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Spanning Eulerian Subdigraphs in Jump Digraphs

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Abstract A jump digraph $J(D)$ of a directed multigraph D has as its vertex set being $A(D)$, the set of arcs of D; where (a, b) is an arc of $J(D)$ if and only if there are vertices u_1, v_1, u_2, v_2 in D such that $a = (u_1, v_1), b = (u_2, v_2)$ and $v_1 \neq u_2$. In this paper, we give a well characterized directed multigraph families $\mathcal{H}_1$ and $\mathcal{H}_2$, and prove that a jump digraph $J(D)$ of a directed multigraph D is strongly connected if and only if $D \notin \mathcal{H}_1$. Specially, $J(D)$ is weakly connected if and only if $D \notin \mathcal{H}_2$. The following results are obtained: (i) There exists a family $\mathcal{D}$ of well-characterized directed multigraphs such that strongly connected jump digraph $J(D)$ of directed multigraph is strongly trail-connected if and only if $D \notin \mathcal{D}$. (ii) Every strongly connected jump digraph $J(D)$ of directed multigraph D is weakly trail-connected, and so is supereulerian. (iii) Every weakly connected jump digraph $J(D)$ of directed multigraph D has a spanning trail.

Keywords supereulerian digraph; line digraph; jump digraph; weakly trail-connected; strongly trail-connected

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1. Introduction

A directed graph D consists of a non-empty finite set $V(D)$ of elements called vertices and a finite set $A(D)$ of ordered pairs of distinct vertices called arcs. Parallel arcs mean pairs of arcs with the same tail and the same head, loop is the arc whose head and tail coincide. A digraph implies that we allow a digraph to have arcs with the same end-vertices, but we do not allow it to contain parallel arcs or loops. When parallel arcs and loops are admissible we speak of directed pseudographs; directed pseudographs without loops are directed multigraphs. An empty graph is one with at least one vertex such that it does not have any arcs. For an arc $a = (x, y)$, the first vertex x is its tail denoted by $t(a)$ and the second vertex y is its head denoted by $h(a)$. Undefined terms and notation will follow [1] and [2].

Let $D = (V(D), A(D))$ be a directed multigraph. A walk in D is an alternating sequence

$$W = x_1 a_1 x_2 \cdots x_{k-1} a_{k-1} x_k \tag{1.1}$$

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with vertices x_i ($1 \le i \le k$) and arcs $a_j = (x_j, x_{j+1})$ ($1 \le j \le k-1$) from D. We say that W in (1.1) is a walk of D from vertex x_1 to vertex x_k, and we also say W is a walk of D from arc a_1 to arc a_{k-1}, denoted by (x_1, x_k)-walk and (a_1, a_{k-1})-walk, respectively. A trail is a walk in which all arcs are distinct. If $1 \le i < j \le k$, we define $W[x_i, x_j]$ to be the subtrail $x_i a_i x_{i+1} a_{i+1} x_{i+2} \cdots x_{j-1} a_{j-1} x_j$. A trail W is an euler (or eulerian) trail if $A(W) = A(D)$, $V(W) = V(D)$ and $x_1 = x_k$. If the vertices of W are distinct, W is a path. The length of path is the number of arcs of path, a k-path is a path of length k with $k+1$ vertices. If the vertices $x_1, x_2, \ldots, x_{k-1}$ are distinct, $k \ge 2$ and $x_1 = x_k$, W is a cycle. A k-cycle is a cycle of length k. A cycle W is a hamiltonian cycle of D if $V(W) = V(D)$. If W is a (v, w)-trail of D and $(u, v), (w, z) \in A(D) - A(W)$, then we use $(u, v)W(w, z)$ to denote the (u, z)-trail of $D[A(W) \cup \{(u, v), (w, z)\}]$. Subdigraphs $(u, v)W$ and $W(w, z)$ are similarly defined.

We often use $G(D)$ for the underlying graph of D, the graph obtained from D by erasing all orientation on the arcs of D. A directed multigraph D is strongly connected if for every pair x and y of distinct vertices in D, there exists an (x, y)-walk and a (y, x)-walk in D. D is weakly connected if $G(D)$ is connected. If $G(D)$ is not connected, then D is not connected. A directed multigraph D is eulerian if itself is an euler trail. D is hamiltonian if D contains a hamiltonian cycle. D is supereulerian if D contains a spanning eulerian subdigraph, or equivalently, a spanning closed trail. A directed multigraph D is weakly trail-connected if for any two vertices x and y of D, D admits a spanning (x, y)-trail or a spanning (y, x)-trail, and D is strongly trail-connected if for any two vertices x and y of D, D contains both a spanning (x, y)-trail and a spanning (y, x)-trail ($x = y$ is allowed).

When we consider the eulerian subdigraph problem, we will discuss whether the directed multigraph D has a spanning trail, a spanning closed trail, or for any vertices $x, y \in V(D)$, there exists a spanning (x, y)-trail. The supereulerian problem in digraphs was considered by Gutin [3]. In particular, Hong et al in [4] and Bang-Jensen and Maddaloni [5] presented several best possible sufficient degree conditions for supereulerian digraphs. Additional researches on supereulerian digraphs can be found in [6–9], among others. The weakly trail-connected and strongly trail-connected problem were considered recently in [10–12].

