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Trail-Connected Digraphs with Given Local Structures*

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For a digraph D, if D contains a spanning closed trail, then D is supereulerian. If for any pairs vertices z and w of D, D contains both a spanning (z, w)-trail and a spanning (w, z)-trail, then D is strongly trail-connected. If D is a strongly trail-connected digraph, then D is a supereulerian digraph. Algefari *et al.* proved that every symmetrically connected digraph and every partially symmetric digraph are supereulerian. In this paper, we prove that every symmetrically connected digraph and every partially symmetric digraph are strongly trail-connected digraphs.

Keywords: Strongly trail-connected digraph; supereulerian digraph; partially symmetric digraph; symmetrically connected digraph.

1. Introduction

We only discuss simple graphs and strict digraphs without loops and parallel arcs in this paper, terminologies and notations not given below follow [7] and [4]. An ordered pair (z, w) denotes an arc oriented from vertex z to vertex w. We use paths, cycles and trails as defined in [7] when the discussion is on an undirected graph G, and to denote directed paths, directed cycles and directed trails when the discussion is on a digraph D. As in [4], the number of the arcs of P is called the length of P.

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For any two vertices $z, w \in V(D)$, a trail (or path, respectively) from z to w is often called a (z, w)-trail (or (z, w)-path, respectively). Let $P_{(v_1,v_p)} = v_1v_2 \cdots v_p$ be a trail (or path) in D. For $1 \leq i < j \leq p$, we define $P_{(v_1,v_p)}[v_i, v_j] = v_iv_{i+1}v_{i+2} \cdots v_{j-1}v_j$ to be the **subtrail** (or **subpath**). A **null graph** is one that does not have any vertices nor arcs.

Let $D = (V(D), A(D))$ be a digraph. For $B \subseteq A(D)$, we use $D - B$ to donote the spanning subdigraph of D with $V(D - B) = V(D)$ and $A(D - B) = A(D) - B$. For $X \subseteq V(D)$, the subdigraph induced by $V(D) - X$ is donoted by $D - X$. The underlying graph of D is denoted by $G(D)$. If $G(D)$ is connected, then D is **weakly connected**. If for any two distinct vertices $z, w \in V(D)$, D has both a (z, w)-path and a (w, z)-path, then D is **strong**. An arc (z, w) is **symmetric** in D if $(w, z), (z, w) \in A(D)$. A path of D is a **symmetric path**, denoted by $\overset{\leftrightarrow}{P}$, if every arc in $A(\overset{\leftrightarrow}{P})$ is symmetric in D. If a digraph D has one vertex, or if D has at least one arc and every arc in $A(D)$ is symmetric, then digraph D is **symmetric**.

Notation 1.1. *For any two vertices $z_1, z_\ell \in V(D)$, if there exists a symmetric path from z_1 to z_ℓ, we will use $\overset{\leftrightarrow}{P}_{(z_1,z_\ell)}$ to denote the symmetric path from vertex z_1 to vertex z_ℓ.*

(i) *To emphasize that every arc in $\overset{\leftrightarrow}{P}_{(z_1,z_\ell)}$ is symmetric, we often express $\overset{\leftrightarrow}{P}_{(z_1,z_\ell)}$ as an arc-disjoint union of a (z_1, z_ℓ)-path and a (z_ℓ, z_1)-path, as denoted by*

$$\overset{\leftrightarrow}{P}_{(z_1,z_\ell)} = z_1z_2 \cdots z_{\ell-1}z_\ell z_{\ell-1} \cdots z_2z_1. \tag{1}$$

(ii) *We use $P_{(z_1,z_\ell)} = z_1z_2 \cdots z_{\ell-1}z_\ell$ to denote the (z_1, z_ℓ)-path, and $P_{(z_\ell,z_1)} = z_\ell z_{\ell-1} \cdots z_2z_1$ to denote the (z_ℓ, z_1)-path contained in the symmetric path in* (1), *respectively.*

(iii) *For any two different i and j with $1 \leq i < j \leq \ell$, define*

$$\overset{\leftrightarrow}{P}_{(z_1,z_\ell)}[z_i, z_j] = z_iz_{i+1} \cdots z_{j-1}z_jz_{j-1} \cdots z_{i+1}z_i$$

to be the symmetric subpath from vertex z_i to vertex z_j of $\overset{\leftrightarrow}{P}_{(z_1,z_\ell)}$.

