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Note

On the extended Clark-Wormold Hamiltonian-like index problem

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A B S T R A C T

For a hamiltonian property $\mathcal{P}$, Clark and Wormold introduced the problem of investigating the value $\mathcal{P}(a,b) = \max\{\min\{n : L^n(G)$ has property $\mathcal{P}\}$: $\kappa'(G) \geq a$ and $\delta(G) \geq b\}$, and proposed a few problems to determine $\mathcal{P}(a,b)$ with $b \geq a \geq 4$ when $\mathcal{P}$ is being hamiltonian, edge-hamiltonian and hamiltonian-connected. Zhan in 1986 proved that the line graph of a 4-edge-connected graph is Hamilton-connected, which implies a solution to the unsettled cases of above-mentioned problem. We consider an extended version of the problem. Let $ess'(G)$ denote the essential edge-connectivity of a graph G, and define $\mathcal{P}'(a,b) = \max\{\min\{n : L^n(G)$ has property $\mathcal{P}\}$: $ess'(G) \geq a$ and $\delta(G) \geq b\}$. We investigate the values of $\mathcal{P}'(a,b)$ when $\mathcal{P}$ is one of these hamiltonian properties. In particular, we show that for any values of $b \geq 1$, $\mathcal{P}'(4,b) \leq 2$ and $\mathcal{P}'(4,b) = 1$ if and only if Thomassen's conjecture that every 4-connected line graph is hamiltonian is valid.

1. The problem

We study finite graphs with undefined terms and notation following those in [1]. Let $L(G)$ denote the **line graph** of a graph G, which is a simple graph with vertex set $E(G)$, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G. For an integer $m > 0$, define $L^0(G) = G$, and the iterated line graph $L^m(G) = L(L^{m-1}(G))$. For discussional convenience, we in this paper denote $\mathcal{G}$ to be the family of all connected nontrivial graphs that are not isomorphic to a path, a cycle or a $K_{1,3}$. To study iterated line graphs, we only consider graphs in $\mathcal{G}$. The iterated line graph index problem is an intensively studied topic in graph theory. Chartrand and Wall in [3] initiated the study of the smallest integer $k \geq 0$, called the hamiltonian index of a graph G, such that the iterated line graph $L^k(G)$ becomes hamiltonian. Other hamiltonian like indices were defined and studied by Clark and Wormald in [7]. More generally, we have the following definition.

Definition 1.1. ([17]) Let $\mathcal{P}$ denote a graphical property and G be a connected graph in $\mathcal{G}$. We also use $\mathcal{P}$ to denote the family of graphs that has property $\mathcal{P}$. Thus a graph $G \in \mathcal{P}$ if and only if G has property $\mathcal{P}$. Define $\mathcal{P}(G)$, the **$\mathcal{P}$-index** of G, as follows:

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$$\mathcal{P}(G) = \begin{cases} \min\{k : L^k(G) \in \mathcal{P}\} & \text{if for some integer } j > 0,\ L^j(G) \text{ has property } \mathcal{P}, \\ \infty & \text{otherwise.} \end{cases}$$

For vertices $u, v \in V(G)$, a (u, v)-path is a path from u to v. A graph G is **edge-hamiltonian** if every edge of G lies in a Hamilton cycle of G; and is **Hamilton-connected** if for every pair of vertices $u, v \in V(G)$, G has a spanning (u, v)-path. Clark and Wormald in [7] initiated the study of the indices for the properties of being hamiltonian, being edge-hamiltonian and being Hamilton-connected, together with several other hamiltonian properties. They proved the existences of the indices of the properties listed above. Additional studies of these indices can also be found in [17], which showed that the above-mentioned hamiltonian-like properties are closed under taking iterated line graphs. In [25], Ryjáček, Woeginger and Xiong indicated that determining the value of the hamiltonian index is a difficult problem. The index problem for graphical properties has been intensively studied, as seen in [3,4,6–8,16,17,22,26–28,25,32,35], among others. Define

$$\mathcal{H} = \{G \in \mathcal{G} : G \text{ is hamiltonian}\},$$
$$\mathcal{E}_h = \{G \in \mathcal{G} : G \text{ is edge-hamiltonian}\},$$
$$\mathcal{H}_c = \{G \in \mathcal{G} : G \text{ is Hamilton-connected}\}.$$

For a hamiltonian property $\mathcal{P}$ and integers $a > 0$ and $b > 0$, Clark and Wormald in [7] define

$$\mathcal{P}(a,b) = \begin{cases} \max\{\min\{n : L^n(G) \in \mathcal{P}\} : G \in \mathcal{G} \text{ with } \kappa'(G) \ge a, \delta(G) \ge b\}, & \text{if such max exists,} \\ \infty & \text{otherwise,} \end{cases} \tag{1}$$

and investigate the values of $\mathcal{P}(a, b)$ when $\mathcal{P}$ represents the properties of being hamiltonian, edge-hamiltoning, pancyclic, edge-pancyclic, vertex-pancyclic, Hamilton-connected and pan-connected, among others. Clark and Wormald in [7] showed that for all the above mentioned properties $\mathcal{P}$,

$$\mathcal{P}(1,1) = \mathcal{P}(1,2) = \mathcal{P}(2,2) = \infty. \tag{2}$$

Clark and Wormald in [7] also proved that for other cases with $b \ge a \ge 3$, $1 \le \mathcal{P}(a, b) \le 3$ except when $b \ge a \ge 4$ and $\mathcal{P} \in \{\mathcal{H}, \mathcal{E}_h, \mathcal{H}_c\}$. The paper [7] ends with the following question: if $\mathcal{P} \in \{\mathcal{H}, \mathcal{E}_h, \mathcal{H}_c\}$, what is the value of $\mathcal{P}(a, b)$ when $b \ge a \ge 4$?