A line digraph $L(D)$ of a directed multigraph D has as its vertex set being $A(D)$, the set of arcs of D; where (a, b) is an arc of $L(D)$ if and only if there are vertices u, v, w in D such that $a = (u, v)$ and $b = (v, w)$ are in $A(D)$. A jump digraph $J(D)$ of a directed multigraph D has as its vertex set being $A(D)$, the set of arcs of D; where (a, b) is an arc of $J(D)$ if and only if there are vertices u_1, v_1, u_2, v_2 in D such that $a = (u_1, v_1), b = (u_2, v_2)$ and $v_1 \ne u_2$. A directed multigraph D with $|V(D)| = n$ and without parallel arcs is a complete digraph if for any two distinct vertices u and v of D, we have $(u, v), (v, u) \in A(D)$, denoted by K_n^*. An out-star is a directed multigraph where there exists a common tail u to all arcs and an in-star is a directed multigraph where there exists a common head v to all arcs. The complement $\overline{D}$ of a digraph D is the digraph with vertex set $V(D)$ in which there are two vertices u, v such that $(u, v) \in A(\overline{D})$

if and only if $(u,v) \notin A(D)$. We observe that complete digraph K_n^* has the following property:

$$\text{For any } u, v \in V(K_n^*),\ K_n^* \text{ has a spanning } (u,v)\text{-trail.} \tag{1.2}$$

For subsets $X, Y \subseteq V(D)$, define

$$(X,Y)_D = \{(x,y) \in A(D) : x \in X, y \in Y\} \text{ and } [X,Y]_D = (X,Y)_{G(D)} = (X,Y)_D \cup (Y,X)_D.$$

If $X = \{x\}$ or $Y = \{y\}$, we often use $(x,Y)_D$ for $(X,Y)_D$ or $(X,y)_D$ for $(X,Y)_D$, respectively. Hence, $(x,y)_D = (\{x\},\{y\})_D$. For a vertex $v \in V(D)$, let $\partial_D^+(v) = (v, V(D)-v)_D$ and $\partial_D^-(v) = (V(D)-v, v)_D$. Thus $d_D^+(v) = |\partial_D^+(v)|$ and $d_D^-(v) = |\partial_D^-(v)|$. If $B \subseteq A(D)$, then $D[B]$ is the subdigraph arc-induced by B of D with vertex set which are incident with at least one arc from B and arc set B. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph vertex-induced by X with vertex set X and arc set, both end-vertices of which are in X.

Let M and M' be two directed multigraphs. Throughout this paper, define $M \cup M'$ to be the directed multigraph with $V(M \cup M') = V(M) \cup V(M')$ and $A(M \cup M') = A(M) \cup A(M')$.

For jump graph problem, Wu and Meng [13] and Liu [14] discussed the hamiltonian and pancyclic jump graph. Clique-transversal sets, clique-perfectness and planarity of jump graph were considered in [15–17], but for jump digraph, there are few results. In this paper, we will discuss the spanning eulerian subdigraph of jump digraph $J(D)$ of a directed multigraph D. In Section 2, we present a well characterized directed multigraph families $\mathcal{H}_1$ and $\mathcal{H}_2$, and prove that a jump digraph $J(D)$ of a directed multigraph D is strongly connected if and only if $D \notin \mathcal{H}_1$. Specially, $J(D)$ is weakly connected if and only if $D \notin \mathcal{H}_2$. In Section 3, we discuss the weakly trail-connected and strongly trail-connected jump digraph of directed multigraph.

2. Strongly (weakly) connected jump digraphs

By the definitions of line digraph and jump digraph, we can obtain that $J(D)$ is a complement digraph of $L(D)$, denoted by $J(D) = \overline{L(D)}$. We first state two useful results. Beineke [18] characterized line digraph as follows:

Theorem 2.1 ([18]) *Let H be a line digraph. If a_1, a_2 and a_3 are any three arcs in H such that $h(a_1) = h(a_2)$ and $t(a_2) = t(a_3)$, then there exists an arc a_4 in H such that $t(a_4) = t(a_1)$ and $h(a_4) = h(a_3)$.*

Lemma 2.2 *Let D be a directed multigraph, $L(D)$ be the line digraph of D. And let $a, b, c \in V(L(D))$. Then each of the following holds.*

(i) If $(a,b), (a,c) \in A(L(D))$, then $(b,c), (c,b) \notin A(L(D))$.

(ii) If $(b,a), (c,a) \in A(L(D))$, then $(b,c), (c,b) \notin A(L(D))$.

(iii) If $(a,b), (c,a) \in A(L(D))$, then $(c,b) \notin A(L(D))$.

Proof Let $a = (u_1, v_1), b = (u_2, v_2), c = (u_3, v_3) \in A(D) = V(L(D))$. By the definition of line digraph of a directed multigraph, $L(D)$ is a digraph, and so

$$L(D) \text{ does not contain loop.} \tag{2.1}$$

If $(a,b),(a,c) \in A(L(D))$, by contradiction, assume first that $(b,c) \in A(L(D))$. Then by the definition of line digraph, we may assume that $v_1 = u_2, v_1 = u_3$ and $v_2 = u_3$. Hence $v_1 = v_2 = u_2 = u_3$, and so b is a loop of $L(D)$, contrary to (2.1). Likewise, if $(c,b) \in A(L(D))$, then a contradiction will be obtained similarly. Hence $(b,c),(c,b) \notin A(L(D))$. This proves (i).

If $(b,a),(c,a) \in A(L(D))$, by contradiction, assume first that $(b,c) \in A(L(D))$. Then by the definition of line digraph, we may assume that $v_2 = u_1, v_3 = u_1$ and $v_2 = u_3$. Hence $v_2 = v_3 = u_1 = u_3$, and so c is a loop of $L(D)$, contrary to (2.1). Likewise, if $(c,b) \in A(L(D))$, then a contradiction will be obtained similarly. Hence $(b,c),(c,b) \notin A(L(D))$. This proves (ii).

If $(a,b),(c,a) \in A(L(D))$, by contradiction, assume that $(c,b) \in A(L(D))$. Then by the definition of line digraph, we may assume that $v_1 = u_2, v_3 = u_1$ and $v_3 = u_2$. Hence $v_1 = v_3 = u_1 = u_2$, and so a is a loop of $L(D)$, contrary to (2.1). Hence $(c,b) \notin A(L(D))$. This proves (iii). This completes the proof of Lemma 2.2. □

Since $J(D) = \overline{L(D)}$, it is routine to obtain the following corollary.

