. Moreover, the arc-maximal (k,ℓ)-digraphs reaching the lower bounds and the upper bounds are respectively characterized.

Keywords Maximum subgraph edge-connectivity · Arc-strong connectivity · Maximum subdigraph arc-strong connectivity · $((k,\ \ell))$-digraphs

Mathematics Subject Classification 05C20 · 05C35 · 05C40

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1 Introduction

Throughout this paper, we consider finite simple graphs and strict digraphs, and normally use G to denote a graph and D a digraph. Undefined terms and notation will follow [5] and [2]. In particular, $\kappa'(G)$ denotes the edge connectivity of a graph G and $\lambda(D)$ denotes the arc-strong connectivity of a digraph D. We shall use (u, v) to denote an arc oriented from u to v in a digraph. If $W \subseteq V(D)$ or if $W \subseteq A(D)$, then $D[W]$ denotes the subdigraph of D induced by W. For $v \in V(D)$, we use $D - v$ for $D[V(D) - v]$. We use $H \subseteq G$ to mean that H is a subgraph of G, and $H \subseteq D$ to mean that H is a subdigraph of D. A simple graph G on n vertices can be viewed as a spanning subgraph of K_n. Let $G^c = K_n - E(G)$ be the **complement of** G. Likewise, let K_n^* be the strict digraph on n vertices such that for every pair of distinct vertices $u, v \in V(K_n^*)$, both $(u, v) \in A(K_n^*)$ and $(v, u) \in A(K_n^*)$. Thus K_n^* is the complete digraph on n vertices. Any strict digraph D on n vertices can be viewed as a spanning subdigraph of K_n^*, that is for every pair of distinct vertices $u, v \in V(K_n^*)$, $(u, v) \in A(D)$, or $(v, u) \in A(D)$, or both $(u, v), (v, u) \in A(D)$, and let $D^c = K_n^* - A(D)$ be the **complement of** D. If $X \subseteq E(\overline{G})$, then $G + X$ is the simple graph with vertex set $V(G)$ and edge set $E(G) \cup X$. We will use $G + e$ for $G + \{e\}$. Likewise, for $X \subseteq A(D^c)$ and $e \in A(D^c)$, we similarly define the strict digraphs $D + X$ and $D + e$, respectively.

As shown in [4], given a family of simple graphs $\mathcal{F}$, and an integer $n > 0$, determining the extremal size of a simple graph on n vertices that does not contain a subgraph isomorphic to a member in $\mathcal{F}$ has been one of the mostly studied problem in graph theory. The theorems of Mental (see Chapter 2 of [5]) and Turán ([14]) on the case when $\mathcal{F} = \{K_k\}$ are well-known. Mader in [9] studied the case when $\mathcal{F} = \{G : \kappa'(G) \geq k - 1\}$, which is related to the study of network subcohesion by Matula [10–12]. Extending the work of Mader and Matula, Boesch and McHugh in [3] studied the case when $\mathcal{F} = \{G : \kappa'(G) \geq k - 1, |V(G)| \geq \ell\}$. For integers n, k, ℓ with $n \geq \ell > k \geq 2$, Boesch and McHugh defines a (k, ℓ)-**graph** to be a simple graph G such that for any subgraph H of G with $|V(H)| \geq \ell$ satisfies $\kappa'(H) \leq k - 1$. A (k, ℓ)-graph G is **an edge-maximal** (k, ℓ)-**graph** if, for any $e \in E(G^c)$, $G + e$ has a subgraph H with $|V(H)| \geq \ell$ and $\kappa'(H) \geq k$. Let $\mathcal{E}(n, k, \ell)$ denote the collection of all edge-maximal (k, ℓ)-graphs of order n and $\mathcal{E}(k, l) = \cup_{n \geq \ell} \mathcal{E}(n, k, \ell)$. For integers $n \geq l \geq k + 1$, define

$$\begin{aligned} \Phi(n, k, \ell) &= \max\{|E(G)| : G \in \mathcal{E}(n, k, \ell)\}, \\ \mathcal{MM}(n, k, \ell) &= \{G : G \in \mathcal{E}(n, k, \ell) \text{ and } |E(G)| = \Phi(n, k, \ell)\}, \\ \phi(n, k, \ell) &= \min\{|E(G)| : G \in \mathcal{E}(n, k, \ell)\}, \\ \mathcal{SM}(n, k, \ell) &= \{G : G \in \mathcal{E}(n, k, \ell) \text{ and } |E(G)| = \phi(n, k, \ell)\}. \end{aligned} \tag{1}$$

Mader [9] initiated the study of edge-maximal $(k, k + 1)$-graphs. In [3], Boesch and McHugh extended the study to edge-maximal (k, ℓ)-graphs for any $\ell \geq k + 1$. The subject has been studied by quite a few researchers, as seen in [3, 6, 7, 9, 11, 13], among others.

Theorem 1.1 *Let k, n be integers with $n > k + 1 \geq 2$. Each of the following holds.*

(i) (Mader [9]) $\Phi(n,k,k+1) = (n-k)k + \binom{k}{2}$. Furthermore, all graphs in $\mathcal{MM}(n,k,k+1)$ are recursively characterized.

(i) (Lai, Theorem 2 of [6]) $\phi(n,k,k+1) = (n-1)k - \binom{k}{2}\lfloor\frac{n}{k+2}\rfloor$. Furthermore, all graphs in $\mathcal{SM}(n,k,k+1)$ are recursively characterized.

Theorem 1.2 (*F. T. Boesch and J. A. M. McHugh, Theorem* 1 *of* [3]) *Let k,ℓ,n be integers with $n \geq \ell \geq k$, and $s,r \geq 0$ be integers satisfying $n = s(\ell-1)+r$ with $0 \leq r \leq \ell-2$.*

$$\Phi(n,k,\ell) = \begin{cases} \frac{s(\ell-1)(\ell-2)}{2} + (s-1+r)(k-1) & \text{if } 2(k-1) < \ell-1 \text{ and } r < 2(k-1), \\ \frac{s(\ell-1)(\ell-2)}{2} + s(k-1) + \frac{r(r-1)}{2} & \text{if } 2(k-1) < \ell-1 \text{ and } r \geq 2(k-1), \\ \frac{(\ell-1)(\ell-2)}{2} + (n-\ell+1)(k-1) & \text{if } 2(k-1) \geq \ell-1. \end{cases}$$

Furthermore, classes of edge-maximal (k,ℓ)-graphs on n vertices with sizes equaling $\Phi(n,k,\ell)$ are constructed.

The value of $\phi(n,k,\ell)$ is determined in [7], with the extremal graphs characterized using the following definitions.

Definition 1.3 ([7]) Let k, ℓ and r be positive integers with $\ell \geq k+2$ and $r \geq 2$. Let G_1, G_2, ..., G_r be mutually vertex disjoint simple graphs.

(i) Let G_1 and G_2 be vertex-disjoint simple graphs with $\max\{|V(G_1)|,|V(G_2)|\} \geq k$, a (k,ℓ)**-joint of** G_1 **and** G_2 is a simple graph obtained from the disjoint union of G_1 and G_2 by adding k new edges e_1, e_2, ..., e_k to $G_1 \cup G_2$ such that each e_i is incident with a vertex in $V(G_1)$ and a vertex in $V(G_2)$, and such that if the new edges e_1, e_2, ..., e_k are joining two maximal complete subgraphs $K_{r_1} \subseteq G_1$ and $K_{r_2} \subseteq G_2$ where $r_i = 1$ or $k+1 \leq r_i \leq \ell-1$ $(i=1,2)$, then the orders of these subgraphs must satisfy $r_1 + r_2 \geq l$. Denote by $[G_1,G_2]_k^\ell$ the set of all (k,ℓ)-joints of G_1 and G_2. For notational convenience, we denote $[G,G]_k^\ell = \{G\}$.

(ii) Inductively, assume that $r \geq 3$ is an integer and that the (k,ℓ)-joints of of any group of at most $r-1$ graphs have been defined. Let G_1, G_2, ..., G_r be vertex-disjoint simple graphs with max $\{|V(G_1)|,\ldots,|V(G_r)|\} \geq k$. Define a (k,ℓ)-joint of G_1, G_2, ..., G_r by partitioning these graphs into two groups: $G_1, G_2, \ldots, G_m$ and $G_{m+1}, \ldots, G_r$ (say), where $1 \leq m \leq r-1$. A (k,ℓ) **-joint of** G_1, G_2, ..., G_r is a graph in the form $G \in [G',G'']_k^\ell$ for some $G' \in [G_1,G_2,\ldots,G_m]_k^\ell$ and $G'' \in [G_{m+1},G_{m+2},\ldots,G_r]_k^\ell$. Let $[G_1,G_2,\ldots,G_r]_k^\ell$ denote the set of all (k,ℓ)-joints of G_1, G_2, ..., G_r.

Theorem 1.4 (*Lai and Zhang*, [7]) *Let $n \geq \ell \geq k+2 \geq 5$ be integers. Then*

$$\phi(n,k,\ell)=\begin{cases}\frac{(\ell-1)(\ell-2)}{2}+(n-\ell+1)(k-1) & \text{if } \ell\leq n<2k+2,\\ (n-1)(k-1)-\lfloor\frac{n}{k+1}\rfloor\frac{k^2-3k}{2} & \text{if } \ell\leq 2k+2\leq n,\\ (n-2t+1)(k-1)+t(t-1)-\lfloor\frac{n-2t}{k+1}\rfloor\frac{k^2-3k}{2} & \text{if } n\geq\ell=2t\geq 2k+3,\\ (n-2t)(k-1)+t^2-\lfloor\frac{n-2t-1}{k+1}\rfloor\frac{k^2-3k}{2} & \text{if } n\geq\ell=2t+1\geq 2k+3.\end{cases}$$

Furthermore, a graph G is in $\mathcal{SM}(n,k,\ell)$ if and only if one of the following holds.

(i) $\ell\leq n<2k+2$, and either $G\in[K_{l-1},K_1]^{\ell}_{k-1}$ or G has a vertex v of degree $k-1$ such that $G-v\in\mathcal{SM}(n-1,k,1)$.