Zhan in [33] is the first addressing this question. He proved in [33] that the line graph of every 4-edge-connected graph is in $\mathcal{H}_c$. This result implies that if $b \ge a \ge 4$, then $\mathcal{H}(a, b) = \mathcal{H}_c(a, b) = 1$. For an Hamilton-connected graph G and an arbitrary edge $e = uv \in E(G)$, as G has a spanning (u, v)-path P, $E(P) \cup \{e\}$ induces a Hamilton cycle that contains e. Therefore by definition, we have

$$\mathcal{H}_c \subseteq \mathcal{E}_h \subseteq \mathcal{H}, \text{ and so for any positive integers } a \text{ and } b, \mathcal{H}_c(a,b) \ge \mathcal{E}_h(a,b) \ge \mathcal{H}(a,b). \tag{3}$$

Hence Zhan's result gives rise to a complete answer to the question raised in [7], as follows.

Theorem 1.2. *(Zhan [33]) If $b \ge a \ge 4$, then $\mathcal{H}(a, b) = \mathcal{H}_c(a, b) = \mathcal{E}_h(a, b) = 1$.*

We consider an extension of the problem. Let $U, W \subseteq V(G)$ be vertex subsets. Define

$$(U, W)_G = \{uw \in E(G) : u \in U \text{ and } w \in W\}.$$

When $Y = V(G) - X$, then we define $\partial_G(U) = (U, V(G) - U)_G$. An edge cut of G is an edge subset of the form $\partial_G(U)$ for some proper nonempty set U. An edge subset $X = \partial_G(U)$ of G is an **essential edge cut** if either each of the subgraphs $G[U]$ and $G - U$ has at least one edge, or $|X| \ge |E(G)| - 1$. The **essential edge connectivity** of G, denoted $ess'(G)$, is the smallest size of an essential edge cut of G. A graph G is **essentially k-edge-connected** if G is connected and $ess'(G) \ge k$. By definition, it is observed in [28] that the following holds for a connected graph G with $|E(G)| \ge 3$:

$$\kappa(L(G)) = ess'(G). \tag{4}$$

For a connected nontrivial graph G, every essential edge cut of G is also an edge-cut of G. Hence we have $ess'(G) \ge \kappa'(G)$. For a graphical property $\mathcal{P}$ and positive integers a, b, define

$$\mathcal{P}'(a,b) = \begin{cases} \max\{\min\{n : L^n(G) \in \mathcal{P}\} : G \in \mathcal{G} \text{ with } ess'(G) \ge a, \delta(G) \ge b\}, & \text{if such max exists,} \\ \infty & \text{otherwise.} \end{cases} \tag{5}$$

By (1) and (5) and as $ess'(G) \ge \kappa'(G)$, it is known that $\mathcal{P}'(a, b) \ge \mathcal{P}(a, b)$ for any property $\mathcal{P}$. By definition, if a graph G satisfies both $\delta(G) \ge k$ and $ess'(G) \ge k$, then G does not have an edge cut whose size is less than $ess'(G)$, and so we must have $ess'(G) = \kappa'(G)$ in this case. Thus

$$\text{for all } b \geq a \geq 1, \mathcal{P}'(a,b) = \mathcal{P}(a,b). \tag{6}$$

As $\delta(G) \geq \kappa'(G)$ for any graph G, we observe that when $a > b$, $\mathcal{P}(a,b)$ does not exist. However, it is meaningful to discuss $\mathcal{P}'(a,b)$ even when $a > b$. Unlike the behavior of $\mathcal{P}(a,b)$, the study of $\mathcal{P}'(a,b)$ is related to the following fascinating conjecture of Thomassen:

Conjecture 1.3. *(Thomassen [30]) Every 4-connected line graph is hamiltonian.*

In this research, we shall investigate the values of $\mathcal{P}'(a,b)$ when $\mathcal{P} \in \{\mathcal{H}, \mathcal{E}_h, \mathcal{H}_c\}$. As (6) has suggested some relationship between $\mathcal{P}'(a,b)$ and $\mathcal{P}(a,b)$ when $b \geq a \geq 1$, we reformulate the results in [7] together with Theorem 1.2 as follows.