Corollary 2.3 *Let D be a directed multigraph, $L(D)$ and $J(D)$ be the line digraph and the jump digraph of D, respectively. Let $a,b,c \in V(J(D))$. Then each of the following holds.*

(i) If $(a,b),(a,c) \notin A(J(D))$, then $(b,c),(c,b) \in A(J(D))$.

(ii) If $(b,a),(c,a) \notin A(J(D))$, then $(b,c),(c,b) \in A(J(D))$.

(iii) If $(a,b),(c,a) \notin A(J(D))$, then $(c,b) \in A(J(D))$.

The rest of this section is devoted to the characterization of strongly (weakly) connected jump digraph. We start with an example.

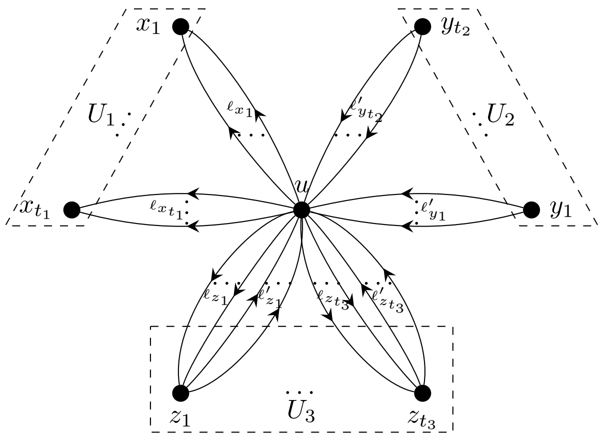

Figure 1 The digraph family $\mathcal{H}$.

Example 2.4 Let t_1, t_2 and t_3 be three nonnegative integers. Let $\{u\}, U_1 = \{x_1, x_2, \ldots, x_{t_1}\}$, $U_2 = \{y_1, y_2, \ldots, y_{t_2}\}$ and $U_3 = \{z_1, z_2, \ldots, z_{t_3}\}$ be mutually disjoint vertex sets with $t_1+t_2+t_3 \geq 1$. Let $\ell_{x_i}, \ell'_{y_j}, \ell_{z_k}$ and ℓ'_{z_k} with $1 \leq i \leq t_1, 1 \leq j \leq t_2$ and $1 \leq k \leq t_3$ be nonnegative integers. We construct a directed multigraph family $\mathcal{H} = \mathcal{H}(t_1,t_2,t_3)$ such that a directed multigraph $D = D(t_1,t_2,t_3) \in \mathcal{H}$ if and only if $V(D) = \{u\} \cup U_1 \cup U_2 \cup U_3$ and $A(D)$ consists of exactly the arcs described in (H1)–(H4) below. (See Figure 1 for an illustration.)

(H1) $D[U_1]$, $D[U_2]$ and $D[U_3]$ are empty graphs.

(H2) For any $x_i \in U_1$, $y_j \in U_2$ and $z_k \in U_3$, $(u, x_i), (y_j, u), (u, z_k), (z_k, u) \in A(D)$.

(H3) For any $x_i \in U_1$, $y_j \in U_2$ and $z_k \in U_3$, $|(u, x_i)_D| = \ell_{x_i}, |(y_j, u)_D| = \ell'_{y_j}, |(u, z_k)_D| = \ell_{z_k}$ and $|(z_k, u)_D| = \ell'_{z_k}$.

(H4) $(u, U_2)_D = \emptyset$, $(U_1, u)_D = \emptyset$, $[U_1, U_2]_D = \emptyset$, $[U_1, U_3]_D = \emptyset$ and $[U_2, U_3]_D = \emptyset$.

Proposition 2.5 *Let $D \in \mathcal{H}$ be a directed multigraph defined as in Example 2.4, and let $L(D)$ and $J(D)$ be the line digraph and the jump digraph of D, respectively, and let $B^+ = \partial^+_D(u)$ and $B^- = \partial^-_D(u)$. With the notation used in Example 2.4, each of the following holds.*

(i) $|B^+| = \sum_{i=1}^{t_1} \ell_{x_i} + \sum_{k=1}^{t_3} \ell_{z_k}$ *and* $|B^-| = \sum_{j=1}^{t_2} \ell'_{y_j} + \sum_{k=1}^{t_3} \ell'_{z_k}$.

(ii) $A(D) = B^+ \cup B^- = V(L(D)) = V(J(D))$.

(iii) $D[B^+]$ *is an out-star and* $D[B^-]$ *is an in-star,* $L(D)[B^+]$ *and* $L(D)[B^-]$ *are empty graphs, and so* $J(D)[B^+]$ *and* $J(D)[B^-]$ *are complete digraphs.*

(iv) *For any* $a \in B^-$ *and any* $b \in B^+$, $(a, b) \in A(L(D))$ *and* $(B^-, B^+)_{J(D)} = \emptyset$.

(v) $|(B^+, B^-)_{L(D)}| = \sum_{k=1}^{t_3} \ell_{z_k}\ell'_{z_k}$ *and* $|(B^+, B^-)_{J(D)}| = (\sum_{i=1}^{t_1} \ell_{x_i} + \sum_{k=1}^{t_3} \ell_{z_k})(\sum_{j=1}^{t_2} \ell'_{y_j} + \sum_{k=1}^{t_3} \ell'_{z_k}) - \sum_{k=1}^{t_3} \ell_{z_k}\ell'_{z_k}$.

(vi) *If* $B^- = \emptyset$ *or* $B^+ = \emptyset$, *then* $J(D)$ *is a complete digraph, and so* $J(D)$ *is strongly connected.*

(vii) *If* $B^- \neq \emptyset$ *and* $B^+ \neq \emptyset$, *then* $J(D)$ *is not strongly connected. Specially,* $J(D)$ *is weakly connected if and only if* $t_1 \neq 0$ *or* $t_2 \neq 0$ *or* $t_3 \neq 1$.