(ii) $\ell\leq 2k+2\leq n$, and either $G\in[K_{k+1},K_{k+1}]^{\ell}_{k-1}$ or $G\in[H_1,H_2]^{\ell}_{k-1}$ such that $H_1,H_2\in\{K_1,K_{k+1}\}\cup\mathcal{SM}(n,k,\ell)$, $|V(H_i)|\in\{1,k+1\}$ or for each $i\in\{1,2\}$, $|V(H_i)|\geq 2k+2$ and $\lfloor\frac{|V(H_1)|}{k+1}\rfloor+\lfloor\frac{|V(H_2)|}{k+1}\rfloor=\lfloor\frac{n}{k+1}\rfloor$.

(iii) $n\geq l=2t\geq 2k+3$, and either $G\in[K_t,K_t]^{\ell}_{k-1}$ or $G\in[H_1,H_2]^{\ell}_{k-1}$ such that $H_1\in\{K_1,K_{k+1}\}$, $H_2\in\mathcal{SM}(|V(H_2)|,k,\ell)$ with $|V(H_2)|\geq 2t$, and $\lfloor\frac{|V(H_1)|}{k+1}\rfloor+\lfloor\frac{|V(H_2)|-2t}{k+1}\rfloor=\lfloor\frac{n-2t}{k+1}\rfloor$.

(iv) $n\geq l=2t+1\geq 2k+3$ and either $G\in[K_{t+1},K_t]^{\ell}_{k-1}$ or $G\in[H_1,H_2]^{\ell}_{k-1}$ such that $H_1\in\{K_1,K_{k+1}\}$, $H_2\in\mathcal{SM}(|V(H_2)|,k,\ell)$ with $|V(H_2)|\geq 2t+1$, and $\lfloor\frac{|V(H_1)|}{k+1}\rfloor+\lfloor\frac{|V(H_2)|-2t-1}{k+1}\rfloor=\lfloor\frac{n-2t-1}{k+1}\rfloor$.

It is natural to investigate the corresponding problems for strict digraphs. Given integers ℓ and k with $\ell\geq k\geq 2$, a strict digraph D with $|V(D)|\geq\ell>k\geq 2$ is a (k,ℓ)**-digraph** if for any $H\subseteq D$ with $|V(H)|\geq\ell$ satisfies $\lambda(H)\leq k-1$. A (k,ℓ)-digraph D is an **arc-maximal** (k,ℓ) **-digraph** if, for any $e\in A(D^c)$, $D+e$ has a subgraph H with $|V(H)|\geq\ell$ and $\lambda(H)\geq k$. Let $\mathcal{A}(n,k,\ell)$ be the family of all arc-maximal (k,ℓ)-digraphs on n vertices and $\mathcal{A}(k,\ell)=\cup_{n\geq\ell}\mathcal{A}(n,k,\ell)$. Define

$$\begin{aligned} f(n,k,\ell)&=\min\{|A(D)|:D\in\mathcal{A}(n,k,\ell)\},\\ F(n,k,\ell)&=\max\{|A(D)|:D\in\mathcal{A}(n,k,\ell)\},\\ \mathcal{SA}(n,k,\ell)&=\{D:D\in\mathcal{A}(n,k,\ell),\text{ and }|A(D)|=f(n,k,\ell)\},\\ \mathcal{MA}(n,k,\ell)&=\{D:D\in\mathcal{A}(n,k,\ell),\text{ and }|A(D)|=F(n,k,\ell)\}.\end{aligned}\tag{2}$$

Theorem 1.5 *Let k, n be integers with $n>k\geq 2$.*

(i) (Anderson, Lai, Lin and Xu, Theorem 1.2 of [1]) $F(n,k,k+1)=nk-\frac{k(k+1)}{2}+\frac{n(n-1)}{2}$. Moreover, all digraphs in $\mathcal{MA}(n,k,k+1)$ are recursively characterized.

(ii) (Lin, Fan, Lai and Xu, Theorem 1.4 of [8]) $f(n,k,k+1)=(n-1)k-\frac{k(k-1)}{2}\lfloor\frac{n}{k+2}\rfloor+\frac{n(n-1)}{2}$. Moreover, all digraphs in $\mathcal{SA}(n,k,k+1)$ are recursively characterized.

The main purpose of this research is, for any $n \geq \ell \geq k+2$, to determine the values of $F(n,k,\ell)$ and $f(n,k,\ell)$ and to characterize the extremal digraphs. Motivated by the comparisons of Theorems 1.1 and 1.5, we have found a relationship between edge-maximal (k,ℓ)-graphs and arc-maximal (k,ℓ)-digraphs, which allows us to characterize digraphs in $\mathcal{SA}(n,k,\ell)$ and $\mathcal{MA}(n,k,\ell)$.

In the next section, we investigate properties of arc-maximal $(k,\, l)$-digraphs, to be applied in our arguments. In Section 3, we focus on the discussion of a relationship between extremal (k,ℓ)-graphs and extremal (k,ℓ)-digraphs. To obtain our main results and determine the values of $F(n,k,\ell)$ and $f(n,k,\ell)$, complete and refined characterizations of graphs in $\mathcal{MM}(n,k,\ell)$ and $\mathcal{SM}(n,k,\ell)$ are respectively presented in Section 3, which extend Theorems 1.2 and 1.4. The main results are the following, which will be proved in the last section.

Theorem 1.6 *Let n,k,ℓ be integers with $n \geq \ell \geq k+2 \geq 5$, and $s,r \geq 0$ be integers satisfying $n = s(\ell-1)+r$ with $0 \leq r \leq \ell-2$. Then*

$$F(n,k,\ell)=\begin{cases}\dfrac{s(\ell-1)(\ell-2)}{2}+(s-1+r)(k-1)+\dfrac{n(n-1)}{2} & \text{if } r<2(k-1)<\ell-1,\\ \dfrac{s(\ell-1)(\ell-2)}{2}+s(k-1)+\dfrac{r(r-1)}{2}+\dfrac{n(n-1)}{2} & \text{if } 2(k-1)<\ell-1 \text{ and } r>2(k-1),\\ \dfrac{s(\ell-1)(\ell-2)}{2}+s(k-1)+\dfrac{r(r-1)}{2}+\dfrac{n(n-1)}{2} & \text{if } r=2(k-1)<\ell-1,\\ \dfrac{s(\ell-1)(\ell-2)}{2}+(n-\ell+1)(k-1)+\dfrac{n(n-1)}{2} & \text{if } 2(k-1)>\ell-1,\\ \dfrac{s(\ell-1)(\ell-2)}{2}+(n-\ell+1)(k-1)+\dfrac{n(n-1)}{2} & \text{if } 2(k-1)=\ell-1.\end{cases}$$

Furthermore, all the bounds are best possible.

Theorem 1.7 *Let n,k,ℓ be integers with $n \geq \ell \geq k+2 \geq 5$, and $s,r \geq 0$ be integers satisfying $n = s(\ell-1)+r$ with $0 \leq r \leq \ell-2$. Define $q_1 = \lfloor \frac{n}{k+1} \rfloor$ and $q_2 = \lfloor \frac{n-\ell}{k+1} \rfloor$. Then*

$$f(n,k,\ell)=\begin{cases}\dfrac{(\ell-1)(\ell-2)}{2}+(n-\ell+1)(k-1)+\dfrac{n(n-1)}{2} & \text{if } \ell \leq n<2k+2,\\ (n-1)(k-1)-\lfloor\dfrac{n}{k+1}\rfloor\dfrac{k^2-3k}{2}+\dfrac{n(n-1)}{2} & \text{if } \ell \leq 2k+2 \leq n,\\ (n-2t+1)(k-1)+t(t-1)-\lfloor\dfrac{n-2t}{k+1}\rfloor\dfrac{k^2-3k}{2}+\dfrac{n(n-1)}{2} & \text{if } n \geq \ell=2t \geq 2k+3,\\ (n-2t)(k-1)+t^2-\lfloor\dfrac{n-2t-1}{k+1}\rfloor\dfrac{k^2-3k}{2}+\dfrac{n(n-1)}{2} & \text{if } n \geq \ell=2t+1 \geq 2k+3.\end{cases}$$

Furthermore, all the bounds are best possible.

2 Properties of (k,ℓ)-Digraphs

Let n, k, ℓ be positive integers. By Theorem 1.5, we assume throughout the paper, that $\ell \geq k+2$. It follows from the definition of a (k,ℓ)-digraph that

$$\text{every } (k,\ell)\text{-digraph D with } |V(D)| \geq \ell \geq k+2 \text{ satisfies } \lambda(D) \leq k-1. \tag{3}$$

Following [2], for a digraph D and vertex subsets $X, Y \subseteq V(D)$, define $(X,Y)_D = \{(x,y) \in A(D) : x \in X, y \in Y\}$ and

$$\partial_D^+(X) = (X, V(D)-X)_D \text{ and } \partial_D^-(X) = (V(D)-X, X)_D.$$

For a vertex $v \in V(D)$, we define

$$N_D^+(v) = \{u \in V(D) : (v,u) \in A(D)\} \text{ and } N_D^-(v) = \{u \in V(D) : (u,v) \in A(D)\}.$$

Definition 2.1 Let k, ℓ and r be positive integers with $\ell \geq k+2$ and $r \geq 2$. Let D_1, D_2, ..., D_r be mutually vertex disjoint strict digraphs.

(i) A (k,ℓ) **-joint from** D_1 **to** D_2 is a strict digraph obtained from the disjoint union of D_1 and D_2 by adding k new arcs (called the **forward arcs**) e_1, e_2, $\cdots$, e_k, each of which is oriented from a vertex in D_1 to a vertex in D_2, and by adding all possible arcs (called the **backward arcs**) of the form $\{(u,v) : v \in V(D_1) \text{ and } u \in V(D_2)\}$, in such a way that if the new arcs e_1, e_2, ..., e_k are joining two maximal complete subdigraphs $K^*_{r_1} \subseteq D_1$ and $K^*_{r_2} \subseteq D_2$, then the orders of these subdigraphs must satisfy $r_1 + r_2 \geq \ell$. We use $[D_1, D_2]_k^\ell$ to denote the family of all (k,ℓ) **-joints of** D_1 **and** D_2, consisting of all (k,ℓ)-joints from D_1 to D_2 as well as all (k,ℓ)-joints from D_2 to D_1. Thus $[D_1, D_2]_k^\ell = [D_2, D_1]_k^\ell$.