Theorem 1.4. *(Clark and Wormald [7], Zhan [33]) For $\mathcal{P} \in \{\mathcal{H}, \mathcal{E}_h, \mathcal{H}_c\}$, we have the following.*

$$P(a,b) = \begin{cases} \infty & \text{if } 1 \leq a \leq b \leq 2, \\ 3 & \text{if } \mathcal{P} \in \{\mathcal{E}_h, \mathcal{H}_c\} \text{ with } a = 1 \text{ and } b = 3, \\ 2 & \text{if } \mathcal{P} = \mathcal{H} \text{ and both } a = 1 \text{ and } b = 3, \\ 2 & \text{if } 2 \leq a \leq b \leq 3, \text{ or if } 1 \leq a \leq 3 < 4 \leq b, \\ 1 & \text{if } b \geq a \geq 4. \end{cases} \tag{7}$$

The following is the main theorem of this research.

Theorem 1.5. *If $\mathcal{P} \in \{\mathcal{H}, \mathcal{E}_h, \mathcal{H}_c\}$, then each of the following holds.*
(i) If $b \geq a \geq 1$, then $\mathcal{P}'(a,b) = \mathcal{P}(a,b)$.
(ii) For $a = 2$, $\mathcal{P}'(2,1) = \infty$.
(iii) For $a = 3$, $\mathcal{P}'(3,1) = \mathcal{P}'(3,2) = 3$.
(iv) If $a \geq 4$ and $b \geq 1$, then $\mathcal{P}'(4,b) \leq 2$. Furthermore, $\mathcal{P}'(4,b) = 1$ if and only if Conjecture 1.3 is valid.
(v) If $a \geq 6$ and $b \geq 1$, or if $a = 5$ and $b \geq 4$, then $\mathcal{P}'(a,b) = 1$.

In the next section, we summarize and develop former results and needed tools in our arguments to prove the main results. The main results will be validated in the last section.

2. Preliminaries

Given a trail $T = v_0 e_1 v_1 \ldots e_{n-1} v_{n-1} e_n v_n$ in a graph G, we often refer this trail as a (v_0, v_n)-trail to emphasize the end vertices, or as an (e_1, e_n)-trail to emphasize the end edges. The vertices $v_1, v_2, \ldots, v_{n-1}$ are the **internal vertices** of T. As a vertex may occur more than once in a trail, when either v_0 or v_n occurs in the trail as a v_i with $0 < i < n$, it is also an internal vertex by definition. A trail T of G is **internally dominating** if every edge of G is incident with an internal vertex of T, is **spanning** if T is internally dominating with $V(T) = V(G)$. A graph G is **spanning trailable** if for any pair of edges $e', e'' \in E(G)$, G has a spanning (e', e'')-trail. If H is an eulerian subgraph (a closed trail) of G, then every vertex of H is an internal vertex. Thus H is **dominating** if $E(G - V(H)) = \emptyset$. Harary and Nash-Williams discovered a close relationship between dominating eulerian subgraphs and hamiltonian line graphs.

Theorem 2.1. *(Harary and Nash-Williams, [10]) Let G be a connected graph with at least three edges. The line graph $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

Following the same idea of Theorem 2.1, the conclusions in the next proposition have been observed.

Proposition 2.2. *Let G be a connected graph with at least three edges.*
(i) The line graph $L(G)$ has a Hamilton path if and only if G has an internally dominating trail.
(ii) (Shao [28], see also Theorem 1.5 of [20]) The line graph $L(G)$ is Hamilton-connected if and only if for any edges $e, e' \in E(G)$, G has an internally dominating (e, e')-trail. In particular, if G is spanning trailable, then $L(G)$ is Hamilton-connected.

Let G be a graph, and define $\tau(G)$ to be the maximum number of edge-disjoint spanning trees in G. For each integer $i \geq 0$, define

$$D_i(G) = \{v \in V(G) : d_G(v) = i\}.$$

Thus $O(G) := \cup_{s \geq 0} D_{2s+1}(G)$ is the set of all odd degree vertices of G. A graph G is **eulerian** if G is connected with $O(G) = \emptyset$; and is **supereulerian** if it contains a spanning eulerian subgraph. For a subset $Y \subseteq E(G)$, the **contraction** G/Y is the graph obtained from G by identifying the two ends of each edge in Y and then by deleting the resulting loops. If H is a subgraph of G, we often use G/H for $G/E(H)$. For a vertex $v \in V(G/X)$, we define $PI_G(v)$ to be the contraction preimage of v in G.

A graph G is called **collapsible** if for any $R \subseteq V(G)$ with $|R|$ is even, G has a spanning subgraph S_R with $O(S_R) = R$. The following theorem briefs some of the properties of collapsible graphs.

Theorem 2.3. *Let $k \geq 1$ be an integer and G be a graph. Each of the following holds.*
(i) (Catlin [2]) (Corollary 1 of [2]) If G has a spanning tree of which every edge lies in a cycle of length 3 in G, then G is collapsible. In particular, cycles of length at most 3 are collapsible.
(ii) (Gusfield [9] and Kundu [15]) If $\kappa'(G) \geq 2k$, then $\tau(G) \geq k$.