Proof By $D \in \mathcal{H}$ and Example 2.4, $\partial^+_D(u) = (u, U_1)_D \cup (u, U_3)_D$ and $\partial^-_D(u) = (U_2, u)_D \cup (U_3, u)_D$, hence $|B^+| = |(u, U_1)_D \cup (u, U_3)_D| = \sum_{i=1}^{t_1} \ell_{x_i} + \sum_{k=1}^{t_3} \ell_{z_k}$ and $|B^-| = |(U_2, u)_D \cup (U_3, u)_D| = \sum_{j=1}^{t_2} \ell'_{y_j} + \sum_{k=1}^{t_3} \ell'_{z_k}$. Since $V(D) = \{u\} \cup U_1 \cup U_2 \cup U_3$, and by Example 2.4, we have $A(D) = B^+ \cup B^- = V(L(D)) = V(J(D))$. Thus (i) and (ii) hold.

Since $B^+ = \partial^+_D(u)$ and $B^- = \partial^-_D(u)$, it follows that $D[B^+]$ is an out-star and $D[B^-]$ is an in-star, and so $L(D)[B^+]$ and $L(D)[B^-]$ are empty graphs. Since $J(D) = \overline{L(D)}$, it follows that $J(D)[B^+]$ and $J(D)[B^-]$ are complete digraphs. Thus (iii) holds.

For any $a \in B^-$ and any $b \in B^+$, let $a = (x, u)$ and $b = (u, y)$ with $x \in U_2 \cup U_3$ and $y \in U_1 \cup U_3$. Then $(a, b) \in A(L(D))$. Since $J(D) = \overline{L(D)}$, we have $(a, b) \notin A(J(D))$, and so $(B^-, B^+)_{J(D)} = \emptyset$. Thus (iv) holds.

Since $(u, U_3)_D \subset B^+$ and $(U_3, u)_D \subset B^-$, we have $|(B^+, B^-)_{L(D)}| = \sum_{k=1}^{t_3} \ell_{z_k}\ell'_{z_k}$. Since $J(D) = \overline{L(D)}$, we have $|(B^+, B^-)_{J(D)}| = (\sum_{i=1}^{t_1} \ell_{x_i} + \sum_{k=1}^{t_3} \ell_{z_k})(\sum_{j=1}^{t_2} \ell'_{y_j} + \sum_{k=1}^{t_3} \ell'_{z_k}) - \sum_{k=1}^{t_3} \ell_{z_k}\ell'_{z_k}$. Thus (v) holds.

If $B^- = \emptyset$, then $t_2 = 0$ and $t_3 = 0$. Since $D \in \mathcal{H}$ and $t_1 + t_2 + t_3 \geq 1$, we have $t_1 \geq 1, B^+ \neq \emptyset$ and $V(D) = \{u\} \cup U_1$. By (ii), we have $A(D) = B^+$, and so $D = D[B^+]$. By (iii), $J(D) = J(D[B^+]) = J(D)[B^+]$ is a complete digraph, and so $J(D)$ is strongly connected. If $B^+ = \emptyset$, then $t_1 = 0$ and $t_3 = 0$. Since $D \in \mathcal{H}$ and $t_1 + t_2 + t_3 \geq 1$, it follows that $t_2 \geq 1, B^- \neq \emptyset$ and $V(D) = \{u\} \cup U_2$. By (ii), we have $A(D) = B^-$, and so $D = D[B^-]$. By (iii), $J(D) = J(D[B^-]) = J(D)[B^-]$ is a complete digraph, and so $J(D)$ is strongly connected. Thus (vi) holds.

If $B^- \neq \emptyset$ and $B^+ \neq \emptyset$, then by (iv), $(B^-, B^+)_{J(D)} = \emptyset$, and so $J(D)$ is not strongly connected. Specially, if $t_1 = 0, t_2 = 0$ and $t_3 = 1$, then $\sum_{i=1}^{t_1} \ell_{x_i} = 0, \sum_{j=1}^{t_2} \ell'_{y_j} = 0, \ell_{z_1} \geq 1$ and $\ell'_{z_1} \geq 1$. By (v),

$$\begin{aligned}|(B^+, B^-)_{J(D)}| =& \Big(\sum_{i=1}^{t_1} \ell_{x_i} + \sum_{k=1}^{t_3} \ell_{z_k}\Big)\Big(\sum_{j=1}^{t_2} \ell'_{y_j} + \sum_{k=1}^{t_3} \ell'_{z_k}\Big) - \sum_{k=1}^{t_3} \ell_{z_k}\ell'_{z_k} \\ =& \sum_{k=1}^{t_3} \ell_{z_k} \sum_{k=1}^{t_3} \ell'_{z_k} - \sum_{k=1}^{t_3} \ell_{z_k}\ell'_{z_k} \\ =& \ell_{z_1}\ell'_{z_1} - \ell_{z_1}\ell'_{z_1} = 0.\end{aligned}$$

Hence $(B^+, B^-)_{J(D)} = \emptyset$. By (iv), and so $J(D)$ is not connected.

Next, we will assume that $t_1 \neq 0$ or $t_2 \neq 0$ or $t_3 \neq 1$ to prove that $J(D)$ is weakly connected. If $t_1 \geq 1$, then $\sum_{i=1}^{t_1} \ell_{x_i} \geq 1$. Since $B^- \neq \emptyset$, by (i), $\sum_{j=1}^{t_2} \ell'_{y_j} + \sum_{k=1}^{t_3} \ell'_{z_k} \geq 1$. If $\sum_{k=1}^{t_3} \ell'_{z_k} = 0$, then $\sum_{j=1}^{t_2} \ell'_{y_j} \geq 1$ and $\sum_{k=1}^{t_3} \ell_{z_k} = 0$. By (v),

$$|(B^+, B^-)_{J(D)}| = \Big(\sum_{i=1}^{t_1} \ell_{x_i} + \sum_{k=1}^{t_3} \ell_{z_k}\Big)\Big(\sum_{j=1}^{t_2} \ell'_{y_j} + \sum_{k=1}^{t_3} \ell'_{z_k}\Big) - \sum_{k=1}^{t_3} \ell_{z_k}\ell'_{z_k} = \sum_{i=1}^{t_1} \ell_{x_i} \sum_{j=1}^{t_2} \ell'_{y_j} \geq 1,$$