Following Bang-Jasen and Gutin [4], for any two subsets $W, Z \subseteq V(D)$, the following notation will be used:

$$(W, Z)_D = \{(w, z) \in A(D) : w \in W \text{ and } z \in Z\}.$$

When $Z = V(D) - W$, we let

$$\partial_D^+(W) = (W, V(D) - W)_D \quad \text{and} \quad \partial_D^-(W) = (V(D) - W, W)_D.$$

Digraph $D[W]$ denotes the subdigraph induced by W. The **out-neighborhood** of a vertex z, denoted by $N_D^+(z)$, is $\{z' \in V(D) : (z, z') \in A(D)\}$; the **in-neighborhood** of a vertex z, denoted by $N_D^-(z)$, is $\{z' \in V(D) : (z', z) \in A(D)\}$. Let D be a digraph, for any two vertices $z, w \in V(D)$, D is **weakly hamiltonian-connected** if D admits a spanning (z, w)-path or a spanning (w, z)-path; and D is **strongly**

hamiltonian-connected if D contains both a spanning (z, w)-path and a spanning (w, z)-path. It is obvious that if D is strongly hamiltonian-connected, then it is hamiltonian. In 1980, Thomassen [11] discussed strongly (weakly) hamiltonian-connected tournaments. D is a **weakly trail-connected** digraph if for any two vertices z and w of D, D admits a spanning (z, w)-trail or a spanning (w, z)-trail, and a digraph D is **strongly trail-connected** if for any two vertices z and w of D, D contains both a spanning (z, w)-trail and a spanning (w, z)-trail. Liu *et al.* in [10] considered weakly (strongly) trail-connected digraphs. A digraph D is **supereulerian** if D contains a spanning closed trail.

Supereulerian digraphs have been studied by a number of authors (cf. Algefari *et al.* [1–3], Bang-Jensen *et al.* [5, 6], Dong *et al.* [8], Hong *et al.* [9], Zhang *et al.* [12], and related references). In particular, Algefari *et al.* [1] defined two kinds of digraphs and proved that they are supereulerian.

Definition 1.1 ([1]). Let D be a digraph such that either D has one vertex or D has at least one arc. If D has a symmetric (z, w)-path for any two vertices z and w of D, then D is called a symmetrically connected digraph.

Definition 1.2 ([1]). Let c be an integer with $c \geq 2$, D be a weakly connected digraph and $\{H_1, H_2, \ldots, H_c\}$ be the set of all maximal symmetrically connected subdigraphs of D. If for any proper nonempty subset $\mathcal{J}$ of $\{H_1, H_2, \ldots, H_c\}$, there exist a subdigraph $H_i \in \mathcal{J}$ and a vertex w of H_i, and an $H_j \notin \mathcal{J}$ such that

$$N_D^+(w) \cap V(H_j) \neq \emptyset \quad \text{and} \quad N_D^-(w) \cap V(H_j) \neq \emptyset,$$

then D is partially symmetric.

Theorem 1.2 (Algefari *et al.* [1], Theorem 1.3). *Every symmetrically connected digraph and every partially symmetric digraph are supereulerian.*

By the definition of strongly (weakly) trail-connected digraph, if D is a strongly trail-connected digraph, we know that D is weakly trail-connected and supereulerian. Liu *et al.* in [10] characterized weakly (strongly) trail-connected tournaments. The purpose of this paper is to prove that symmetrically connected digraph and partially symmetric digraph are strongly trail-connected.

2. Main Results

Let M and M' be two digraphs. Throughout this paper, we define the **union** $M \cup M'$ (or **intersection** $M \cap M'$, respectively) of M and M' to be the digraph with $V(M \cup M') = V(M) \cup V(M')$ and $A(M \cup M') = A(M) \cup A(M')$ (or $V(M \cap M') = V(M) \cap V(M')$ and $A(M \cap M') = A(M) \cap A(M')$, respectively). By definition, it is possible that $M \cap M'$ is the null graph.

Let v, v', w and w' be vertices of D. If $T_{(v,w)}$ is a (v, w)-trail of D and $(v', v), (w, w') \in A(D) - A(T_{(v,w)})$, then we use $(v', v)T_{(v,w)}(w, w')$ to denote the (v', w')-trail $D[A(T_{(v,w)}) \cup \{(v', v), (w, w')\}]$. The subdigraphs $(v', v)T_{(v,w)}$ and $T_{(v,w)}(w, w')$ are similarly defined.

Theorem 2.1. *Every symmetrically connected digraph is strongly trail-connected.*

Proof. Let D be a symmetrically connected digraph. Toward a contradiction, suppose that D is not a strongly trail-connected digraph. By Theorem 1.2, D is supereulerian, and therefore D is a strong digraph. Hence D has a nontrivial strongly trail-connected subdigraph. Let C be a strongly trail-connected subdigraph in D with

$$|V(C)| \text{ is maximized among all strongly trail-connected subdigraph in } D. \quad (2)$$

Since D is symmetrically connected digraph but not strongly trail-connected, hence there are a vertex z of C and a vertex w of $D - C$ such that $(z, w), (w, z) \in A(D)$. Now we consider $|V(C)| < |V(D)|$ and the subdigraph $D[V(C) \cup \{w\}]$, denoted by D'.