Lemma 2.4. *(Li et al., Proposition 2.3 of [21]) Let $k \geq 1$ be an integer, and let $\mathcal{T}_k = \{G : \tau(G) \geq k\}$. Then $\mathcal{T}_k$ satisfies each of the following.*
(C1) $K_1 \in \mathcal{T}_k$.
(C2) If $G \in \mathcal{T}_k$ and $e \in E(G)$, then $G/e \in \mathcal{T}_k$.
(C3) Let H be a subgraph of G. If $H, G/H \in \mathcal{T}_k$, then $G \in \mathcal{T}_k$.

Definition 2.5. Let $e = uv$ be an edge of G. Define $G(e)$ to be the graph obtained from G by replacing $e = uv$ with a path uv_ev, where v_e is a new vertex not in $V(G)$. We say that $G(e)$ is formed by performing an **elementary subdivision** on $e \in E(G)$. For an edge subset $X \subseteq E(G)$, we use $G(X)$ to denote the graph formed by performing an elementary subdivision on each edge in X. When $X = \{e_1, e_2\}$, we also use $G(e_1, e_2)$ for $G(X)$.

As defined in [23,19], a graph G is **strongly spanning trailable** if for any $e, e' \in E(G)$, $G(e, e')$ has a $(v_e, v_{e'})$-trail T with $V(G) = V(T) - \{v_e, v_{e'}\}$. By definition, every strongly spanning trailable graph is spanning trailable. As observed in [24] (also in Chapter 1 of [31]), there exist graphs that are spanning trailable but not strongly spanning trailable.

Lemma 2.6. *Let G be a connected graph. Then each of the following holds.*
(i) (Lei et al., Theorem 2.2 (iv) of [19]) Suppose that $\tau(G) \geq 2$. For any $e', e'' \in E(G)$, $G(e', e'')$ has a spanning $(v_{e'}, v_{e''})$-trail if and only if $\{e', e''\}$ is not an edge-cut of G. Moreover, if $\{e', e''\}$ is an edge-cut of G and G_1, G_2 are the two components of $G - \{e', e''\}$, then for any $i \in \{1, 2\}$, G has an (e', e'')-trail containing all vertices in $V(G_i)$.
(ii) (Proposition 1.1 of [20]) If $G(e', e'')$ has a spanning $(v_{e'}, v_{e''})$-trail, then G has a spanning (e', e'')-trail.
(iii) If $\tau(G) \geq 2$ and $ess'(G) \geq 3$, then $L(G)$ is Hamilton-connected.

Proof. It remains to prove (iii). Suppose that $\tau(G) \geq 2$ and $ess'(G) \geq 3$. By Proposition 2.2 (ii), we shall show that G is spanning trailable. Let $e', e'' \in E(G)$. If $\{e', e''\}$ is not an edge-cut of G, then Lemma 2.6 (i) and (ii) imply that G has a spanning (e', e'')-trail. If $\{e', e''\}$ is an edge-cut of G, then as $ess'(G) \geq 3$, there exists a vertex v of degree 2 in G incident with both e' and e'', and so by Lemma 2.6 (i), G has a spanning (e', e'')-trail. $\square$

3. Proof of Theorem 1.5

Theorem 1.5 will be justified in this section. The arguments will utilize the **symmetric difference** of two sets X and Y, which is defined as

$$X \triangle Y = X \cup Y - (X \cap Y).$$

We have the following observations.

Observation 3.1. *Let G be a graph and let $u, v \in V(G)$ be two distinct vertices.*
(i) If $\{u, v\}$ is a vertex cut of G, then G does not have a spanning (u, v)-path.
(ii) If $e = uv \in E(G)$ and $\{u, v\}$ is a vertex cut of G, then G does not have a Hamilton cycle containing e.
(iii) If G is Hamilton-connected, then $\kappa(G) \geq 3$.

Shao [29] proves some useful properties for essential edge-connectivity of line graphs.

Theorem 3.2. *(Shao, Theorem 1.3 of [29]) Let $G \in \mathcal{G}$ be a connected graph with $|E(G)| \geq 4$. If $D_2(G) = \emptyset$, then $ess'(L(G)) \geq 2ess'(G) - 2$.*

Lemma 3.3. *Let G be a connected graph with $|E(G)| \geq 4$, $ess'(G) \geq 1$ and $\delta(G) \geq 3$. Then $ess'(L(G)) \geq \min\{ess'(G) + 1, 4\}$.*

Proof. By Theorem 3.2, if $ess'(G) \geq 3$, then $ess'(L(G)) \geq 2ess'(G) - 2 \geq \min\{ess'(G) + 1, 4\}$. Hence we assume that $ess'(G) \in \{1, 2\}$. Since $\delta(G) \geq 3$, we have $\delta(L(G)) \geq 4$. As $|V(L(G))| = |E(G)| \geq 4$, we have $|E(L(G))| \geq 8$. Hence we may assume that

$L(G)$ has two connected nontrivial components L_1 and L_2, with $V(L(G)) = V(L_1) \cup V(L_2)$ and $V(L_1) \cap V(L_2) = \emptyset$, such that $F = (V(L_1), V(L_2))_{L(G)}$ is a minimum essential edge-cut of $L(G)$.