and so $(B^+, B^-)_{J(D)} \neq \emptyset$. By (iii), $J(D)$ is weakly connected. If $\sum_{k=1}^{t_3} \ell'_{z_k} \geq 1$, then $\sum_{k=1}^{t_3} \ell_{z_k} \geq 1$. Since $\sum_{k=1}^{t_3} \ell_{z_k} \sum_{k=1}^{t_3} \ell'_{z_k} \geq \sum_{k=1}^{t_3} \ell_{z_k}\ell'_{z_k}$, by (v),

$$\begin{aligned}|(B^+, B^-)_{J(D)}| =& \Big(\sum_{i=1}^{t_1} \ell_{x_i} + \sum_{k=1}^{t_3} \ell_{z_k}\Big)\Big(\sum_{j=1}^{t_2} \ell'_{y_j} + \sum_{k=1}^{t_3} \ell'_{z_k}\Big) - \sum_{k=1}^{t_3} \ell_{z_k}\ell'_{z_k} \\ \geq& \Big(\sum_{i=1}^{t_1} \ell_{x_i} + \sum_{k=1}^{t_3} \ell_{z_k}\Big) \sum_{k=1}^{t_3} \ell'_{z_k} - \sum_{k=1}^{t_3} \ell_{z_k} \sum_{k=1}^{t_3} \ell'_{z_k} \\ \geq& \sum_{i=1}^{t_1} \ell_{x_i} \sum_{k=1}^{t_3} \ell'_{z_k} \geq 1.\end{aligned}$$

Thus $(B^+, B^-)_{J(D)} \neq \emptyset$. By (iii), $J(D)$ is weakly connected.

If $t_2 \geq 1$, then $\sum_{j=1}^{t_2} \ell'_{y_j} \geq 1$. Since $B^+ \neq \emptyset$, by (i), $\sum_{i=1}^{t_1} \ell_{x_i} + \sum_{k=1}^{t_3} \ell_{z_k} \geq 1$. If $\sum_{k=1}^{t_3} \ell_{z_k} = 0$, then $\sum_{i=1}^{t_1} \ell_{x_i} \geq 1$ and $\sum_{k=1}^{t_3} \ell'_{z_k} = 0$. By (v),

$$|(B^+, B^-)_{J(D)}| = \Big(\sum_{i=1}^{t_1} \ell_{x_i} + \sum_{k=1}^{t_3} \ell_{z_k}\Big)\Big(\sum_{j=1}^{t_2} \ell'_{y_j} + \sum_{k=1}^{t_3} \ell'_{z_k}\Big) - \sum_{k=1}^{t_3} \ell_{z_k}\ell'_{z_k} = \sum_{i=1}^{t_1} \ell_{x_i} \sum_{j=1}^{t_2} \ell'_{y_j} \geq 1,$$

and so $(B^+, B^-)_{J(D)} \neq \emptyset$. By (iii), $J(D)$ is weakly connected. If $\sum_{k=1}^{t_3} \ell_{z_k} \geq 1$, then $\sum_{k=1}^{t_3} \ell'_{z_k} \geq 1$. Since $\sum_{k=1}^{t_3} \ell_{z_k} \sum_{k=1}^{t_3} \ell'_{z_k} \geq \sum_{k=1}^{t_3} \ell_{z_k}\ell'_{z_k}$, it follows by (v),

$$\begin{aligned}|(B^+, B^-)_{J(D)}| =& \Big(\sum_{i=1}^{t_1} \ell_{x_i} + \sum_{k=1}^{t_3} \ell_{z_k}\Big)\Big(\sum_{j=1}^{t_2} \ell'_{y_j} + \sum_{k=1}^{t_3} \ell'_{z_k}\Big) - \sum_{k=1}^{t_3} \ell_{z_k}\ell'_{z_k} \\ \geq& \sum_{k=1}^{t_3} \ell_{z_k} \Big(\sum_{j=1}^{t_2} \ell'_{y_j} + \sum_{k=1}^{t_3} \ell'_{z_k}\Big) - \sum_{k=1}^{t_3} \ell_{z_k} \sum_{k=1}^{t_3} \ell'_{z_k}\end{aligned}$$

$$\geq \sum_{k=1}^{t_3} \ell_{z_k} \sum_{j=1}^{t_2} \ell'_{y_j} \geq 1,$$

and so $(B^+, B^-)_{J(D)} \neq \emptyset$. By (iii), $J(D)$ is weakly connected.

Assume that $t_3 = 0$. Since $B^+ \neq \emptyset$ and $B^- \neq \emptyset$, by (i), $\sum_{i=1}^{t_1} \ell_{x_i} \geq 1$ and $\sum_{j=1}^{t_2} \ell'_{y_j} \geq 1$, and by (v),

$$|(B^+, B^-)_{J(D)}| = \sum_{i=1}^{t_1} \ell_{x_i} \sum_{j=1}^{t_2} \ell'_{y_j} \geq 1.$$

Thus $(B^+, B^-)_{J(D)} \neq \emptyset$. By (iii), $J(D)$ is weakly connected.