Consider any two vertices u and v of D'. Since C is a strongly trail-connected subdigraph of D, if $u, v \in V(C)$, then there exist a spanning (u, v)-trail $T_{(u,v)}$ and a spanning (v, u)-trail $T_{(v,u)}$ of C. Thus, $T_{(u,v)} \cup (w, z) \cup (z, w)$ is a spanning (u, v)-trail and $T_{(v,u)} \cup (w, z) \cup (z, w)$ is a spanning (v, u)-trail of D'. If $u = w$ and $v \in V(C)$, since $z \in V(C)$, C contains a spanning (z, v)-trail $T_{(z,v)}$ and a spanning (v, z)-trail $T_{(v,z)}$. Thus, $(u, z) \cup T_{(z,v)}$ is a spanning (u, v)-trail and $T_{(v,z)} \cup (z, u)$ is a spanning (v, u)-trail of D'. The analyses above imply that D' is also a strongly trail-connected subdigraph of D, contrary to (2). Therefore, a symmetrically connected digraph is strongly trail-connected. This proves the theorem. □

Next, we construct a new digraph from several strongly trail-connected digraphs and prove that the new digraph is strongly trail-connected.

Definition 2.1. For a given integer t with $t \geq 2$, let $S_1, S_2, \ldots, S_t$ be strongly trail-connected digraphs, and y_i^1, y_i^2, y_i^3 be three vertices ($y_i^{j_1} = y_i^{j_2}$ is allowed, $j_1, j_2 \in \{1, 2, 3\}$) in $V(S_i)$ for $1 \leq i \leq t$. We shall define a digraph M with $V(M) = V(S_1) \cup V(S_2) \cup \cdots \cup V(S_t)$ and $A(M) = \bigcup_{i=1}^{t} (A(S_i) \cup (y_i^1, y_{i+1}^2) \cup (y_{i+1}^3, y_i^1))$ (Consider modulo t) (See Fig. 1).

Theorem 2.2. *Let M be a digraph defined as in Definition* 2.1. *Then M is strongly trail-connected.*

Proof. Let $S_1, S_2, \ldots, S_t$ be strongly trail-connected digraphs, and y_i^1, y_i^2, y_i^3 be three vertices in $V(S_i)$ for $1 \leq i \leq t$. Thus, S_i contains a spanning (z, w)-trail $T^i_{(z,w)}$ for any two vertices z and w of S_i. Let u and v be any two vertices of M, we want to prove that M contains a spanning (u, v)-trail and a spanning (v, u)-trail. Let $T' = \bigcup_{i=1}^{t} ((y_i^1, y_{i+1}^2) \cup T^{i+1}_{(y_{i+1}^2, y_{i+1}^3)} \cup (y_{i+1}^3, y_i^1))$.

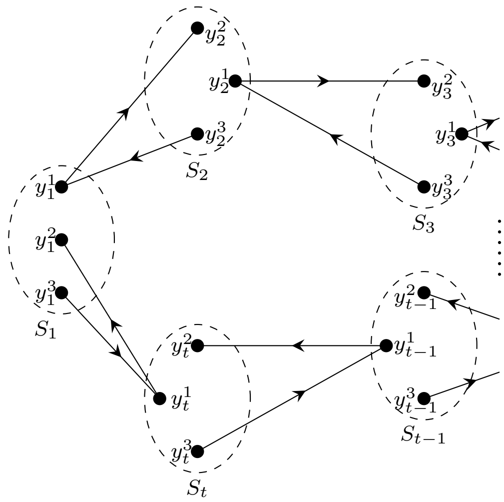

Fig. 1. A digraph defined as in Definition 2.1.

If $u, v \in V(S_k)$ for some index k with $1 \leq k \leq t$, then

$$\big(T' - (y_{k-1}^1, y_k^2) - A(T^k_{(y_k^2, y_k^3)}) - (y_k^3, y_{k-1}^1)\big) \cup T^k_{(u,v)}$$

is a spanning (u, v)-trail of M. Similarly,

$$\big(T' - (y_{k-1}^1, y_k^2) - A(T^k_{(y_k^2, y_k^3)}) - (y_k^3, y_{k-1}^1)\big) \cup T^k_{(v,u)}$$

is a spanning (v, u)-trail of M.