(ii) Inductively, assume that $r \geq 3$, and that for some integer m with $1 \leq m \leq r-1$, both $[D_1, D_2, \ldots, D_m]_k^\ell$, the family of all the (k,ℓ)-joints of D_1, D_2, ..., D_m and $[D_{m+1}, D_{m+2}, \ldots, D_r]_k^\ell$, the family of all the (k,ℓ)-joints of D_{m+1}, D_{m+2}, ..., D_r have been obtained. Define a (k,ℓ)**-joint of** D_1, D_2, ..., D_m, D_{m+1} ..., D_r to be a strict digraph in $[D', D'']_k^\ell$, where $D' \in [D_1, D_2, \ldots, D_m]_k^\ell$ and $D'' \in [D_{m+1}, D_{m+2}, \ldots, D_r]_k^\ell$. We use $[D_1, D_2, \ldots, D_r]_k^\ell$ denote the family of all (k,ℓ)-joints of D_1, D_2, ..., D_r. Thus if $(D_1', D_2', ..., D_r')$ is an r-tuple formed by permuting the components of $(D_1, D_2, \ldots, D_r)$, then $[D_1', D_2', \ldots, D_r']_k^\ell = [D_1, D_2, \ldots, D_r]_k^\ell$.

Following the definition of arc-strong connectivity in [2], a digraph D satisfies $\lambda(D) \geq k$ if and only if for any nonempty proper subset $X \subset V(D)$, we always have $|\partial_D^+(X)| \geq k$. Recall that $\mathcal{A}(n,k,\ell)$ is the collection of all arc-maximal (k,ℓ)-digraphs on n vertices.

Lemma 2.2 *Suppose that $D \in \mathcal{A}(k,\ell)$. For any proper nonempty subset $X \subset V(D)$*

such that $|\partial_D^+(X)| \le k-1$, *each of the following holds*:

(i) $(X, V(D)-X)_{D^c} \neq \emptyset$.
(ii) $|\partial_D^+(X)| = k-1$.
(iii) $(V(D)-X, X)_D = \{(y,x) : \text{for any } y \in V(D)-X \text{ and for any } x \in X\}$.

Proof Let $n = |V(D)|$ and $Y = V(D)-X$. Suppose that $(X,Y)_{D^c} = \emptyset$. Then the arcs in $\partial_D^+(X)$ induce an underlying complete bipartite digraph with a vertex bipartite $\{X, Y\}$. It follows from $|\partial_D^+(X)| \le k-1$ that we must have

$$|X|(n-|X|) = |X||Y| = \partial_D^+(X) \le k-1, \text{ and } |X|+|Y| = n \ge k+1.$$

Thus the minimum of $|X|(n-|X|)$ must be attained at the boundary point of the domain $1 \le |X| \le n-1$, and so $k \le n-1 \le |X|(n-|X|) \le k-1$, a contradiction. This proves (i).

By (i), there exists an arc $e = (x,y) \in (X,Y)_{D^c}$. Since $D \in \mathcal{A}(n,k,\ell)$, $D+e$ has a subdigraph H with $|V(H)| \ge \ell$ and $\lambda(H) \ge k$. As D is a (k,ℓ)-digraph, we must have $e \in A(H)$ and so $x, y \in V(H)$. It follows that $e \in \partial_H^+(X \cap V(H))$ and so $\partial_H^+(X \cap V(H)) - \{e\} \subseteq \partial_D^+(X)$. Thus

$$k-1 \ge |\partial_D^+(X)| \ge |\partial_H^+(X \cap V(H)) - \{e\}| \ge k-1,$$

implying that $|\partial_D^+(X)| = k-1$. This proves (ii).

We argue by contradiction to prove (iii) and assume that for some $x \in X$ and $y \in Y$, the arc $(y,x) \notin A(D)$. Then as $D \in \mathcal{A}(k,\ell)$, it follows that $D+(y,x)$ has a subdigraph H' with $|H'| \ge \ell$ and $\lambda(H') \ge k$. Since D is a (k,ℓ)-digraph, we must have $(y,x) \in A(H')$. Hence both $X \cap V(H') \neq \emptyset$ and $Y \cap V(H') \neq \emptyset$. As $\partial_{H'}^+(X \cap V(H')) \subseteq \partial_D^+(X)$, we have $k \le |\partial_{H'}^+(X \cap V(H'))| \le |\partial_D^+(X)| \le k-1$, a contradiction. □

Lemma 2.3 *Let* $D \in \mathcal{A}(k,\ell)$, *and let* $X \subset V(D)$ *be a proper nonempty subset satisfying either* $|\partial_D^+(X)| = k-1$ *or* $|\partial_D^-(X)| = k-1$. *Each of the following holds*:

(i) For any $e \in A(D[X]^c)$, if H is a subdigraph of $D+e$ with $|V(H)| \ge \ell$ and $\lambda(H) \ge k$, then H must be a subdigraph of $D[X]+e$ with $e \in A(H)$.
(ii) If $D[X]$ is not a complete digraph, then $|X| \ge \ell$ and $D[X] \in \mathcal{A}(k,\ell)$. On the other hand, if $|X| \ge \ell$, then $D[X] \in \mathcal{A}(k,\ell)$.
(iii) If $|X| \le \ell - 1$ and $\ell \ge k+2$, then $D[X]$ is a complete digraph. Moreover, if $k \ge 3$, then either $|X| = 1$ or $|X| \ge k+1$.

Proof Let $n = |V(D)|$. We justify each of the conclusions.

(i) For any $e \in A(D[X]^c)$, suppose that $D+e$ has a subdigraph H of $D+e$ with $|V(H)| \ge \ell$ and $\lambda(H) \ge k$. Since D is a (k,ℓ)-digraph, H cannot be a subdigraph of D and so we must have $e \in A(H)$. Thus $X \cap V(H) \neq \emptyset$. If $V(H) - X \neq \emptyset$, then

$$\partial^+_H(X \cap V(H)) = (V(H) \cap X, V(H) - X)_H \subseteq (X, D - X)_D,$$

and

$$\partial^-_H(X \cap V(H)) = (V(H) - X, V(H) \cap X)_H \subseteq (D - X, X)_D.$$

It follows that $k \le \lambda(H) \le \min\{|\partial^+_H(X \cap V(H))|, |\partial^-_H(X \cap V(H))|\} \le k - 1$, a contradiction. Hence H must be a subdigraph of $D[X] + e$. This proves (i).

(ii) Suppose $D[X]$ is not a complete digraph. As $D \in \mathcal{A}(k, \ell)$, for any $e \in A(D[X]^c)$, $D + e$ has a subdigraph H with both $|V(H)| \ge \ell$ and $\lambda(H) \ge k$. By Lemma 2.3(i), H must be a subdigraph of $D[X] + e$ and so $|X| \ge |V(H)| \ge \ell$. By definition, $D[X] \in \mathcal{A}(k, \ell)$. If $|X| \ge \ell$, then as D is a (k, ℓ)-digraph, $D[X]$ cannot be a complete digraph, and so we also have $A(D[X]^c) \neq \emptyset$, which implies that $D[X] \in \mathcal{A}(k, \ell)$. This justifies Lemma 2.3(ii).

To prove (iii), we assume that $|X| \le \ell - 1$ and $\ell \ge k + 2$. If $D[X]$ is not complete, then there is an arc $e \in A(D[X]^c) \subseteq A(D^c)$. As D is an arc-maximal (k, ℓ)-digraph, $D + e$ has a subdigraph H with $|V(H)| \ge \ell$ and $\lambda(H) \ge k$. By Lemma 2.3 (i), H must be a subdigraph of $D[X] + e$, and so $\ell - 1 \ge |X| \ge |V(H)| \ge \ell$, a contradiction. Hence $D[X]$ must be a complete digraph.

In the rest of the arguments, we by symmetry assume both $k \ge 3$ and $|\partial^+_D(X)| = (X, D - X)_D = k - 1$ to prove the other conclusions of Lemma 2.3(iii). Let $r_1 = |X|$. Then $D[X] = K^*_{r_1}$. Suppose that $1 < r_1 \le k$.

Claim 1 *There exists an arc $e = (u, u') \in A(D^c)$ such that $u \in X$, $u' \in V(D) - X$ and u is not incident with at least one arc in $\partial^+_D(X)$.*

If there is a vertex x in X incident with all arcs joining X to $V(D) - X$ in $\partial^+_D(X)$, then since $|X| = r_1 > 1$, there must be a vertex $u \in X - \{x\}$, which is not incident with any arc in $\partial^+_D(X)$, and so Claim 1 holds in this case. Thus we assume that no vertex in X is incident with all arcs in $\partial^+_D(X)$.

Let $r_2 = |V(D) - X|$. By contradiction, we assume that every vertex in X is joining to all vertices in $V(D) - X$. Then $r_1 r_2 = |\partial^+_D(X)| = k - 1$. As $2 \le r_1 \le k$ and $r_1 + r_2 \ge \ell \ge k + 2$, we have $r_2 \ge \ell - r_1 \ge 2$. Since $r_1 \ge 2$ and $r_2 \ge 2$, $\frac{r_1 + r_2}{r_1 r_2} = \frac{1}{r_2} + \frac{1}{r_1} \le 1$. Thus, $r_1 r_2 \ge r_1 + r_2$. It follows that $k - 1 = r_1 r_2 \ge r_1 + r_2 = |V(D)| \ge \ell$, a contradiction. This proves the claim.

By Claim 1, there exists an arc $e = (u, u') \in A(D^c)$ satisfying the conclusion of Claim 1. Since $D \in \mathcal{A}(k, \ell)$, $D + e$ has a subdigraph H with $|V(H)| \ge \ell$ and $\lambda(H) \ge k$. Since D is a (k, ℓ)-digraph, we must have $e \in A(H)$. Since $\lambda(H) \ge k$ and $e \in A(H)$, it follows that all the $k - 1$ arcs in $\partial_D(X)$ must be in $A(H)$. Let $H_1 = H \cap D[X]$ and $H_2 = H \cap (D - X)$.

Claim 2 *Each of the following holds.*

(i) $|V(H_1)| \ge 2$ and $H_1 = D[X]$.