Let $c = |F|$ and denote $F = \{f_1, f_2, ..., f_c\}$. Then $1 \le c \le 2$. For each $i \in \{1, 2, ..., c\}$, denote $f_i = e_i e'_i$ for edges $e_i, e'_i \in E(G)$, with $e_i \in V(L_1)$ and $e'_i \in V(L_2)$. Thus we may assume that there exist distinct vertices $u_i, v_i, w_i \in V(G)$ such that $e_i = u_i v_i$ and $e'_i = v_i w_i$. Since $\delta(G) \ge 3$, there must be an edge $e''_i = z_i v_i \in E(G) - \{e_i, e'_i\}$. As F is an essential edge cut of $L(G)$, it follows by definition that $\{e_1, e_2, ..., e_c\}$ is an essential edge cut of G. Thus $ess'(G) \le c$.

Suppose that $c = 1$. Then e_1 is an essential cut edge of G. As $e_1, e'_1, e''_1 \in E(G)$, we by symmetry may assume that $e''_i \in V(L_2)$, and so $e_1 e'_1, e_1 e''_1 \in (V(L_1), V(L_2))_{L(G)} = F$. It follows that $ess'(L(G)) = c \ge 2$.

Suppose now that $c = ess'(G) = 2$. Then we may assume that $e_1 \ne e_2$ and $\{e_1, e_2\}$ is an essential edge cut of G. With the notation above, we have $e_1, e'_1, e''_1, e_2, e'_2, e''_2 \in E(G) = V(L(G))$ with e_1, e_2, e'_1, e''_1 being mutually distinct edges in G. If $e'_1 \ne e'_2$, then F contains three distinct edges $e_1 e'_1, e_1 e''_1, e_2 e'_2$, and so we have $2 = |F| \ge |\{e_1 e'_1, e_1 e''_1, e_2 e'_2\}| = 3$, a contradiction. Assume that $e'_1 = e'_2$. If $e''_2 \in V(L_1)$, then $2 = |F| \ge |\{e_1 e'_1, e_1 e''_1, e_1 e'_2\}| = 3$, and if $e''_2 \in V(L_2)$, then $2 = |F| \ge |\{e_1 e'_1, e_1 e_2, e_2 e''_2\}| = 3$. A contradiction occurs in any case. Hence we must have $c \ge 3$. This proves the lemma. □

As $\mathcal{P}'(a, b) \ge \mathcal{P}(a, b)$, it follows from (2) that

$$\mathcal{P}'(1, 1) = \mathcal{P}'(1, 2) = \mathcal{P}'(2, 2) = \infty. \tag{8}$$

The following lemma shows an upper bound of $\mathcal{P}'(a, b)$ when $a \ge 3$.

Lemma 3.4. *Let $G \in \mathcal{G}$ be a connected graph with $|E(G)| \ge 4$ and $ess'(G) \ge 3$. Then $L^3(G)$ is Hamilton-connected. Thus for any $a \ge 3$ and $b \ge 1$, $\mathcal{H}'_c(a, b) \le 3$.*

Proof. As $G \in \mathcal{G}$ and by the definition of iterated line graphs, we have $|E(L^i(G))| \ge 4$ for $i \ge 1$. As $ess'(G) \ge 3$, we have $ess'(L(G)) \ge \kappa'(L(G)) \ge \kappa(L(G)) \ge 3$. Thus by Theorem 3.2 and $\delta(L(G)) \ge \kappa(L(G)) \ge 3$, we have $\kappa'(L^2(G)) = ess'(L^2(G)) \ge 4$, and so by Theorem 2.3 (ii), $\tau(L^2(G)) \ge 2$. It follows from Lemma 2.6 (iii) that $L^3(G)$ is Hamilton-connected. □

3.1. Justification of Theorem 1.5 (i), (ii) and (iii)

It is straightforward that Theorem 1.5 (i) is a consequence of (6). It suffices to prove Theorem 1.5 (ii) and (iii).

Proposition 3.5. *For any integer $k > 0$, there exists an infinite family $\mathcal{G}_1(k)$ of connected graphs such that every $G \in \mathcal{G}_1(k)$ satisfies $ess'(G) = 2$, $\delta(G) = 1$ and $L^k(G)$ is not hamiltonian. Thus $\mathcal{H}'(2, 1)$ cannot be bounded above by a finite number.*

Proof. Let s_1, s_2 be nonnegative integers, w_1, w_2 be two distinct vertices, and for $i \in \{1, 2\}$, $X_i = \{x^i_1, x^i_2, ..., x^i_{s_i}\}$ be a set of vertices, and for $j \in \{1, 2, 3\}$, $P_j = v^j_1 ... v^j_{k+1}$ be a path of length k, such that the sets $\{w_1, w_2\}$, X_1, X_2 and $V(P_1), V(P_2)$ and $V(P_3)$ are mutually disjoint. Define $G = G(k, s_1, s_2)$ to be the graph with

$$\begin{aligned} V(G) &= \{w_1, w_2\} \cup X_1 \cup X_2 \cup V(P_1) \cup V(P_2) \cup V(P_3), \\ E(G) &= E(P_1) \cup E(P_2) \cup E(P_3) \cup \{w_1 v^j_1 : 1 \le j \le 3\} \cup \{w_2 v^j_{k+1} : 1 \le j \le 3\} \\ &\quad \cup \{w_1 x^1_s : 1 \le s \le s_1\} \cup \{w_2 x^2_s : 1 \le s \le s_2\}. \end{aligned}$$