If $u \in V(S_k)$ and $v \in V(S_\ell)$ for indices k and ℓ such that $1 \leq k < \ell \leq t$, then

$$\left(T' - \left\{\bigcup_{j=k}^{\ell} \big(A(T^j_{(y_j^2, y_j^3)}) \cup (y_j^3, y_{j-1}^1)\big)\right\} - (y_{k-1}^1, y_k^2)\right)$$
$$\cup T^k_{(u, y_k^1)} \cup \bigcup_{j=k+1}^{\ell-1} T^j_{(y_j^2, y_j^1)} \cup T^\ell_{(y_\ell^2, v)}$$

is a spanning (u, v)-trail of M, where the addition operation takes modulo t. Similarly,

$$\left(T' - \left\{\bigcup_{j=k}^{\ell} \big((y_{j-1}^1, y_j^2) \cup A(T^j_{(y_j^2, y_j^3)})\big)\right\} - (y_k^3, y_{k-1}^1)\right)$$
$$\cup T^\ell_{(v, y_\ell^3)} \cup \bigcup_{j=k+1}^{\ell-1} T^j_{(y_j^1, y_j^3)} \cup T^k_{(y_k^1, u)}$$

is a spanning (v, u)-trail of M. This proves the theorem. □

Next, we will prove that every partially symmetric digraph is strongly trail-connected. Before the proof is given, we define a useful digraph from a weakly connected digraph.

Definition 2.2. For a given integer c such that $c \geq 2$, let D be a weakly connected digraph and $\{H_1, H_2, \ldots, H_c\}$ be the set of all maximal symmetrically connected subdigraphs of D. We shall define a digraph D' in the following steps.

(i) Contracting all maximal symmetrically connected subdigraphs $H_1, H_2, \ldots, H_c$ as vertices of D' with $V(D') = \{h_1, h_2, \ldots, h_c\}$.
(ii) For any two different vertices $h_i, h_j \in V(D')$, $(h_i, h_j) \in A(D')$ if and only if H_i has a vertex w satisfying

$$N_D^+(w) \cap V(H_j) \neq \emptyset \quad \text{and} \quad N_D^-(w) \cap V(H_j) \neq \emptyset.$$

Lemma 2.1. *For a weakly connected digraph D, let D' be a digraph constructed from D as in Definition 2.2. Then D is a partially symmetric digraph if and only if the digraph D' is a strong digraph.*

Proof. If D is a partially symmetric digraph and D has maximal symmetrically connected subdigraphs $H_1, H_2, \ldots, H_c$, let $\mathcal{H} = \{H_1, H_2, \ldots, H_c\}$. And let D' be constructed by Definition 2.2 with $V(D') = \{h_1, h_2, \ldots, h_c\}$.

On the one hand, we show that if D is a partially symmetric digraph, then D' is a strong digraph. Toward a contradiction, suppose that D' is not a strong digraph, then there exists a subset $\mathcal{J}' = \{h_{i_1}, h_{i_2}, \ldots, h_{i_s}\} \subset V(D')$ such that $|\partial_{D'}^+(\mathcal{J}')| = 0$, where $i_1, i_2, \ldots, i_s \in \{1, 2, \ldots, c\}$. By Definition 2.2 and $\mathcal{J}'$, we can get the subset $\mathcal{J}$ of $\mathcal{H}$ with $\mathcal{J} = \{H_{i_1}, H_{i_2}, \ldots, H_{i_s}\}$. Since $|\partial_{D'}^+(\mathcal{J}')| = 0$, hence there are no vertex w of $H_{i_q} \in \mathcal{J}$ with $1 \leq q \leq s$, and $H_j \in \mathcal{H} - \mathcal{J}$ with $j \in \{i_{s+1}, i_{s+2}, \ldots, i_c\}$ such that

$$N_D^+(w) \cap V(H_j) \neq \emptyset \quad \text{and} \quad N_D^-(w) \cap V(H_j) \neq \emptyset.$$

It can be obtained from Definition 1.2 that D is not a partially symmetric digraph. This proves that D' is strong.

On the other hand, for any subset $\mathcal{J}_1' \subset V(D')$, let $\mathcal{J}_1' = \{h_{i_1'}, h_{i_2'}, \ldots, h_{i_{s'}'}\} \subset V(D')$ and let $\mathcal{J}_2' = \{h_{j_1}, h_{j_2}, \ldots, h_{j_{s''}}\} = V(D') - \mathcal{J}_1'$. As D' is a strong digraph, hence there exists $(h_{i_r'}, h_{j_p}) \in A(D')$ such that $h_{i_r'} \in \mathcal{J}_1'$ and $h_{j_p} \in \mathcal{J}_2'$. By Definition 2.2, this implies that for any $\mathcal{J}_1 = \{H_{i_1'}, H_{i_2'}, \ldots, H_{i_{s'}'}\} \subset \mathcal{H}$, there exist at least one vertex w of $H_{i_r'}$ for $1 \leq r \leq s'$ and an H_{j_p} of $\mathcal{H} - \mathcal{J}_1$ for $1 \leq p \leq s''$ such that

$$N_D^+(w) \cap V(H_{j_p}) \neq \emptyset \quad \text{and} \quad N_D^-(w) \cap V(H_{j_p}) \neq \emptyset.$$