(ii) Each vertex in $X - \{u\}$ is incident with exactly one of the $k-1$ arcs in $\partial_D^+(X)$.

(iii) The vertex u is not incident with any arc in $\partial_D^+(X)$.

As $k \geq 3$ and as $e = (u, u') \in A(D^c)$ satisfies the conclusion of Claim 1, there is at least one vertex $v \in V(H_1) - \{u\}$. Hence $|V(H_1)| \geq 2$. By $\delta^+(H) \geq \lambda(H) \geq k$, we have

$$k|V(H_1)| \leq \sum_{v \in V(H_1)_D} d_H^+(v) \leq |A(H_1)| + |\partial_{D+e}^+(V(H_1))| \leq |V(H_1)|(|V(H_1)| - 1) + k,$$

and so $(|V(H_1)| - 1)k \leq |V(H_1)|(|V(H_1)| - 1)$. As $|V(H_1)| \geq 2$, we have $|V(H_1)| = k = r_1 = |X|$, implying Claim 2(i).

Since $|X| \leq k$ and $|\partial_H^+(X)| \geq \lambda(H) \geq k$, it follows by $H_1 = D[X]$ that Claim 2(ii) must hold. As $|\partial_D^+(X)| = k - 1$, Claim 2(iii) follows from Claim 2(ii). This proves Claim 2.

Since $r_2 \geq 2$, there is an arc $e' \in (X, V(D) - X)_{D^c}$ not incident with u. By the assumption of $D \in \mathcal{A}(n, k, \ell)$, $D + e'$ has a subdigraph H' with $|V(H')| \geq \ell$ and $\lambda(H') \geq k$. Since e' is not incident with u and by Claim 2(iii), we conclude that u has outdegree $k - 1$ in $D + e'$ and so $u \notin V(H')$.

Let $H_1' = H' \cap D[X]$, $H_2' = H' \cap (D - X)$. By Claim 2(ii), we have $H_1' = D[X - u]$ and so $|V(H_1')| = k - 1$. It follows that

$$k(k-1) = k|V(H_1')| \leq \sum_{v \in V(H_1')} d_{H'}^+(v) \leq |V(H_1')|(|V(H_1')| - 1) + k = (k-1)(k-2) + k,$$

forcing $k = 2$, contrary to the assumption that $k \geq 3$. This justifies (iii) and completes the proof of the lemma. □

Theorem 2.4 *Let k and ℓ be integers with $\ell \geq k + 2$ and and $k \geq 3$. A strict digraph D with $n = |V(D)| \geq \ell$ is in $\mathcal{A}(k, \ell)$ if and only if $D \in [H_1, H_2]_{k-1}^{\ell}$ for some digraphs H_1 and H_2 satisfying one of the following.*

(i) $H_1 = K_{r_1}^*$, $H_2 = K_{r_2}^*$ with $r_1 + r_2 \geq \ell \geq k + 2$ and for $i \in \{1, 2\}$, either $r_i = 1$ or $k + 1 \leq r_i \leq \ell - 1$, or

(ii) $H_1 \in \mathcal{A}(k, l)$ and $H_2 = K_{r_2}^*$ with $r_2 = 1$ or $k + 1 \leq r_2 \leq \ell - 1$, or

(iii) $H_1, H_2 \in \mathcal{A}(k, \ell)$.

Proof We first assume that $D \in \mathcal{A}(k, \ell)$ to show one of the conclusions must hold. Since $n \geq \ell \geq k + 2$, by (3), there must be a nonempty proper subset $X \subset V(D)$ such that $|\partial_D^+(X)| < k$. Let $Y = V(D) - X$. By Lemma 2.2(iii), all possible arcs from Y to X are in $A(D)$.

For each $W \in \{X, Y\}$, by Lemma 2.3, either $|W| \geq \ell$ and $D[W] \in A(k, \ell)$ or $D[W]$ is complete digraph with $|W| = 1$ or $k + 1 \leq |W| \leq \ell - 1$. If both $D[X]$ and $D[Y]$ are complete digraphs, then Theorem 2.4(i) must hold. If one of $D[X]$ and $D[Y]$ is a complete digraph, and the other is not, then Theorem 2.4(ii) holds. If both

$D[X]$ and $D[Y]$ are not complete digraphs, then Theorem 2.4(iii) follows.

We now assume that $D \in [H_1, H_2]_{k-1}^{\ell}$ and $n = |V(D)|$, such that H_1 and H_2 satisfy one of Theorem 2.4 (i), (ii) and (iii) to show that $D \in \mathcal{A}(k, \ell)$. We argue by contradiction and assume that

$$\text{D is a counterexample with } |\text{V(D)}| \text{ minimized.} \tag{4}$$

Without loss of generality, we assume that D is a (k, ℓ)-joint from H_1 to H_2. Let $W = (V(H_1), V(H_2))_D$. Then by Lemma 2.2, $|W| = k - 1$. Since $n \geq \ell \geq k + 2$, it follows by (3) that $\lambda(D) \leq k - 1$ and D cannot be a complete digraph. Hence $A(D^c) \neq \emptyset$.

Claim 3 *Every subdigraph H of D with $|V(H)| \geq \ell$ must have $\lambda(H) \leq k - 1$.*

If for some $i \in \{1, 2\}$, H is a subdigraph of H_i, then as $|V(H_i)| \geq |V(H)| \geq \ell$, we conclude that $H_i \in \mathcal{A}(k, \ell)$. Thus by (3), $\lambda(H) \leq k - 1$. Therefore, we assume that both $V(H) \cap V(H_1) \neq \emptyset$ and $V(H) \cap V(H_2) \neq \emptyset$. Thus $\lambda(H) \leq |(V(H) \cap V(H_1), V(H) \cap V(H_2))_H| \leq |W| = k - 1$. This proves Claim 3.

By Claim 3, D is a (k, ℓ)-digraph. By (4), $D \notin \mathcal{A}(k, \ell)$, and so there must be an arc $e \in A(D^c)$ such that

$$D + e \text{ is also a } (k, \ell)- \text{ digraph.} \tag{5}$$

If for some $i \in \{1, 2\}$, we have $e \in A(H_i^c)$, then H_i is not a complete digraph and so by Theorem 2.4 (ii) or (iii), $H_i \in \mathcal{A}(k, \ell)$. By definition, $H_i \cup e$ contains a subdigraph H_i' with $|V(H_i')| \geq \ell$ and $\lambda(H_i') \geq k$, contrary to (5). Therefore, we conclude that $e \notin A(H_1^c) \cup A(H_2^c)$. As D is a (k, ℓ)-joint from H_1 to H_2, we have $e = (u_1, u_2)$ for some vertices $u_1 \in V(H_1)$ and $u_2 \in V(H_2)$. If Theorem 2.4(i) holds, then as for each $i \in \{1, 2\}$, $r_i = 1$ or $r_i \geq k + 1$, it follows that $\lambda(D + e) \geq k$, contrary to (5). Hence we assume that Theorem 2.4 (ii) or (iii) must hold. It follows by Theorem 2.4 (ii) or (iii) that

$$\text{both } |V(H_1)| \geq k + 1 \text{ and } |V(H_2)| \geq k + 1. \tag{6}$$

By (5) and by (3), there must be a proper nonempty subset $X \subseteq V(D + e)$ such that $|\partial_{D+e}^{+}(X)| \leq k - 1$. Let $X_1 = X \cap V(H_1)$ and $X_2 = X \cap V(H_2)$.

Claim 4 *Each of the following holds.*

(i) $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$. In particular, $X \neq V(H_1)$ and $X \neq V(H_2)$.
(ii) For some $i \in \{1, 2\}$, we must have $V(H_i) \subset X$ and $V(H_{3-i}) - X \neq \emptyset$.

Let $i \in \{1, 2\}$. If $X \cap V(H_i) = \emptyset$, then $X \subseteq V(H_{3-i})$, and so by Definition 1.3 and by (6), if $i = 1$, then $k - 1 \geq |\partial_{D+e}^{+}(X)| \geq |(X, V(H_1))_D| \geq |V(H_1)| \geq k + 1$; and if $i = 2$, $k - 1 \geq |\partial_{D+e}^{+}(X)| \geq |(V(H_2), X)_D| \geq |V(H_2)| \geq k + 1$. These contradictions justify Claim 4(i).

If $V(H_i) - X_i \neq \emptyset$, for both $i \in \{1,2\}$, then by Theorem 2.4 (ii) or (iii), H_i is either in $\mathcal{A}(k,\ell)$ or is a complete digraph with order at least $k+1$. It follows by Claim 4(i), by Lemma 2.2 and by $k \geq 2$ that

$$k-1 \geq |\partial^+_{D+e}(X)| \geq \sum_{i=1}^{2} |\partial^+_{D+e}(X)| \geq 2(k-1) \geq k.$$

This contradiction justifies Claim 4(ii).

By Claim 4, we may assume that both $X_1 = V(H_1)$ and $X_2 \neq \emptyset$. Then $V(H_2) - X_2 \neq \emptyset$ as X is a proper subset of $V(D)$. It follows that in this case, $k-1 \geq |\partial^+_{D+e}(X)| \geq |\partial_{H_2}(X_2)|$. If H_2 is a complete digraph of order at least $k+1$, then $k-1 \geq |\partial_{H_2}(X_2)| \geq k+1$, a contradiction. Therefore, by Theorem 2.4 (ii) or (iii), we conclude by Lemma 2.2 that

$$H_2 \in \mathcal{A}(k,\ell),\ (V(H_1), V(H_2))_{D+e} \subseteq (V(H_1), X_2)_{D+e}, \text{ and } |\partial_{H_2}(X_2)| = k-1. \tag{7}$$

Let $L = H_2[X_2]$. Since $H_2 \in \mathcal{A}(k,\ell)$, by Lemma 2.3, either $|X_2| \geq \ell$ and $L \in \mathcal{A}(k,\ell)$ or $|X_2| \leq \ell - 1$ and $L = K^*_{r'}$ with $r' = 1$ or $r' \geq k+1$. Let $D' = D - (V(H_2) - X_2)$. Then by (7), we conclude that $D' \in [H_1, L]^{\ell}_{k-1}$ and so by (4), $D' \in \mathcal{A}(k,\ell)$. It follows that $D' + e$ contains a subdigraph H' such that $|V(H')| \geq \ell$ and $\lambda(H') \geq k$. As H' is also a subdigraph of D, this is a contradiction to (5). This completes the proof of Lemma. □

Corollary 2.5 *Let k and ℓ be integers with $\ell \geq k+2$ and and $k \geq 3$. Let D be a strict digraph with $n = |V(D)| \geq \ell$. The following are equivalent.*

(i) $D \in \mathcal{A}(k,\ell)$.