Hence $G - (X_1 \cup X_2)$ can be viewed as a subdivision of $K_{2,3}$. By the definition of iterated line graphs, we observe that $L^{k-1}(G)$ can be contracted to a $K_{2,3}$ in which every vertex in $D_2, K_{2,3}$ has a nontrivial contraction preimage. It follows by Theorem 2.1 that $L^k(G)$ is not hamiltonian. □

By Proposition 3.5, we conclude that for any $\mathcal{P} \in \{\mathcal{H}, \mathcal{E}_h, \mathcal{H}_c\}$, $\mathcal{P}'(2, 1) = \infty$. This proves Theorem 1.5 (ii).

To prove Theorem 1.5 (iii), we start with a proposition showing the lower bounds.

Proposition 3.6. *For any integer $k > 0$, each of the following holds.*

(i) There exists an infinite family of connected graphs $\mathcal{F}_1$ such that for any $G \in \mathcal{F}_1$, $ess'(G) = 3$, $\delta(G) = 1$ and $L^2(G)$ is not hamiltonian. Thus $\mathcal{H}'_c(3, 1) \ge \mathcal{E}'_h(3, 1) \ge \mathcal{H}'(3, 1) \ge 3$.

(ii) There exists an infinite family of connected graphs $\mathcal{F}_2$ such that for any $G \in \mathcal{F}_2$, $ess'(G) = 3$, $\delta(G) = 2$ and $L^2(G)$ is not hamiltonian. Thus $\mathcal{H}'_c(3, 2) \ge \mathcal{E}'_h(3, 2) \ge \mathcal{H}'(3, 2) \ge 3$.

Proof. Let $P(10)$ denote the Petersen graph with $E = E(P(10))$ and $V(P(10)) = \{v_i : 1 \le i \le 10\}$. As in Definition 2.5, $P(10)(E)$ is the graph formed from $P(10)$ by performing an elementary subdivision on each edge in E.

(i) For each i with $1 \le i \le 10$, let $J_i \cong K_{1,d_i}$ be a star with $d_i \ge 2$ and with w_i being the only vertex of degree d_i in J_i. Obtain a graph $G = P(10)(d_i : 1 \le i \le 10)$ from $P(10)(E)$ by identifying v_i with w_i, for each i with $1 \le i \le 10$. Define $\mathcal{F}_1$

to be the graph family such that $G \in \mathcal{F}_1$ if and only if $G = P(10)(d_i : 1 \le i \le 10)$ for some integers $d_i \ge 2$. Thus for each $G \in \mathcal{F}_1$, $ess'(G) = 3$ and $\delta(G) = 1$. As $L(G)$ is contractible to the Petersen graph with every vertex in the contraction having a nontrivial contraction preimage, it follows by Theorem 2.1 that $L^2(G)$ is not hamiltonian. This, together with (3), implies that $\mathcal{H}'_c(3,1) \ge \mathcal{E}'_h(3,1) \ge \mathcal{H}'(3,1) \ge 3$.

(ii) For each i with $1 \le i \le 10$, let $J_i \cong K_{2,d_i}$ be a star with $d_i \ge 3$ and with w_i being one of the two vertex of degree d_i in J_i. Obtain a graph $G = P(10)'(d_i : 1 \le i \le 10)$ from $P(10)(E)$ by identifying v_i with w_i, for each i with $1 \le i \le 10$. Define $\mathcal{F}_2$ to be the graph family such that $G \in \mathcal{F}_2$ if and only if $G = P(10)'(d_i : 1 \le i \le 10)$ for some integers $d_i \ge 3$. Thus for each $G \in \mathcal{F}_2$, $ess'(G) = 3$ and $\delta(G) = 2$. As $L(G)$ is contractible to the Petersen graph with every vertex in the contraction having a nontrivial contraction preimage, it follows by Theorem 2.1 that $L^2(G)$ is not hamiltonian. This, together with (3), implies that $\mathcal{H}'_c(3,2) \ge \mathcal{E}'_h(3,2) \ge \mathcal{H}'(3,2) \ge 3$. □

By Lemma 3.4, for any positive integer b, we have $\mathcal{H}'(3,b) \le \mathcal{H}'_c(3,b) \le 3$. By Proposition 3.6 (ii) and (iii), we conclude that $\mathcal{H}'_c(3,1) = \mathcal{E}'_h(3,1) = \mathcal{H}'(3,1) = 3$ and $\mathcal{H}'_c(3,2) = \mathcal{E}'_h(3,2) = \mathcal{H}('3,2) = 3$. This completes the proof for Theorem 1.5 (ii).

3.2. Justification of Theorem 1.5 (iv)

While Conjecture 1.3 remains open, there have been many researches done towards the conjecture. The following theorem summarizes some efforts on the hamiltonian properties of 4-connected iterated line graphs.