It follows from Definition 1.2 that D is a partially symmetric digraph. This proves the lemma. □

Lemma 2.2. *Let H_1 be a strongly trail-connected digraph and H_2 be a symmetrically connected digraph. If $V(H_1) \cap V(H_2) \neq \emptyset$, then the union $H_1 \cup H_2$ of H_1 and H_2 is a strongly trail-connected digraph.*

Proof. Let H_1 be a strongly trail-connected digraph and H_2 be a symmetrically connected digraph, by Theorem 2.1, H_2 is also a strongly trail-connected digraph. Let J be a spanning connected symmetric subdigraph of H_2. Denote $D = H_1 \cup H_2$, if $V(D) = V(H_1)$ or $V(D) = V(H_2)$, we are done. In the following, we only need to assume that $V(D) \neq V(H_1)$, $V(D) \neq V(H_2)$ and prove that for any two vertices u and v of D,

$$D \text{ contains a spanning } (u,v)\text{-trail and a spanning } (v,u)\text{-trail.} \tag{3}$$

Since $V(H_1) \cap V(H_2) \neq \emptyset$, let $X_1 = \{x_{11}, x_{12}, \ldots, x_{1c_1}\} = V(H_1) \cap V(H_2)$, we consider the following two cases.

Case 1. $c_1 = 1$.

$X_1 = \{x_{11}\} = V(H_1) \cap V(H_2)$. If $u, v \in V(H_1)$, as H_1 and H_2 are strongly trail-connected digraphs, then H_1 contains a spanning (u,v)-trail $T^{H_1}_{(u,v)}$ and a spanning (v,u)-trail $T^{H_1}_{(v,u)}$, and H_2 contains a spanning (x_{11}, x_{11})-trail $T^{H_2}_{(x_{11},x_{11})}$. Thus, $T^{H_1}_{(u,v)} \cup T^{H_2}_{(x_{11},x_{11})}$ is a spanning (u,v)-trail and $T^{H_1}_{(v,u)} \cup T^{H_2}_{(x_{11},x_{11})}$ is a spanning (v,u)-trail of D. Similarly, we can get the result for $u, v \in V(H_2)$. Hence assume that $u \in V(H_1)$ and $v \in V(H_2)$, then H_1 contains a spanning (u, x_{11})-trail $T^{H_1}_{(u,x_{11})}$ and a spanning (x_{11}, u)-trail $T^{H_1}_{(x_{11},u)}$, and H_2 contains a spanning (x_{11}, v)-trail $T^{H_2}_{(x_{11},v)}$ and a spanning (v, x_{11})-trail $T^{H_2}_{(v,x_{11})}$. Hence $T^{H_1}_{(u,x_{11})} \cup T^{H_2}_{(x_{11},v)}$ is a spanning (u,v)-trail and $T^{H_2}_{(v,x_{11})} \cup T^{H_1}_{(x_{11},u)}$ is a spanning (v,u)-trail of D.

Case 2. $c_1 \geq 2$.

$X_1 = \{x_{11}, x_{12}, \ldots, x_{1c_1}\} = V(H_1) \cap V(H_2)$. In this case, we will consider three subcases to prove that (3) holds for any two vertices u and v of D.

Subcase 2.1. $u, v \in V(H_1)$.

Since H_1 is a strongly trail-connected digraph, hence there exist a spanning (u,v)-trail $T^{H_1}_{(u,v)}$ and a spanning (v,u)-trail $T^{H_1}_{(v,u)}$ of H_1. Let $V(H_2) - X_1 = \{y_{11}, y_{12}, \ldots, y_{1s_1}\}$. Since H_2 is a symmetrically connected digraph, by the Definition 1.1, H_2 has a symmetric path

$$\overset{\leftrightarrow}{P}_{(v_{11},v_{1m_1})} = v_{11}v_{12} \cdots v_{1m_1}v_{1(m_1-1)} \cdots v_{12}v_{11}$$

from v_{11} to v_{1m_1}, where $v_{11} = x_{11}$ and $v_{1m_1} = y_{11}$. Since $x_{11} \in X_1$ and $y_{11} \in V(H_2) - X_1$, there exists a largest integer t_1 with $t_1 > 0$ such that $v_{1(t_1-1)} \in X_1$ and $v_{1t_1}, v_{1(t_1+1)}, \ldots, v_{1m_1} \in V(H_2) - X_1$. By Notation 1.1 (*iii*), we use $\overset{\leftrightarrow}{P}_{(v_{11},v_{1m_1})}[v_{1(t_1-1)}, v_{1m_1}]$ to denote the symmetric subpath of $\overset{\leftrightarrow}{P}_{(v_{11},v_{1m_1})}$ from $v_{1(t_1-1)}$ to v_{1m_1}. Then