(ii) There exist an integer $r \geq 2$ and an r-tuple $(m_1, m_2, ..., m_r)$ of integers with $m_1 + m_2 + \ldots + m_r = n$ satisfying

$$\forall i \in \{1,2,...,r\},\ k+1 \leq m_i \leq \ell - 1 \text{ or } m_i = 1, \text{ and } \max\{m_i : 1 \leq i \leq r\} \geq k+1, \tag{8}$$

such that $D \in [K^*_{m_1}, \ldots, K^*_{m_r}]^{\ell}_{k-1}$.

Proof Assume that (i) holds. We argue by induction on n to prove (ii). If $n = \ell$, then by Lemmas 2.2 and 2.3, $D \in [K^*_{m_1}, K^*_{m_2}]^{l}_{k-1}$ for some integers m_1 and m_2 satisfying $m_1 + m_2 = \ell$ such that for $i \in \{1,2\}$, either $m_i = 1$ or $k+1 \leq m_i \leq \ell - 1$, and such that $\max\{m_1, m_2\} \geq k+1$.

Therefore, we assume $n \geq \ell + 1$ and Corollary 2.5 holds for smaller values of n. Since $D \in \mathcal{A}(n,k,l)$, one of Theorem 2.4 (i), (ii) and (iii) must hold. As Theorem 2.4(i) implies Corollary 2.5, we may assume that either Theorem 2.4(ii) or Theorem 2.4(iii) holds, and so there exist digraphs H_1 and H_2 such that $H_1 \in \mathcal{A}(k,\ell)$ and either $H_2 \cong K_m$ with $m = 1$ or $k+1 \leq m \leq \ell - 1$, of $H_2 \in \mathcal{A}(k,\ell)$. By induction, for $i \in \{1,2\}$, if $H_i \in \mathcal{A}(k,\ell)$, then H_i is a (k,ℓ)-joint of complete digraphs whose order satisfies (8). It follows that Corollary 2.5(ii) must hold.

Conversely, we assume (ii) and argue by induction on r to prove (i). If $r=2$, then Theorem 2.4 (i) implies that Corollary 2.5(i) must hold. Assume that $r \geq 3$ and that (ii) implies (i) when r takes a smaller value. Let $D \in [K^*_{m_1}, \ldots, K^*_{m_r}]^{\ell}_{k-1}$. By Definition 1.3, we may assume that $D \in [H_1, H_2]^{\ell}_{k-1}$, where for some integer s with $1 \leq s < r$, $H_1 \in [K^*_{m_1}, \ldots, K^*_{m_s}]^{\ell}_{k-1}$ and $H_2 \in [K^*_{m_{s+1}}, \ldots, K^*_{m_r}]^{\ell}_{k-1}$. By induction, H_1 and H_2 satisfy Theorem 2.4 (ii) or (iii). It follows by Theorem 2.4 that $D \in \mathcal{A}(k, \ell)$ and so (i) must hold. □

3 Relationship between Extremal Graphs and Extremal Digraphs

Throughout this section, let k, ℓ, n be integers such that $n \geq \ell \geq k+2$. In this section, we shall show a relationship between edge-maximal (k, ℓ)-graphs and arc-maximal (k, ℓ)-digraphs. As in [3], extremal graphs in $\mathcal{MM}(n, k, \ell)$ are not completely characterized, we in this section will also extend the results in [3] and in [7], respectively, to obtain structural characterizations for the proofs of our main results in the last section.

Recall that $\mathcal{E}(n, k, \ell)$ is the family of all edge-maximal (k, ℓ)-graphs of order n, and the graph families $\mathcal{MM}(n, k, \ell)$ and $\mathcal{SM}(n, k, \ell)$ are defined in (1). We start with a lemma in [7].

Lemma 3.1 (*Lai and Zhang*, *Lemma* 4 *and Theorem* 1 *of* [7]). *Let G be a simple graph on n vertices. Then $G \in \mathcal{E}(k, \ell)$ if and only if one of the following holds.*

(i) For some integers r_1 and r_2 satisfying $r_1 + r_2 \geq \ell$ and for each $i \in \{1, 2\}$, either $r_i = 1$ or $k+1 \leq r_i \leq \ell - 1$, $G \in [K_{r_1}, K_{r_2}]^{\ell}_{k-1}$.

(ii) For some $H \in \mathcal{E}(k, \ell)$ and an integer r satisfying either $r = 1$ or $k+1 \leq r \leq \ell - 1$, $G \in [H, K_r]^{\ell}_{k-1}$.

(iii) For some $H_1, H_2 \in \mathcal{E}(k, \ell)$, $G \in [H_1, H_2]^{\ell}_{k-1}$.

Corollary 3.2 below follows Lemma 3.1 with an inductive argument.

Corollary 3.2 *A graph G is in $\mathcal{E}(n, k, \ell)$ if and only if there exist integers $r > 0$ and m_1, m_2, …, m_r satisfying*

$$m_1 + m_2 + \ldots + m_r = n, \ \max\{m_1, \ldots, m_r\} \geq k+1, \quad \text{and} \tag{9}$$

such that $G \in [K_{m_1}, K_{m_2}, \ldots, K_{m_r}]^{\ell}_{k-1}$.

Following [2], the **underlying graph** $UG(D)$ of a digraph D is the graph formed by erasing all the orientations from the arcs in D. Let $D \in \mathcal{A}(n, k, \ell)$. By Corollary 2.5, $D \in [K^*_{m_1}, \ldots, K^*_{m_r}]^{\ell}_{k-1}$ for some integers r, and $m_1, m_2, \ldots, m_r$ satisfying $m_1 + m_2 + \ldots + m_r = n$ and (8). Using the terminology in Definition 2.1, let $B(D)$ denote the set of all the backward arcs arising from the construction of D from the complete digraphs $K^*_{m_1}, \ldots, K^*_{m_r}$. For these values of $m_1, \ldots, m_r$, we define the **corresponding**

graph of D, denoted by $\zeta(D)$, to be the underlying graph of the digraph $D - B(D)$. By Definition 1.3, $\zeta(D) \in [K_{m_1}, \ldots, K_{m_r}]^{\ell}_{k-1}$. By Theorem 1 of [7], $\zeta(D) \in \mathcal{E}(n,k,\ell)$. Conversely, for each graph $G \in [K_{m_1}, \ldots, K_{m_r}]^{\ell}_{k-1}$, there exists a digraph $D \in [K^*_{m_1}, \ldots, K^*_{m_r}]^{\ell}_{k-1}$ such that $G = \zeta(D)$. This digraph D is called an **associated digraph** of G. For each $G \in [K_{m_1}, \ldots, K_{m_r}]^{\ell}_{k-1}$, let $\zeta^{-1}(G)$ denote the collection of all associated digraph of G. The following result is the relationship between edge-maximal (k,ℓ)-graphs and arc-maximal (k,ℓ)-digraphs.

Lemma 3.3 *Let $D \in \mathcal{A}(n,k,\ell)$ and $G = \zeta(D)$ be the corresponding graph of D. Then $|A(D)| = |E(G)| + \frac{n(n-1)}{2}$.*

Proof Let $D \in \mathcal{A}(n,k,l)$. By Lemma 2.5, there exist integers $r > 0$ and $m_1, m_2, \ldots, m_r$ satisfying $m_1 + m_2 + \ldots + m_r = n$ and (8) such that $D \in [K^*_{m_1}, \ldots, K^*_{m_r}]^{\ell}_{k-1}$. We color all forward arcs in red and all backward arcs in blue. By Definition 2.1, $A(D)$ is partitioned into the forward arcs, backward arcs, and arcs in each of the complete digraphs $K^*_{m_i}$, $1 \le i \le r$. For each i with $1 \le i \le r$, and for any pair of distinct vertices $u, v \in V(K^*_{m_i})$, randomly color one arc in blue and the other in red. Let D_1 and D_2 be the subdigraphs of D induced by the red arcs, and by the blue arcs, respectively. By definition, G is isomorphic to $UG(D_1)$, and $UG(D_2) \cong K_n$. It follows that $|A(D)| = |A(D_1)| + |A(D_2)| = |E(G)| + |E(K_n)| = |E(G)| + \frac{1}{2}n(n-1)$. □

To apply Lemma 3.3 in the determination of the extremal sizes of the arc-maximal (k,ℓ)-digraphs, and the characterization of the extremal arc-maximal (k,ℓ)-digraphs, we need to extend the results in [3, 7] and to further the investigation of the structural properties of the extremal (k,ℓ)-graphs in $\mathcal{MM}(n,k,\ell)$ and in $\mathcal{SM}(n,k,\ell)$.

3.1 The Edge-Maximal (k,ℓ)-Graphs with Maximum Sizes

The main purpose of this subsection is to determine the structures of graphs in $\mathcal{MM}(n,k,\ell)$.

Lemma 3.4 *Let $G \in \mathcal{MM}(n,k,\ell)$, let $X \subset E(G)$ be an edge-cut with $|X| = k-1$ and H_i be the component of $G - X$, $1 \le i \le 2$. Let $n_i = |V(H_i)| \ge \ell$. Then $H_i \in \mathcal{MM}(n_i,k,\ell)$.*

Proof By contradiction, assume by symmetry that $H_1 \notin \mathcal{MM}(n_1,k,\ell)$. By Lemma 3.1, $H_1 \in \mathcal{E}(n_1,k,\ell)$. Hence there is some $H' \in \mathcal{MM}(n_1,k,\ell)$ with $|E(H')| > |E(H_1)|$. Choose some $G' \in [H', H_2]^{\ell}_{k-1}$. By Lemma 3.1, $G' \in \mathcal{E}(n,k,\ell)$ with
$|E(G')| = |E(H')| + |E(H_2)| + (k-1) > |E(H_1)| + |E(H_2)| + (k-1) = |E(G)|$, contrary to the fact that $G \in \mathcal{MM}(n,k,\ell)$. This proves the lemma. □

Lemma 3.5 *Let $G \in \mathcal{E}(n,k,\ell)$, and r, s be nonnegative integers such that $n = s(\ell-1) + r$ with $0 \le r \le \ell - 2$. Each of the following holds.*

(i) Suppose that $2(k-1)<\ell-1$ and $r<2(k-1)$. Then $G\in\mathcal{MM}(n,k,\ell)$ if and only if $G\in[K_{m_1},K_{m_2},\ldots,K_{m_{r+s}}]^{\ell}_{k-1}$ such that $|\{i: m_i=\ell-1\}|=s$ and $|\{j: m_j=1\}|=r$.