Theorem 3.7. *Let G be a connected graph. Each of the following holds.*
(i) (Corollary 3.9 of [5]) If $L^2(G)$ is 4-connected, then $L^2(G)$ is hamiltonian.
(ii) (Kriesell, [13]) If $L^2(G)$ is 4-connected, then $L^2(G)$ is Hamilton-connected.
(iii) (Theorem 1.3 of [18]) Let G be a connected graph with $|E(G)| \ge 4$ and $ess'(G) \ge 3$. If every 3-edge-cut of G has at least one edge lying in a short cycle of G, then $L(G)$ is Hamilton-connected.

In fact, Kriesell in [13] proved that every 4-connected line graph of a graph without an induced $K_{1,3}$ is Hamilton-connected, which apparently implies Theorem 3.7 (i) and (ii). As shown in Corollary 1.5 of [18], Theorem 3.7 (iii) is an extension of the above mentioned results in [5] and [13]. By Theorem 3.7 and (3), we observe that

$$\text{for any integer } b \ge 1, \mathcal{H}'(4,b) \le \mathcal{E}'_h(4,b) \le \mathcal{H}'_c(4,b) \le 2. \tag{9}$$

By (4) and (9), we are led to the conclusion that $\mathcal{H}'(4,b) = 1$ if and only if Conjecture 1.3 holds. To complete the justification of Theorem 1.5 (iv), we need the following result of Kučzel and Xiong in [14].

Theorem 3.8. *(Kučel and Xiong [14]) The following are equivalent.*
(i) Every 4-connected line graph is hamiltonian.
(ii) Every 4-connected line graph is Hamilton-connected.

By (9), $\mathcal{H}'(4,b) \le \mathcal{H}'_c(4,b) \le 2$. This, together with Theorem 3.8, has led us to the following observation.

Observation 3.9. *The following statements are equivalent.*
(i) For any positive integer b, $\mathcal{H}'(4,b) = 1$.
(ii) Every 4-connected line graph is hamiltonian.
(iii) Every 4-connected line graph is Hamilton-connected.
(iv) For any positive integer b, $\mathcal{H}'_c(4,b) = 1$.

In fact, assume that Observation 3.9 (i) holds, and so for any positive integer b, $\mathcal{H}'(4,b) = 1$. Then the definition of $\mathcal{H}'(4,b)$ implies that every connected, essentially 4-edge-connected graph has a hamiltonian line graph. By (4), this is equivalent to Observation 3.9 (ii). Next we assume that Observation 3.9 (ii) is valid. Then by Theorem 3.8, Observation 3.9 (iii) follows. By the definition of $\mathcal{H}'_c(4,b)$, we conclude that Observation 3.9 (iii) implies Observation 3.9 (iv). Finally, by (3), we observe that Observation 3.9 (iv) implies Observation 3.9 (i). This completes the justification of Observation 3.9. By (9), (3) and Observation 3.9, Theorem 1.5 (iv) is now validated.

3.3. Justification of Theorem 1.5 (v)

The first result towards Conjecture 1.3 was done by Zhan in [34]. In 2012, Kaiser and Vrána made a breakthrough after Zhan's first result. Later in 2014, Kaiser, Ryjáček and Vrána gave a further improvement, as presented below.

Theorem 3.10. *A graph G is 1-Hamilton-connected if for any vertex subset $S \subseteq V(G)$ with $|S| \le 1$, $G - S$ is Hamilton-connected.*
(i) (Zhan, Theorem 3 in [34]) If $\kappa(L(G)) \ge 7$, then $L(G)$ is Hamilton-connected.

(ii) (Kaiser and Vrána [12]) Every 5-connected claw-free graph with minimum degree at least 6 is hamiltonian.
(iii) (Kaiser, Ryjáček and Vrána [11]) Every 5-connected claw-free graph with minimum degree at least 6 is 1-Hamilton-connected.

Theorem 3.10 has an immediate corollary, which implies Theorem 1.5 (v).

Corollary 3.11. *Each of the following holds.*
(i) For any $a \geq 6$ and $b \geq 1$, $\mathcal{H}'_c(a, b) = 1$.
(ii) For any $a \geq 5$ and $b \geq 4$, $\mathcal{H}'_c(a, b) = 1$.