$$T^{H_1}_{(u,v)} \cup \overset{\leftrightarrow}{P}_{(v_{11},v_{1m_1})}[v_{1(t_1-1)}, v_{1m_1}]$$
$$\cup \Big(J - A\big(H_2[V(\overset{\leftrightarrow}{P}_{(v_{11},v_{1m_1})}[v_{1(t_1-1)}, v_{1m_1}])]\big) - A\big(H_2[X_1]\big)\Big)$$

is a spanning (u, v)-trail of D and

$$T^{H_1}_{(v,u)} \cup \overset{\leftrightarrow}{P}_{(v_{11},v_{1m_1})}[v_{1(t_1-1)}, v_{1m_1}] \\ \cup\Big(J - A\big(H_2[V(\overset{\leftrightarrow}{P}_{(v_{11},v_{1m_1})}[v_{1(t_1-1)}, v_{1m_1}])]\big) - A\big(H_2[X_1]\big)\Big)$$

is a spanning (v, u)-trail of D.

Subcase 2.2. $u \in V(H_1)$ and $v \in V(H_2)$.

If $v \in X_1$, then $v \in V(H_1)$, by the similar analysis as in Subcase 2.1, (3) holds. Hence assume that $v \in V(H_2) - X_1$. Since H_2 is a symmetrically connected digraph, hence H_2 contains a symmetric path

$$\overset{\leftrightarrow}{P}_{(v_{11},v_{1m_2})} = v_{11}v_{12}\cdots v_{1m_2}v_{1(m_2-1)}\cdots v_{12}v_{11}$$

from v_{11} to v_{1m_2}, where $v_{11} = x_{11}$ and $v_{1m_2} = v$. Since $x_{11} \in X_1$, $v \in V(H_2) - X_1$, thus, there exists a largest integer t_2 with $t_2 > 0$ satisfying $v_{1(t_2-1)} \in X_1$ and $v_{1t_2}, v_{1(t_2+1)}, \ldots, v_{1m_2} \in V(H_2) - X_1$. Since H_1 is a strongly trail-connected digraph, there exist a spanning $(u, v_{1(t_2-1)})$-trail $T^{H_1}_{(u,v_{1(t_2-1)})}$ and a spanning $(v_{1(t_2-1)}, u)$-trail $T^{H_1}_{(v_{1(t_2-1)},u)}$ of H_1. By Notation 1.1 (*ii*), we use $P_{(v_{11},v)} = P_{(v_{11},v_{1m_2})}$ and $P_{(v,v_{11})} = P_{(v_{1m_2},v_{11})}$ to denote the (v_{11}, v)-path and (v, v_{11})-path contained in the symmetric path $\overset{\leftrightarrow}{P}_{(v_{11},v_{1m_2})}$, so $P_{(v_{11},v)}[v_{1(t_2-1)}, v]$ is a subpath of $P_{(v_{11},v)}$ from $v_{1(t_2-1)}$ to v, and $P_{(v,v_{11})}[v, v_{1(t_2-1)}]$ is a subpath of $P_{(v,v_{11})}$ from v to $v_{1(t_2-1)}$. Then

$$T^{H_1}_{(u,v_{1(t_2-1)})} \cup P_{(v_{11},v)}[v_{1(t_2-1)}, v] \cup \big(J - A\big(H_2[V(P_{(v_{11},v)}[v_{1(t_2-1)}, v])]\big)\big)$$

is a spanning (u, v)-trail of D and

$$P_{(v,v_{11})}[v, v_{1(t_2-1)}] \cup T^{H_1}_{(v_{1(t_2-1)},u)} \cup \big(J - A\big(H_2[V(P_{(v_{11},v)}[v, v_{1(t_2-1)}])]\big)\big)$$

is a spanning (v, u)-trail of D.

Subcase 2.3. $u, v \in V(H_2)$.

If $u, v \in X_1$, then $u, v \in V(H_1)$, by the similar analysis as in Subcase 2.1, (3) holds. If $u \in X_1$ and $v \in V(H_2) - X_1$, by the similar analysis of the second case as in Subcase 2.2, (3) holds. Finally, we consider the case that $u, v \in V(H_2) - X_1$. Since H_2 is a symmetrically connected digraph, hence there is a symmetric path from u to v, for convenience, let $\overset{\leftrightarrow}{P}_{(u,v)} = u_1u_2\cdots u_{m_3}u_{m_3-1}\cdots u_2u_1$ be a symmetric path as longer as possible with $u_1 = u$ and $u_{m_3} = v$.