If $t_3 \geq 2$, then $\sum_{k=1}^{t_3} \ell_{z_k} \sum_{k=1}^{t_3} \ell'_{z_k} > \sum_{k=1}^{t_3} \ell_{z_k}\ell'_{z_k}$ and by (v),

$$|(B^+, B^-)_{J(D)}| = \Big(\sum_{i=1}^{t_1} \ell_{x_i} + \sum_{k=1}^{t_3} \ell_{z_k}\Big)\Big(\sum_{j=1}^{t_2} \ell'_{y_j} + \sum_{k=1}^{t_3} \ell'_{z_k}\Big) - \sum_{k=1}^{t_3} \ell_{z_k}\ell'_{z_k}$$
$$\geq \sum_{k=1}^{t_3} \ell_{z_k} \sum_{k=1}^{t_3} \ell'_{z_k} - \sum_{k=1}^{t_3} \ell_{z_k}\ell'_{z_k} \geq 1.$$

It follows $(B^+, B^-)_{J(D)} \neq \emptyset$. By (iii), and so $J(D)$ is weakly connected. Thus (vii) holds. □

Definition 2.6 *Let $\mathcal{H}$ be the directed multigraph family as defined in Example 2.4. Define two directed multigraph families as follows. Let $\mathcal{H}_1 = \{D(t_1, t_2, t_3) \in \mathcal{H} :$ with $t_1 + t_3 \geq 1$ and $t_2 + t_3 \geq 1\}$ and let $\mathcal{H}_2 = \mathcal{H}(0, 0, 1)$.*

Theorem 2.7 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two directed multigraph families as defined in Definition 2.6 and $J(D)$ be a jump digraph of directed multigraph D. Then $J(D)$ is strongly connected if and only if $D \notin \mathcal{H}_1$, and $J(D)$ is weakly connected if and only if $D \notin \mathcal{H}_2$.*

Proof If $D \in \mathcal{H}_1$, then by Proposition 2.5 (i) and Example 2.4, $|B^+| = \sum_{i=1}^{t_1} \ell_{x_i} + \sum_{k=1}^{t_3} \ell_{z_k} \geq t_1 + t_3 \geq 1$ and $|B^-| = \sum_{j=1}^{t_2} \ell'_{y_j} + \sum_{k=1}^{t_3} \ell'_{z_k} \geq t_2 + t_3 \geq 1$, by Proposition 2.5 (vii), $J(D)$ is not strongly connected, a contradiction. Hence assume that $J(D)$ is not strongly connected, we want to prove that $D \in \mathcal{H}_1$. Since $J(D)$ is not strongly connected, it follows that $V(J(D))$ can be partitioned into two nonempty vertex-disjoint subsets B_1 and B_2 such that $(B_2, B_1)_{J(D)} = \emptyset$. Since $J(D) = \overline{L(D)}$, we have

$$\text{for any } a \in B_1 \text{ and } b \in B_2,\ (b, a) \in A(L(D)). \tag{2.2}$$

By Lemma 2.2 (i) and (ii), $L(D)[B_1]$ and $L(D)[B_2]$ are empty graphs. Let $B_1 = \{a_1, a_2, \ldots, a_{k_1}\}$ and $B_2 = \{b_1, b_2, \ldots, b_{k_2}\}$, by (2.2), for any $a_i \in B_1$ and $b_j \in B_2$, we can let $a_i = (u, x_i)$ and $b_j = (y_j, u)$. As $B_1 \cup B_2 = A(D)$. By (2.2), we can obtain that $B_1 = \partial_D^+(u) \neq \emptyset$ and $B_2 = \partial_D^-(u) \neq \emptyset$, and so $D \in \mathcal{H}_1$, a contradiction. Hence $J(D)$ is strongly connected if and only if $D \notin \mathcal{H}_1$.

Since $J(D)$ is strongly connected if and only if $D \notin \mathcal{H}_1$, it follows that $J(D)$ is not strongly connected if and only if $D \in \mathcal{H}_1$. Specially, by Proposition 2.5 (vii), $J(D)$ is weakly connected if and only if $t_1 \neq 0$ or $t_2 \neq 0$ or $t_3 \neq 1$, and so $J(D)$ is weakly connected if and only if $D \notin \mathcal{H}_2$. This completes the proof of the theorem. □

3. Spanning eulerian subdigraphs in jump digraphs

In this section, we will identify a directed multigraph family $\mathcal{D}$, and use it to prove our main results. We start with a definition.

Definition 3.1 *Let $\ell \geq 1$ be a positive integer, and let $U = \{x, y, z_1, z_2\}$ be a set of vertices, $a = (z_1, x)$ and $b = (y, z_2)$ ($z_1 = z_2$ is allowed) and let $C_\ell = \{c_i = (x, y) : 1 \leq i \leq \ell\}$ be a set of ℓ parallel arcs. Define a directed multigraph $D(\ell)$ with $V(D(\ell)) = U$ and $A(D(\ell)) = \{a, b\} \cup C_\ell$, and $\mathcal{D}$ to be a family of directed multigraphs by $\mathcal{D} = \{D(\ell) : \ell \geq 1\}$. (See Figures 2 and 3 for illustrations.)*

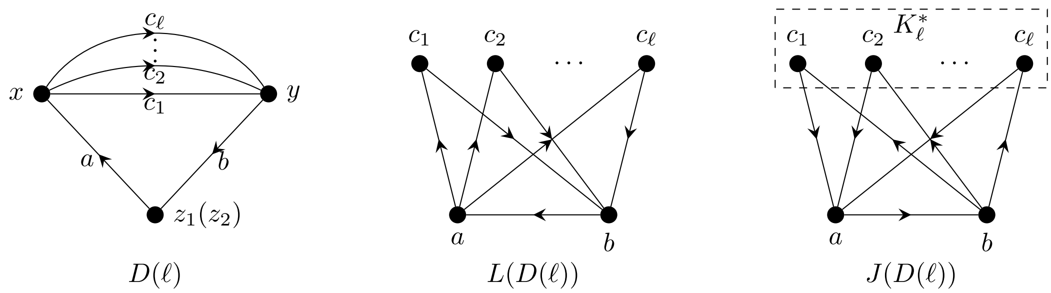

Figure 2 $z_1 = z_2$

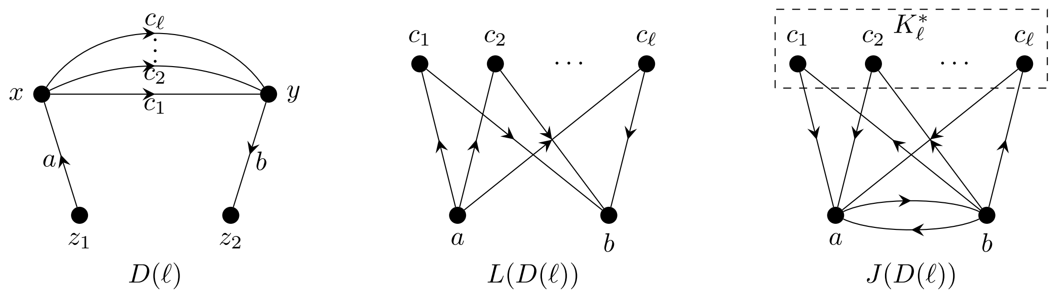

Figure 3 $z_1 \neq z_2$

By Definition 3.1 and by the definitions of line digraph and jump digraph, we have the following.