(ii) Suppose that $2(k-1)<\ell-1$ and $r>2(k-1)$. Then $G\in\mathcal{MM}(n,k,\ell)$ if and only if $G\in[K_{m_1},K_{m_2},\ldots,K_{m_{s+1}}]^{\ell}_{k-1}$ such that $|\{i: m_i=\ell-1\}|=s$ and $|\{j: m_j=r\}|=1$.

(iii) Suppose that $2(k-1)<\ell-1$ and $r=2(k-1)$. Then $G\in\mathcal{MM}(n,k,\ell)$ if and only if either $G\in[K_{m_1},K_{m_2},\ldots,K_{m_{s+1}}]^{\ell}_{k-1}$ such that $|\{i: m_i=\ell-1\}|=s$ and $|\{j: m_j=r\}|=1$ or $G\in[K_{m_1},K_{m_2},...,K_{m_{r+s}}]^{\ell}_{k-1}$ such that $|\{i: m_i=\ell-1\}|=s$ and $|\{j: m_j=1\}|=r$.

(iv) Suppose that $2(k-1)>\ell-1$. Then $G\in\mathcal{MM}(n,k,\ell)$ if and only if $G\in[K_{m_1},K_{m_2},\ldots, K_{m_{n-\ell+2}}]^{\ell}_{k-1}$ such that $|\{i: m_i=\ell-1\}|=1$ and $|\{j: m_j=1\}|=n-\ell+1$.

(v) Suppose that $2(k-1)=\ell-1$. Then $G\in\mathcal{MM}(n,k,\ell)$ if and only if $G\in[K_{m_1},K_{m_2},\ldots, K_{m_{n-t\ell+2t}}]^{\ell}_{k-1}$ such that for some $t\in\{1,2,\ldots,s-1\}$, $|\{i: m_i=\ell-1\}|=t$ and $|\{j: m_j=1\}|=n-t\ell+t$.

Proof By Corollary 3.2 and Theorem 1.2, it suffices to prove the only if part in each of the conclusions of this lemma.

Let $G\in\mathcal{MM}(n,k,\ell)$. If $n=\ell$, by Lemma 3.1, $G\in[K_{\ell-1},K_1]^{\ell}_{k-1}$, and so the lemma holds. Assume that $n>\ell$ and the lemma holds for smaller values of n. By Lemma 3.1, $G\in[H_1,H_2]^{\ell}_{k-1}$ where for each $i\in\{1,2\}$, H_i is a complete graph or H_i is an edge-maximal (k,ℓ)-graph. By symmetry, set $n_1=|V(H_1)|$ and $n_2=|V(H_2)|$ with $n_1\le n_2$. For each $i\in\{1,2\}$, let $n_i=s_i(\ell-1)+r_i$ where s_i, r_i are nonnegative integers and $0\le r_i\le\ell-2$.

Case 1 $\ell-1>2(k-1)>r_1+r_2$.

We shall show that in this case, Lemma 3.5(i) holds. Then $s=s_1+s_2$ and $r=r_1+r_2$. We justify the lemma with discussions for different values of n_1. If $n_1=1$, then $s_1=0$, $r_1=1$, $r_2\le 2(k-1)-1<\ell-1$ and $n_2\ge\ell$. By Lemma 3.1, we conclude that $H_1=K_1$ and $H_2\in\mathcal{MM}(n_2,k,\ell)$. By induction, $H_2\in[K_{m_1},K_{m_2},...,K_{m_{r_2+s_2}}]^{\ell}_{k-1}$ such that $|\{i: m_i=\ell-1\}|=s_2$ and $|\{j: m_j=1\}|=r_2$. It follows that Lemma 3.5(i) must hold.

Assume that $n_1=\ell-1$. Then $s_1=1$, $r_1=0$, $r_2=r<2(k-1)$ and $n_2\ge\ell-1$. By Lemma 3.1, $H_1=K_{\ell-1}$ and either $H_2=K_{\ell-1}$ or $H_2\in\mathcal{MM}(n_2,k,\ell)$. In the case when $H_2\in\mathcal{MM}(n_2,k,\ell)$, by induction, $H_2\in[K_{m_1},K_{m_2},...,K_{m_{r_2+s_2}}]^{\ell}_{k-1}$ such that $|\{i: m_i=\ell-1\}|=s_2$ and $|\{j: m_j=1\}|=r_2$. Hence Lemma 3.5(i) holds again in either case of H_2.

Now assume that $n_1\ge\ell$. Then $n_2\ge n_1\ge\ell$. By Lemma 3.1, $H_1\in\mathcal{MM}(n_1,k,\ell)$ and $H_2\in\mathcal{MM}(n_2,k,\ell)$. By induction, for each $t\in\{1,2\}$, $H_t\in[K_{m_1},K_{m_2},...,K_{m_{r_i+s_i}}]^{\ell}_{k-1}$ such that $|\{i: m_i=\ell-1\}|=s_i$ and $|\{j: m_j=1\}|=r_i$. If follows that Lemma 3.5(i) holds in this case also.

Finally we assume that $2 \leq n_1 \leq \ell - 2$. If $n_2 \leq \ell - 1$, then by Lemma 3.1, we conclude that both $H_1 = K_{n_1}$ and $H_2 = K_{n_2}$. Direct computation yields that

$$\begin{aligned}\Phi(n,k,\ell) = |E(G)| = |E(H_1)| + |E(H_2)| + k - 1 = \frac{n_1(n_1-1)}{2} \\ + \frac{n_2(n_2-1)}{2} + k - 1 < \Phi(n,k,\ell),\end{aligned} \tag{10}$$

a contradiction. Hence we must have $n_2 \geq \ell$. By Lemma 3.1, $H_1 = K_{n_1}$ and $H_2 \in \mathcal{MM}(n_2,k,\ell)$. Once again, direct computation shows that the contradiction of (10) is obtained. This completes the proof for Case 3.1.

Case 2 $\ell - 1 > r_1 + r_2 > 2(k-1)$.

We shall show that in this case, Lemma 3.5(ii) holds. Thus $s = s_1 + s_2$ and $r = r_1 + r_2$. Assume that $r_1 r_2 \neq 0$. By Lemmas 3.1 and 3.4, for each $i \in \{1,2\}$, either $H_i = K_{r_i}$ and $|E(H_i)| = \frac{r_i(r_i-1)}{2}$, or $H_i \in \mathcal{MM}(n_1,k,\ell)$ and $|E(H_i)| = \Phi(n_i,k,\ell)$. Direct computation yields that $\Phi(n,k,\ell) = |E(G)| = |E(H_1)| + |E(H_2)| + k - 1 < \Phi(n,k,\ell)$, a contradiction. Therefore, we must have $r_1 r_2 = 0$, and so we may assume that $r_1 = 0$ and $r_2 > 2(k-1)$. By Lemmas 3.1 and 3.4, for each $i \in \{1,2\}$, either $H_i = K_{\ell-1}$ or $H_i \in \mathcal{MM}(n_i,k,\ell)$. By induction, if $H_1 \in \mathcal{MM}(n_1,k,\ell)$, then $H_1 \in [K_{m_1}, K_{m_2}, \ldots, K_{m_{r_1+s_1}}]_{k-1}^{\ell}$ such that $|\{i : m_i = \ell - 1\}| = s_1$ and $|\{j : m_j = 1\}| = r_1$, and H_2 satisfies the conclusions of Lemma 3.5(ii). This implies that Lemma 3.5(ii) must hold, and completes the proof for Case 2.

Case 3 $r_1 + r_2 = 2(k-1) < \ell - 1$.

We shall show that in this case, Lemma 3.5(iii) holds. Then $r = r_1 + r_2 = 2(k-1)$ and $s_1 + s_2 = s$. Assume first that $r_1 r_2 \neq 0$. Then both $r_1 < 2(k-1)$ and $r_2 < 2(k-1)$. By induction, for each $i \in \{1,2\}$, H_i satisfies the conclusion of Lemma 3.5(i). It follows that G also satisfies the conclusion of Lemma 3.5(iii). Hence we may assume that $r_1 = 0$ and $r_2 = 2(k-1)$. By Lemmas 3.1 and 3.4, we conclude that either $H_1 = K_{\ell-1}$ or $H_1 \in \mathcal{MM}(n_1,k,\ell)$, and $H_2 \in \mathcal{MM}(n_2,k,\ell)$. By induction, H_1 satisfies the conclusions of Lemma 3.5(i) and H_2 satisfies the conclusions of Lemma 3.5(iii). It follows that G satisfies the conclusions of Lemma 3.5(iii).

Case 4 $2(k-1) < \ell - 1 \leq r_1 + r_2$.

In this case, $s > s_1 + s_2$. By Lemmas 3.1 and 3.4, for each $i \in \{1,2\}$, either $H_i = K_{r_i}$ or $H_i \in \mathcal{MM}(n_i,k,\ell)$ and so $|E(H_i)| = \Phi(n_i,k,\ell)$. By induction and direct computation, we conclude that $\Phi(n,k,\ell) = |E(G)| = |E(H_1)| + |E(H_2)| + k - 1 < \Phi(n,k,\ell)$, a contradiction. Hence this case cannot occur.

Case 5 $2(k-1) > \ell - 1$.