If $V(\overset{\leftrightarrow}{P}_{(u,v)}) \cap X_1 \neq \emptyset$, then there exist a smallest integer s with $1 \leq s < m_3$ and a largest integer t_3 with $1 < t_3 \leq m_3$ such that $u_{s+1} \in X_1$ and $u_1, u_2, \ldots, u_s \in V(H_2) - X_1$, and $u_{t_3-1} \in X_1$ and $u_{t_3}, u_{t_3+1}, \ldots, u_{m_3} \in V(H_2) - X_1$. Since H_1 is a strongly trail-connected digraph, there exist a spanning (u_{s+1}, u_{t_3-1})-trail $T^{H_1}_{(u_{s+1},u_{t_3-1})}$ and a spanning (u_{t_3-1}, u_{s+1})-trail $T^{H_1}_{(u_{t_3-1},u_{s+1})}$ of H_1. We use $P_{(u,v)}[u, u_{s+1}]$ and $P_{(u,v)}[u_{t_3-1}, v]$ to denote the subpaths of $P_{(u,v)}$ from vertex u to vertex u_{s+1} and from vertex u_{t_3-1} to vertex v, and we use $P_{(v,u)}[u_{s+1}, u]$ and

$P_{(v,u)}[v, u_{t_3-1}]$ to denote the subpaths of $P_{(v,u)}$ from vertex u_{s+1} to vertex u and from vertex v to vertex u_{t_3-1}. Thus,

$$P_{(u,v)}[u, u_{s+1}] \cup T^{H_1}_{(u_{s+1}, u_{t_3-1})} \cup P_{(u,v)}[u_{t_3-1}, v]$$
$$\cup \big(J - A\big(H_2[V(P_{(u,v)}[u, u_{s+1}])]\big) - A\big(H_2[V(P_{(u,v)}[u_{t_3-1}, v])]\big)\big)$$

is a spanning (u, v)-trail of D, and

$$P_{(v,u)}[v, u_{t_3-1}] \cup T^{H_1}_{(u_{t_3-1}, u_{s+1})} \cup P_{(v,u)}[u_{s+1}, u]$$
$$\cup \big(J - A\big(H_2[V(P_{(v,u)}[u_{s+1}, u])]\big) - A\big(H_2[V(P_{(v,u)}[v, u_{t_3-1}])]\big)\big)$$

is a spanning (v, u)-trail of D.

If $V(\overset{\leftrightarrow}{P}_{(u,v)}) \cap X_1 = \emptyset$, since H_2 is symmetrically connected digraph, hence there exist a vertex $x_{1c'_1}$ of X_1 for $1 \leq c'_1 \leq c_1$, a vertex u_j of $P_{(u,v)}$ for $1 \leq j \leq m_3$ and a symmetric path $\overset{\leftrightarrow}{P}_{(x_{1c'_1}, u_j)}$ from $x_{1c'_1}$ to u_j such that $V(\overset{\leftrightarrow}{P}_{(x_{1c'_1}, u_j)}) \cap V(\overset{\leftrightarrow}{P}_{(u,v)}) = \{u_j\}$ and $V(\overset{\leftrightarrow}{P}_{(x_{1c'_1}, u_j)}) \cap V(H_1) = \{x_{1c'_1}\}$. Then

$$P_{(u,v)} \cup T^{H_1}_{(x_{1c'_1}, x_{1c'_1})} \cup \overset{\leftrightarrow}{P}_{(x_{1c'_1}, u_j)} \cup \big(J - A\big(H_2[V(P_{(u,v)})]\big) - A\big(H_2[V(\overset{\leftrightarrow}{P}_{(x_{1c'_1}, u_j)})]\big)\big)$$

is a spanning (u, v)-trail of D, and

$$P_{(v,u)} \cup T^{H_1}_{(x_{1c'_1}, x_{1c'_1})} \cup \overset{\leftrightarrow}{P}_{(x_{1c'_1}, u_j)} \cup \big(J - A\big(H_2[V(P_{(v,u)})]\big) - A\big(H_2[V(\overset{\leftrightarrow}{P}_{(x_{1c'_1}, u_j)})]\big)\big)$$

is a spanning (v, u)-trail of D. This proves the lemma. □

Theorem 2.3. *Let D be a partially symmetric digraph. Then D is strongly trail-connected.*

Proof. If D is a partially symmetric digraph, and digraph D has maximal symmetrically connected subdigraphs $H_1, H_2, \ldots, H_c$, let $\mathcal{H} = \{H_1, H_2, \ldots, H_c\}$. By contradiction, suppose that D is not a strongly trail-connected digraph. By Theorem 1.2, D is a strong digraph. Therefore, D has a nontrivial strongly trail-connected subdigraph. Let C be a strongly trail-connected subdigraph in D with

$$|V(C)| \text{ is maximized among all strongly trail-connected subdigraph in } D. \quad (4)$$