The line digraph $L(D(\ell))$ of $D(\ell)$ is a digraph with

$$V(L(D(\ell))) = \{a, b, c_1, c_2, \ldots, c_\ell\},$$

$$A(L(D(\ell))) = \begin{cases} \{(b, a)\} \cup \{(a, c_i), (c_i, b) : 1 \leq i \leq \ell\}, & \text{if } z_1 = z_2, \\ \{(a, c_i), (c_i, b) : 1 \leq i \leq \ell\}, & \text{if } z_1 \neq z_2. \end{cases}$$

The jump digraph $J(D(\ell))$ of $D(\ell)$ is a digraph with

$$V(J(D(\ell))) = \{a, b, c_1, c_2, \ldots, c_\ell\},$$

$$A(J(D(\ell))) = \begin{cases} \{(a,b)\} \cup \{(c_i,a),(b,c_i) : 1 \le i \le \ell\} \cup \{(c_i,c_j),(c_j,c_i) : 1 \le i \ne j \le \ell\}, \\ \qquad \text{if } z_1 = z_2, \\ \{(a,b),(b,a)\} \cup \{(c_i,a),(b,c_i) : 1 \le i \le \ell\} \cup \{(c_i,c_j),(c_j,c_i) : 1 \le i \ne j \le \ell\}, \\ \qquad \text{if } z_1 \ne z_2. \end{cases}$$

By the definition of $\mathcal{D}$, we have the following proposition.

Proposition 3.2 *Let $D(\ell) \in \mathcal{D}$ be a directed multigraph as defined in Definition 3.1 and let $\mathcal{H}_1$ be a directed multigraph family as defined in Definition 2.6, and let $L(D(\ell))$ and $J(D(\ell))$ be the line digraph and the jump digraph of $D(\ell)$, respectively. Then each of the following holds.*

(i) $D(\ell) \notin \mathcal{H}_1$, and so $J(D(\ell))$ is strongly connected.

(ii) $J(D(\ell))$ is weakly trail-connected.

(iii) $J(D(\ell))$ is not strongly trail-connected.

(iv) If there exists a directed multigraph H such that $J(H) \cong J(D(\ell))$, then $H \cong D(\ell)$.

Proof Let $\mathcal{H}$ be a directed multigraph family as defined in Example 2.4. By the definitions of $D(\ell)$ and $\mathcal{H}$, we have $D(\ell) \notin \mathcal{H}$ and $\mathcal{H}_1 \subset \mathcal{H}$. Hence $D(\ell) \notin \mathcal{H}_1$. By Theorem 2.7, $J(D(\ell))$ is strongly connected. Hence (i) holds.

Since $C_\ell = \{c_1, c_2, \ldots, c_\ell\}$ and $J(D(\ell))[C_\ell]$ is a complete digraph, by (1.2), $J(D(\ell))[C_\ell]$ contains a spanning closed trail T_1 and a spanning (c_1,c_2)-trail $T_{(c_1,c_2)}$. For any two vertices $a_1, a_2 \in V(J(D(\ell)))$, we want to prove that $J(D(\ell))$ has either a spanning (a_1,a_2)-trail or a spanning (a_2,a_1)-trail.

Suppose that $a_1 = a_2$. Since $a, b \notin C_\ell$, we have $(a,b) \notin A(J(D(\ell))[C_\ell])$. Thus

$$(a,b)(b,c_1)T_{(c_1,c_2)}(c_2,a)$$

is a spanning closed trail of $J(D(\ell))$.

Assume now that $a_1 \ne a_2$. If $\{a_1,a_2\} = \{a,b\}$, then $(a_1,c_1)T_{(c_1,c_2)}(c_2,a_2)$ is a spanning (a_1,a_2)-trail of $J(D(\ell))$ when $a_1 = b$ and $a_2 = a$, and $(a_2,c_1)T_{(c_1,c_2)}(c_2,a_1)$ is a spanning (a_2,a_1)-trail of $J(D(\ell))$ when $a_1 = a$ and $a_2 = b$. If $a_1, a_2 \in C_\ell$, then $(a_2,a)(a,b)(b,a_1) \cup T_1$ is a spanning (a_2,a_1)-trail of $J(D(\ell))$. If $a_1 = a$ and $a_2 \in C_\ell$, then $(a_1,b)(b,a_2) \cup T_1$ is a spanning (a_1,a_2)-trail of $J(D(\ell))$. Likewise, if $a_1 \in C_\ell$ and $a_2 = a$, then $J(D(\ell))$ has a spanning (a_2,a_1)-trail. If $a_1 = b$ and $a_2 \in C_\ell$, then $T_1 \cup (a_2,a)(a,a_1)$ is a spanning (a_2,a_1)-trail of $J(D(\ell))$. Likewise, if $a_1 \in C_\ell$ and $a_2 = b$, then $J(D(\ell))$ has a spanning (a_1,a_2)-trail. Hence, $J(D(\ell))$ is weakly trail-connected. Thus (ii) holds.

Since $J(D(\ell))$ does not contain spanning (a,b)-trail, it follows that $J(D(\ell))$ is not strongly trail-connected. Thus (iii) holds.