We shall show that in this case, Lemma 3.5(iv) holds. Assume first that $n_1 \geq \ell$. Then $n_2 \geq n_1 \geq \ell$. By Lemma 3.4, for each $i \in \{1,2\}$, $H_i \in \mathcal{MM}(n_i,k,\ell)$ and so $|E(H_i)| = \Phi(n_i,k,\ell)$. Direct computations shows that $\Phi(n,k,\ell) = |E(G)| = |\sum_{i=1}^{2} E(H_i)| + (k-1) = \sum_{i=1}^{2} \Phi(n_i,k,\ell) + k - 1 < \Phi(n,k,\ell)$, a contradiction. Hence $n_1 < \ell$.

If $n_1 = 1$, then $n_2 \geq \ell$. By Lemma 3.4, $H_2 \in \mathcal{MM}(n_2,k,\ell)$, and so by induction,

H_2 satisfies the conclusion of Lemma 3.5(iv). As $n = n_2 + 1$, this implies that G satisfies the conclusion of Lemma 3.5(iv). Next we assume that $2 \le n_1 \le \ell - 1$ and $n_2 \ge \ell$. By Lemma 3.1 and 3.4, $H_1 = K_{n_1}$ and $H_2 \in \mathcal{MM}(n_2, k, \ell)$. Thus by direct computation, $\Phi(n,k,\ell) = |E(G)| = \frac{n_1(n_1-1)}{2} + \Phi(n_2,k,\ell) + k - 1 < \Phi(n,k,\ell)$, a contradiction. Therefore we may assume that $2 \le n_1 \le \ell - 1$ and $n_2 \le \ell - 1$. by Lemma 3.1, we have $H_1 = K_{n_1}$ and $H_2 = K_{n_2}$. Direct computation shows the contradiction of (10) is obtained. This completes the proof for this case.

Case 6 $2(k-1) = \ell - 1$.

We shall show that in this case, Lemma 3.5(v) holds. Suppose first that $n_1 \ge \ell - 1$. Then $n_2 \ge n_1 \ge \ell - 1$, and $s \ge s_1 + s_2$. By Lemma 3.1 and 3.4, for each $i \in \{1,2\}$, $H_i \in \mathcal{MM}(n_i,k,\ell)$ and so $|E(H_i)| = \Phi(n_i,k,\ell)$. By induction, there exist integers t_1 and t_2 with $t_i \in \{1,2,...,s_i - 1\}$, such that H_i satisfies the conclusion of Lemma 3.5(v). Let $t = t_1 + t_2$. Then $t \le (s_1 - 1) + (s_2 - 1) \le s - 1$. This implies that G also satisfies the conclusion of Lemma 3.5(v). Hence we assume that $n_1 < \ell - 1$.

If $n_1 = 1$, then $H_1 = K_1$ and $n_2 \ge \ell$. By Lemma 3.1 and 3.4, $H_2 \in \mathcal{MM}(n_2,k,\ell)$. By induction, H_2 satisfies the conclusion of Lemma 3.5(v). As $H_1 = K_1$, G also satisfies the conclusion of Lemma 3.5(v). Finally we assume that $2 \le n_1 \le \ell - 2$, and so $H_1 = K_{n_1}$. If $n_2 \le \ell - 1$, then by Lemma 3.1, $H_2 = K_{n_2}$, and so the contradiction of (10) is obtained. Hence we must have $n_2 \ge \ell$. By induction, and by Lemma 3.1 and 3.4, $H_2 \in \mathcal{MM}(n_2,k,\ell)$, whence direct computation yields $\Phi(n,k,\ell) = |E(G)| = \frac{n_1(n_1-1)}{2} + \Phi(n_2,k,\ell) + k - 1 < \Phi(n,k,\ell)$, a contradiction. This completes the proof of the lemma. □

3.2 The Edge-Maximal (k,ℓ)-Graphs with Minimum Sizes

In this subsection, we will determine the structures of graphs in $\mathcal{SM}(n,k,\ell)$.

Lemma 3.6 *Let $G \in \mathcal{E}(n,k,\ell)$ and $s, r \ge 0$ be integers satisfying $n = s(\ell-1) + r$ with $0 \le r \le \ell - 2$. Let $q_1 = \lfloor \frac{n}{k+1} \rfloor$ and $q_2 = \lfloor \frac{n-\ell}{k+1} \rfloor$. Each of the following holds.*

(i) Suppose that $\ell \le n < 2k+2$. Then $G \in \mathcal{SM}(n,k,\ell)$ if and only if $G \in [K_{m_1}, K_{m_2}, \ldots, K_{m_{n-\ell+2}}]^{\ell}_{k-1}$ such that $|\{i : m_i = \ell - 1\}| = 1$ and $|\{j : m_j = 1\}| = n - \ell + 1$.

(ii) Suppose that $\ell \le 2k+2 \le n$. Then $G \in \mathcal{SM}(n,k,\ell)$ if and only if $G \in [K_{m_1}, K_{m_2}, \ldots, K_{m_{n-q_1 k}}]^{\ell}_{k-1}$ such that $|\{i : m_i = k+1\}| = q_1$ and $|\{j : m_j = 1\}| = n - q_1(k+1)$.

(iii) Suppose that for some integer $t > 0$, $n \ge \ell = 2t \ge 2k+3$. Then $G \in \mathcal{SM}(n,k,\ell)$ if and only if $G \in [K_{m_1}, K_{m_2}, \ldots, K_{m_{n-\ell-q_2 k+2}}]^{\ell}_{k-1}$ such that $|\{i : m_i = t\}| = 2$, $|\{i' : m_{i'} = k+1\}| = q_2$ and $|\{j : m_j = 1\}| = n - \ell - q_2(k+1)$.

(iv) Suppose that for some integer $t > 0$, $n \ge \ell = 2t+1 \ge 2k+3$. Then $G \in \mathcal{SM}(n,k,\ell)$ if and only if $G \in [K_{m_1}, K_{m_2}, \ldots, K_{m_{n-\ell-q_2 k+2}}]^{\ell}_{k-1}$ such that

$|\{i : m_i = t\}| = |\{i' : m_{i'} = t+1\}| = 1, \quad |\{i'' : m_{i''} = k+1\}| = q_2$ and $|\{j : m_j = 1\}| = n - \ell - q_2(k+1)$.

Proof By Corollary 3.2 and Theorem 1.4, it suffices to prove the only if part in each of the conclusions of this lemma. In the different relationships among n, ℓ and k below, we always assume that $G \in \mathcal{SM}(n,k,\ell)$. We shall argue by induction on n to prove the lemma.

(i) Suppose that $\ell \le n < 2k+2$. If $n = \ell$, then by Lemma 3.1, $G \in [K_{l-1}, K_1]^{\ell}_{k-1}$ and so Lemma 3.6(i) holds. Assume that $n > \ell$. By Theorem 1.4(i) that G has a vertex v of degree $k-1$ such that $G - v \in \mathcal{SM}(n-1,k,\ell)$. By induction, $G - v \in [K_{m_1}, K_{m_2}, \ldots, K_{m_{(n-1)-\ell+2}}]^{\ell}_{k-1}$ such that $|\{i : m_i = \ell - 1\}| = 1$ and $|\{j : m_j = 1\}| = (n-1) - \ell + 1$. This implies that G satisfies the conclusion of Lemma 3.6(i).

(ii) Suppose that $\ell \le 2k+2 \le n$. If $n = 2k+2$, then by Theorem 1.4(ii), $G \in [K_{k+1}, K_{k+1}]^{\ell}_{k-1}$ and so Lemma 3.6(ii) holds. Assume that $n > 2k+2$. Then by Theorem 1.4(ii), $G \in [H_1, H_2]^{\ell}_{k-1}$ for some graphs H_1, H_2 such that, for each $i \in \{1,2\}$ either $H_i \in \{K_1, K_{k+1}\} \cup \mathcal{SM}(n,k,\ell)$ and $|V(H_i)| \in \{1, k+1\}$ or $|V(H_i)| \ge 2k+2$ and $\lfloor \frac{|V(H_1)|}{k+1} \rfloor + \lfloor \frac{|V(H_2)|}{k+1} \rfloor = \lfloor \frac{n}{k+1} \rfloor = q_1$. Let $|V(H_1)| = n_1$ and $|V(H_2)| = n_2$. If $n_i \in \{1, k+1\}$, then $H_i = K_{n_i}$. If $n_i > k+1$, then by induction with $q_{1,i} = \lfloor \dfrac{n_i}{k+1} \rfloor (k+1)$, $H_i \in [K_{m_1}, K_{m_2}, \ldots, K_{m_{n_i - q_{1,i}k}}]^{\ell}_{k-1}$ such that $|\{i : m_i = k+1\}| = q_{1,i}$ and $|\{j : m_j = 1\}| = n_i - q_{1,i}(k+1)$. This, together with $q_{1,1} + q_{1,2} = q_1$, implies that $G \in [K_{m_1}, K_{m_2}, \ldots, K_{m_{n-q_1 k}}]^{\ell}_{k-1}$ such that $|\{i : m_i = k+1\}| = q_1$ and $|\{j : m_j = 1\}| = n - q_1(k+1)$, and so Lemma 3.6(ii) follows.

(iii) Suppose that for some integer $t > 0$, $n \ge \ell = 2t \ge 2k+3$. If $n = \ell$, then by Theorem 1.4(iii), $G \in [K_t, K_t]^{\ell}_{k-1}$, and so Lemma 3.6(iii) holds. Assume that $n > \ell$. Then by Theorem 1.4(iii), $G \in [H_1, H_2]^{\ell}_{k-1}$ for some graphs H_1, H_2 such that $H_1 \in \{K_1, K_{k+1}\}$, $H_2 \in \mathcal{SM}(|V(H_2)|, k, \ell)$ with $|V(H_2)| \ge 2t$, and $\lfloor \frac{|V(H_1)|}{k+1} \rfloor + \lfloor \frac{|V(H_2)| - 2t}{k+1} \rfloor = \lfloor \frac{n-2t}{k+1} \rfloor = q_2$. Define

$$|V(H_2)| = n_2, \text{ and } q_{2,2} = \lfloor \frac{n_2 - 2t}{k+1} \rfloor (k+1). \tag{11}$$

By induction, $H_2 \in [K_{m_1}, K_{m_2}, \ldots, K_{m_{n_2 - \ell - q_{2,2}k + 2}}]^{\ell}_{k-1}$ such that $|\{i : m_i = t\}| = 2$, $|\{i' : m_{i'} = k+1\}| = q_{2,2}$ and $|\{j : m_j = 1\}| = n_2 - \ell - q_{2,2}(k+1)$. This, together with $\lfloor \frac{|V(H_1)|}{k+1} \rfloor + q_{2,2} = q_2$, implies that Lemma 3.6(iii) must hold.