For any H_i of $\mathcal{H}$ $(1 \leq i \leq c)$, if $V(H_i) \cap V(C) \neq \emptyset$, we can claim that $V(H_i) \subseteq V(C)$; otherwise, if $V(H_i) \nsubseteq V(C)$, by Lemma 2.2, $C \cup H_i$ is a strongly trail-connected subdigraph of D, contrary to (4). Since D is not strongly trail-connected, hence $|V(C)| < |V(D)|$ and there exists some H_i with $V(H_i) \cap V(C) = \emptyset$. Assume that $V(C) = V(H_1) \cup V(H_2) \cup \cdots \cup V(H_b)$ for $1 \leq b \leq c-1$. By Definition 2.2, we can construct the digraph D' with $V(D') = \{h_1, h_2, \ldots, h_b, h_{b+1}, \ldots, h_c\}$ and the digraph C' with $V(C') = \{h_1, h_2, \ldots, h_b\}$, where C' is obtained by contracting $H_1, H_2, \ldots, H_b$ as b vertices. By Lemma 2.1, D' is a strong digraph. Therefore, there exist two vertices $h_k, h_{k'} \in V(C')$ and a vertex $h_{i_1} \in V(D') - V(C')$ satisfying $(h_k, h_{i_1}) \in A(D')$. As D' is a strong digraph, hence D' has an $(h_{i_1}, h_{k'})$-path denoted by $P_{(h_{i_1}, h_{k'})} = h_{i_1} h_{i_2} \dots h_{i_p}$, where $h_{i_p} = h_{k'}$. Since $h_{i_1} \in V(D') - V(C')$ and $h_{k'} \in V(C')$, hence there exists q with $1 < q \leq p$, satisfying $h_{i_q} \in V(C')$ and $V(P_{(h_{i_1}, h_{k'})}[h_{i_1}, h_{i_{q-1}}]) \subseteq V(D') - V(C')$. Thus,

$$h_k h_{i_1} h_{i_2} \dots h_{i_{q-1}} h_{i_q} \tag{5}$$

is a path (or a cycle when $h_k = h_{i_q}$) of D' from h_k to h_{i_q} such that $h_k, h_{i_q} \in V(C')$ and $h_{i_1}, \dots, h_{i_{q-1}} \in V(D') - V(C')$. Next, we consider the subdigraph $D[V(C) \cup V(H_{i_1}) \cup V(H_{i_2}) \cup \cdots \cup V(H_{i_{q-1}})]$ of D. Digraphs $H_{i_1}, H_{i_2}, \dots H_{i_{q-1}}$ are symmetrically connected subdigraphs of D, and so they are strongly trail-connected by Theorem 2.1. By (4), C is strongly trail-connected. By Definition 2.2 and (5), for $1 \leq k \leq b$ there exists at least one vertex y_k^1 of H_k such that $N_D^+(y_k^1) \cap V(H_{i_1}) \neq \emptyset$ and $N_D^-(y_k^1) \cap V(H_{i_1}) \neq \emptyset$. Let $y_{i_1}^2 \in N_D^+(y_k^1) \cap V(H_{i_1})$ and $y_{i_1}^3 \in N_D^-(y_k^1) \cap V(H_{i_1})$. And there is at least one vertex $y_{i_{q-1}}^1$ of $H_{i_{q-1}}$ such that $N_D^+(y_{i_{q-1}}^1) \cap V(H_{i_q}) \neq \emptyset$ and $N_D^-(y_{i_{q-1}}^1) \cap V(H_{i_q}) \neq \emptyset$. Let $y_{i_q}^2 \in N_D^+(y_{i_{q-1}}^1) \cap V(H_{i_q})$ and $y_{i_q}^3 \in N_D^-(y_{i_{q-1}}^1) \cap V(H_{i_q})$. And for any s with $2 \leq s \leq q-1$, there exists at least one vertex $y_{i_{s-1}}^1$ of $H_{i_{s-1}}$ such that $N_D^+(y_{i_{s-1}}^1) \cap V(H_{i_s}) \neq \emptyset$ and $N_D^-(y_{i_{s-1}}^1) \cap V(H_{i_s}) \neq \emptyset$. Let $y_{i_s}^2 \in N_D^+(y_{i_{s-1}}^1) \cap V(H_{i_s})$ and $y_{i_s}^3 \in N_D^-(y_{i_{s-1}}^1) \cap V(H_{i_s})$. By analysis above, $D[V(C) \cup V(H_{i_1}) \cup V(H_{i_2}) \cup \cdots \cup V(H_{i_{q-1}})]$ has a spanning subdigraph which is isomorphic to the digraph defined as Definition 2.1. Thus, by Theorem 2.2, $D[V(C) \cup V(H_{i_1}) \cup V(H_{i_2}) \cup \cdots \cup V(H_{i_{q-1}})]$ is a strongly trail-connected subdigraph with more vertices than $V(C)$, contrary to (4). Hence D is strongly trail-connected. This proves the theorem. □

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