4. Remark

The cases we cannot determine the values of $\mathcal{P}'(a, b)$ are those when $a = 4$ and $1 \leq b \leq 5$, or $a = 5$ and $1 \leq b \leq 3$. Using the same arguments as in Subsection 3.2, it is clear that if Conjecture 1.3 holds, then $\mathcal{P}'(a, b) = 1$ for all these above-mentioned unsettled values of a and b.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, New York, 2008.
[2] P.A. Catlin, A reduction method to find spanning eulerian subgraphs, J. Graph Theory 12 (1988) 29–44.
[3] G. Chartrand, C.E. Wall, On the hamiltonian index of a graph, Studia Sci. Math. Hung. 8 (1973) 43–48.
[4] Z.-H. Chen, Y. Hong, J.-L. Lin, Z.-S. Tao, The Hamiltonian index of graphs, Discrete Math. 309 (2009) 288–292.
[5] Z.-H. Chen, H.Y. Lai, H.-J. Lai, G. Weng, Jackson's conjecture on eulerian subgraphs, in: Y. Alavi, et al. (Eds.), Combinatorics, Graph Theory, Algorithms and Applications, World Scientific, River Edge, NJ, 1994, pp. 53–58.
[6] Z.-H. Chen, H.-J. Lai, L. Xiong, H. Yan, M. Zhan, Hamilton-connected indices of graphs, Discrete Math. 309 (2009) 4819–4827.
[7] L.K. Clark, N.C. Wormald, Hamiltonian like indices of graphs, Ars Comb. 15 (1983) 131–148.
[8] R. Gould, On line graphs and the hamiltonian index, Discrete Math. 34 (1981) 111–117.
[9] D. Gusfield, Connectivity and edge-disjoint spanning trees, Inf. Process. Lett. 16 (1983) 87–89.
[10] F. Harary, C.St.J.A. Nash-Williams, On Eulerian and hamiltonian graphs and line graphs, Can. Math. Bull. 8 (1965) 701–709.
[11] T. Kaiser, Z. Ryjáček, P. Vrána, On 1-Hamilton-connected claw-free graphs, Discrete Math. 321 (2014) 1–11.
[12] T. Kaiser, P. Vrána, Hamilton cycles in 5-connected line graphs, Eur. J. Comb. 33 (2012) 924–947.
[13] M. Kriesell, All 4-connected line graphs of claw free graphs are Hamiltonian-connected, J. Comb. Theory, Ser. B 82 (2001) 306–315.
[14] R. Kučzel, L. Xiong, Every 4-connected Line Graph is Hamiltonian if and only if it is Hamiltonian Connected, in R. Kučzel: Hamiltonian properties of graphs, Ph.D. Thesis, U.W.B. Pilsen, 2004.
[15] S. Kundu, Bounds on the number of disjoint spanning trees, J. Comb. Theory, Ser. B 17 (1974) 199–203.
[16] H.-J. Lai, On the hamiltonian index, Discrete Math. 69 (1988) 43–53.
[17] H.-J. Lai, Y. Shao, Problems related to hamiltonian line graphs, AMS/IP Stud. Adv. Math. 39 (2007) 149–159.
[18] H.-J. Lai, Y. Shao, G. Yu, M. Zhan, Hamiltonian connectedness in 3-connected line graphs, Discrete Appl. Math. 157 (2009) 982–990.
[19] L. Lei, X. Li, X. Ma, M. Zhan, H.-J. Lai, Chvátal-Erdős conditions and almost spanning trails, Bull. Malays. Math. Sci. Soc. 43 (2020) 4373–4391.
[20] D. Li, H.-J. Lai, M. Zhan, Eulerian subgraphs and hamilton-connected line graphs, Discrete Appl. Math. 145 (2005) 422–428.
[21] P. Li, H.-J. Lai, Y. Liang, Characterization of removable elements with respect to having k disjoint bases in a matroid, Discrete Appl. Math. 160 (2012) 2445–2451.
[22] H. Li, H.-J. Lai, Y. Wu, S. Zhu, Panconnected index of graphs, Discrete Math. 340 (2017) 1092–1097.
[23] P. Li, K. Wang, M. Zhan, H.-J. Lai, Strongly spanning trailable graphs with short longest paths, Ars Comb. 137 (2018) 3–39.
[24] J. Liu, A. Yu, K. Wang, H.-J. Lai, Degree sum and hamiltonian-connected line graphs, Discrete Math. 341 (2018) 1363–3179.
[25] Z. Ryjáček, G.J. Woeginger, L. Xiong, Hamiltonian index is NP-complete, Discrete Appl. Math. 159 (2011) 246–250.
[26] M.L. Sarazin, On the hamiltonian index of a graph, Discrete Math. 122 (1993) 373–376.
[27] M.L. Sarazin, A simple upper bound for the hamiltonian index of a graph, Discrete Math. 134 (1994) 85–91.
[28] Y. Shao, Claw-free Graphs and Line Graphs, Ph. D. Dissertation, West Virginia University, 2005.
[29] Y. Shao, Essential edge connectivity of line graphs, Discrete Math. 341 (2018) 3441–3446.
[30] C. Thomassen, Reflections on graph theory, J. Graph Theory 10 (1986) 309–324.
[31] K. Wang, Supereulerian Properties in Graphs and Hamiltonian Properties in Line Graphs, Ph.D. Dissertation, West Virginia University, USA, 2015.
[32] W. Xiong, Z. Zhang, H.-J. Lai, Spanning 3-connected index of graphs, J. Comb. Optim. 27 (2014) 199–208.
[33] S.M. Zhan, Hamiltonian-connectedness of line graphs, Ars Comb. 22 (1986) 89–95.
[34] S.M. Zhan, On Hamiltonian line graphs and connectivity, Discrete Math. 89 (1991) 89–95.
[35] L. Zhang, E. Eschen, H.-J. Lai, Y. Shao, The s-Hamiltonian index, Discrete Math. 308 (2008) 4779–4785.