(iv) Suppose that for some integer $t > 0$, $n \ge \ell = 2t+1 \ge 2k+3$. If $n = \ell$, then by Theorem 1.4(iii), $G \in [K_t, K_{t+1}]^{\ell}_{k-1}$, and so Lemma 3.6(iv) holds. Assume that $n > \ell$. Then by Theorem 1.4(iv), $G \in [H_1, H_2]^{\ell}_{k-1}$ for some graphs H_1 and H_2 such that $H_1 \in \{K_1, K_{k+1}\}$, $H_2 \in \mathcal{SM}(|V(H_2)|, k, \ell)$ with

$|V(H_2)| \geq 2t+1$, and $\lfloor \frac{|V(H_1)|}{k+1} \rfloor + \lfloor \frac{|V(H_2)|-2t-1}{k+1} \rfloor = \lfloor \frac{n-2t-1}{k+1} \rfloor = q_2$. Define n_2 and $q_{2,2}$ as in (11). By induction, $H_2 \in [K_{m_1}, K_{m_2}, \ldots, K_{m_{n_2-\ell-q_{2,2}k+2}}]^{\ell}_{k-1}$ such that $|\{i : m_i = t\}| = |\{i' : m_{i'} = k+1\}| = 1$, $|\{i'' : m_{i''} = k+1\}| = q_{2,2}$ and $|\{j : m_j = 1\}| = n_2 - \ell - q_{2,2}(k+1)$. This, together with $\lfloor \frac{|V(H_1)|}{k+1} \rfloor + q_{2,2} = q_2$, implies that Lemma 3.6(iv) must hold. This completes the proof of the lemma. □

4 The Main Results

We now present and prove the main results in this section. Recall that the digraph families $\mathcal{MA}(n,k,\ell)$ and $\mathcal{SA}(n,k,\ell)$, and the functions $F(n,k,\ell)$ and $f(n,k,\ell)$ are defined in (2).

Theorem 4.1 *Let n,k,ℓ be integers with $n \geq \ell \geq k+2 \geq 5$, and $s,r \geq 0$ be integers satisfying $n = s(\ell-1)+r$ with $0 \leq r \leq \ell-2$. Then*

$$F(n,k,\ell) = \begin{cases} \frac{s(\ell-1)(\ell-2)}{2} + (s-1+r)(k-1) + \frac{n(n-1)}{2} & \text{if } r < 2(k-1) < \ell-1, \\ \frac{s(\ell-1)(\ell-2)}{2} + s(k-1) + \frac{r(r-1)}{2} + \frac{n(n-1)}{2} & \text{if } 2(k-1) < \ell-1 \text{ and } r > 2(k-1), \\ \frac{s(\ell-1)(\ell-2)}{2} + s(k-1) + \frac{r(r-1)}{2} + \frac{n(n-1)}{2} & \text{if } r = 2(k-1) < \ell-1, \\ \frac{s(\ell-1)(\ell-2)}{2} + (n-\ell+1)(k-1) + \frac{n(n-1)}{2} & \text{if } 2(k-1) > \ell-1, \\ \frac{s(\ell-1)(\ell-2)}{2} + (n-\ell+1)(k-1) + \frac{n(n-1)}{2} & \text{if } 2(k-1) = \ell-1. \end{cases}$$

Furthermore, a digraph D is in $\mathcal{MA}(n,k,\ell)$ if and only if one of the following holds.

(i) $r < 2(k-1) < \ell-1$, and $D \in [K^*_{m_1}, K^*_{m_2}, \ldots, K^*_{m_{n-\ell+2}}]^{\ell}_{k-1}$ such that $|\{i : m_i = \ell-1\}| = 1$ and $|\{j : m_j = 1\}| = n-\ell+1$.

(ii) $2(k-1) < \ell-1$ and $r > 2(k-1)$, and $D \in [K^*_{m_1}, K^*_{m_2}, \ldots, K^*_{m_{s+1}}]^{\ell}_{k-1}$ such that $|\{i : m_i = \ell-1\}| = s$ and $|\{j : m_j = r\}| = 1$.

(iii) $r = 2(k-1) < \ell-1$, and either $D \in [K^*_{m_1}, K^*_{m_2}, \ldots, K^*_{m_{s+1}}]^{\ell}_{k-1}$ such that $|\{i : m_i = \ell-1\}| = s$ and $|\{j : m_j = r\}| = 1$; or $D \in [K^*_{m_1}, K^*_{m_2}, \ldots, K^*_{m_{s+r}}]^{\ell}_{k-1}$ such that $|\{i : m_i = \ell-1\}| = s$ and $|\{j : m_j = 1\}| = r$.

(iv) $2(k-1) > \ell-1$, and $D \in [K^*_{m_1}, K^*_{m_2}, \ldots, K^*_{m_{n-\ell+2}}]^{\ell}_{k-1}$ such that $|\{i : m_i = \ell-1\}| = 1$ and $|\{j : m_j = 1\}| = n-\ell+1$.

(v) $2(k-1) = \ell-1$, and for some integer t with $1 \leq t \leq s$, $D \in [K^*_{m_1}, K^*_{m_2}, \ldots, K^*_{m_{n-t\ell+2t}}]^{\ell}_{k-1}$ such that for some integer $t \in \{1,2,\ldots,s-1\}$, $|\{i : m_i = \ell-1\}| = t$ and $|\{j : m_j = 1\}| = n - t\ell + t$.

Theorem 4.2 *Let n, k, ℓ be integers with $n \geq \ell \geq k+2 \geq 5$, and $s, r \geq 0$ be integers satisfying $n = s(\ell-1)+r$ with $0 \leq r \leq \ell-2$. Define $q_1 = \lfloor \frac{n}{k+1} \rfloor$ and $q_2 = \lfloor \frac{n-\ell}{k+1} \rfloor$. Then*

$$f(n,k,\ell) = \begin{cases} \frac{(\ell-1)(\ell-2)}{2} + (n-\ell+1)(k-1) + \frac{n(n-1)}{2} & \text{if } \ell \leq n < 2k+2, \\ (n-1)(k-1) - \lfloor \frac{n}{k+1} \rfloor \frac{k^2-3k}{2} + \frac{n(n-1)}{2} & \text{if } \ell \leq 2k+2 \leq n, \\ (n-2t+1)(k-1) + t(t-1) - \lfloor \frac{n-2t}{k+1} \rfloor \frac{k^2-3k}{2} + \frac{n(n-1)}{2} & \text{if } n \geq \ell = 2t \geq 2k+3, \\ (n-2t)(k-1) + t^2 - \lfloor \frac{n-2t-1}{k+1} \rfloor \frac{k^2-3k}{2} + \frac{n(n-1)}{2} & \text{if } n \geq \ell = 2t+1 \geq 2k+3. \end{cases}$$

Furthermore, a digraph D is in $\mathcal{SA}(n,k,\ell)$ if and only if one of the following holds.

(i) $\ell \leq n < 2k+2$, and $D \in [K^*_{m_1}, K^*_{m_2}, \ldots, K^*_{m_{n-\ell+2}}]^{\ell}_{k-1}$ such that $|\{i : m_i = \ell-1\}| = 1$ and $|\{j : m_j = 1\}| = n-\ell+1$.

(ii) $\ell \leq 2k+2 \leq n$, and $D \in [K^*_{m_1}, K^*_{m_2}, \ldots, K^*_{m_{n-q_1 k}}]^{\ell}_{k-1}$ such that $|\{i : m_i = k+1\}| = q_1$ and $|\{j : m_j = 1\}| = n - q_1(k+1)$.

(iii) $n \geq l = 2t \geq 2k+3$, and $D \in [K^*_{m_1}, K^*_{m_2}, \ldots, K^*_{m_{n-\ell-q_2 k+2}}]^{\ell}_{k-1}$ such that $|\{i : m_i = t\}| = 2$, $|\{i' : m_{i'} = k+1\}| = q_2$ and $|\{j : m_j = 1\}| = n-\ell-q_2(k+1)$.

(iv) $n \geq l = 2t+1 \geq 2k+3$, and $D \in [K^*_{m_1}, K^*_{m_2}, \ldots, K^*_{m_{n-\ell-q_2 k+2}}]^{\ell}_{k-1}$ such that $|\{i : m_i = t\}| = |\{i' : m_{i'} = t+1\}| = 1$, $|\{i'' : m_{i''} = k+1\}| = q_2$ and $|\{j : m_j = 1\}| = n-\ell-q_2(k+1)$.

Proof of Theorems 4.1 and 4.2 Let $D \in \mathcal{A}(n,k,\ell)$. By Corollary 2.5, there exist integers $r > 0$ and $m_1, m_2, \ldots, m_r$ satisfying (8) such that $D \in [K^*_{m_1}, \ldots, K^*_{m_r}]^{\ell}_{k-1}$. Let $G = \zeta(D)$ be the corresponding graph of D. By Lemma 3.3, $G \in \mathcal{E}(n,k,l)$.

Suppose that $D \in \mathcal{MA}(n,k,\ell)$. We claim that $G \in \mathcal{MM}(n,k,\ell)$. If not, then there must be a graph $G' \in \mathcal{MM}(n,k,\ell)$ with $|E(G')| > |E(G)|$. Let $D' \in \zeta^{-1}(G')$ be an associate digraph of G'. Then by Lemma 3.3, $|A(D)| < |A(D')|$, contrary to the assumption that $D \in \mathcal{MA}(n,k,\ell)$. With a similar argument, we conclude that

$$D \in \mathcal{MA}(n,k,\ell) \text{ if and only if} \tag{12}$$

$$D \in \mathcal{SA}(n,k,\ell) \text{ if and only if} \tag{13}$$

Thus Theorem 4.1 follows from (12), Theorem 1.2 and Lemma 3.5; and Theorem 4.2 follows from (13), Theorem 1.4 and Lemma 3.6. □

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