If there exists a directed multigraph H such that $J(H) \cong J(D(\ell))$, then $L(H) \cong L(D(\ell))$. Let $V(L(H)) = \{a', b', c'_1, c'_2, \ldots, c'_\ell\}$. As $L(H) \cong L(D(\ell))$, we have

$$A(L(H)) = \begin{cases} \{(b',a')\} \cup \{(a',c'_i),(c'_i,b') : 1 \le i \le \ell\}, & \text{if } h(b') = t(a'), \\ \{(a',c'_i),(c'_i,b') : 1 \le i \le \ell\}, & \text{if } h(b') \ne t(a'). \end{cases}$$

By the definition of line digraph, we may assume that $a' = (z'_1, x')$ and $b' = (y', z'_2)(z'_1 = z'_2$

is allowed) and denote $C'_\ell = \{c'_i = (x', y') : 1 \le i \le \ell\}$. Hence $V(H) = \{z'_1, z'_2, x', y'\}$ and $A(H) = V(L(H)) = \{a', b', c'_1, c'_2, \ldots, c'_\ell\}$. Thus $H \cong D(\ell)$, and so (iv) holds. This completes the proof of Proposition 3.2. □

Theorem 3.3 *Let $J(D)$ be a jump digraph of directed multigraph D and $J(D)$ be strongly connected, and let $\mathcal{D}$ be a directed multigraph family as defined in Definition 3.1. Then $J(D)$ is strongly trail-connected if and only if $D \notin \mathcal{D}$.*

Proof If $D \in \mathcal{D}$, then by Proposition 3.2 (iii), $J(D)$ is not strongly trail-connected. Hence we may assume that $D \notin \mathcal{D}$ to prove that $J(D)$ is strongly trail-connected.

We argue by contradiction and assume that there exist two vertices $a, b \in V(J(D))$ such that $J(D)$ does not have a spanning (a, b)-trail. Let

$$T = a_1a_2 \cdots a_t \text{ is an } (a, b)\text{-trail of } J(D) \text{ with } |V(T)| \text{ is maximum, where } a_1 = a \text{ and } a_t = b. \tag{3.1}$$

Since no spanning (a, b)-trail exists in $J(D)$, we have $V(J(D)) - V(T) \neq \emptyset$.

If there exist a vertex $c \in V(J(D)) - V(T)$ and a vertex $a_i \in V(T)$ such that $(c, a_i), (a_i, c) \in A(J(D))$, then

$$T'' = T \cup \{(c, a_i), (a_i, c)\}$$

is an (a, b)-trail of $J(D)$ with $|V(T'')| > |V(T)|$, contrary to (3.1). Hence assume that for any vertex $c \in V(J(D)) - V(T)$ and any vertex $a_i \in V(T)$,

$$|[c, a_i]_{J(D)}| \le 1. \tag{3.2}$$

Next, we consider two cases in the following.

Case 1. There exists a vertex $c \in V(J(D)) - V(T)$ such that

$$(c, V(T))_{J(D)} = \emptyset \text{ or } (V(T), c)_{J(D)} = \emptyset.$$

Assume first that $(c, V(T))_{J(D)} = \emptyset$, by Corollary 2.3 (i),

$$\text{for any two distinct vertices } a_i, a_j \in V(T), (a_i, a_j), (a_j, a_i) \in A(J(D)). \tag{3.3}$$

Since $J(D)$ is strongly connected, there exist two vertices $c', c'' \in V(J(D)) - V(T)$ such that $(V(T), c')_{J(D)} \neq \emptyset$ and $(c'', V(T))_{J(D)} \neq \emptyset$. Since $J(D)$ is strongly connected, it follows that $J(D)$ has a (c', c'')-trail $T_1 = c_1c_2 \cdots c_q$, where $c_1 = c'$ and $c_q = c''$. If $V(T_1) \cap V(T) = \emptyset$, then let $T' := T_1$; if $V(T_1) \cap V(T) \neq \emptyset$, then let q_1 be a minimum integer such that $c_{q_1} \in V(T_1) - V(T)$ and $(c_{q_1}, V(T))_{J(D)} \neq \emptyset$, hence let $T' := T_1[c_1, c_{q_1}]$ and $c'' := c_{q_1}$. Thus $J(D)$ has a (c', c'')-trail T' satisfying $V(T') \cap V(T) = \emptyset$. Let $a_{t_1}, a_{t_2} \in V(T)$ such that $(a_{t_1}, c'), (c'', a_{t_2}) \in A(J(D))$. Next, we consider (c'', a_{t_1}), if $(c'', a_{t_1}) \in A(J(D))$, then

$$T'' = T \cup \{(a_{t_1}, c'), (c'', a_{t_1})\} \cup T'$$

is an (a, b)-trail of $J(D)$ with $|V(T'')| > |V(T)|$, contrary to (3.1); if $(c'', a_{t_1}) \notin A(J(D))$, then $a_{t_1} \neq a_{t_2}$. By $(c, V(T))_{J(D)} = \emptyset$ and $(c'', a_{t_2}) \in A(J(D))$, we have $c \neq c''$. Since

$(c, a_{t_1}), (c, a_{t_2}), (c'', a_{t_1}) \notin A(J(D))$ and $J(D) = \overline{L(D)}$, we have $(c, a_{t_1}), (c, a_{t_2})$, $(c'', a_{t_1}) \in A(L(D))$. By Lemma 2.1, $(c'', a_{t_2}) \in A(L(D))$, and so $(c'', a_{t_2}) \notin A(J(D))$, a contrary to $(c'', a_{t_2}) \in A(J(D))$. Likewise, if $(V(T), c)_{J(D)} = \emptyset$, then a contradiction will be obtained similarly.

Case 2. For any vertex $c \in V(J(D)) - V(T)$, $(c, V(T))_{J(D)} \neq \emptyset$ and $(V(T), c)_{J(D)} \neq \emptyset$.

Let $V(J(D)) - V(T) = \{c_1, c_2, \ldots, c_\ell\}$. For any $c_i \in V(J(D)) - V(T)$, note that $(c_i, V(T))_{J(D)} \neq \emptyset$ and $(V(T), c_i)_{J(D)} \neq \emptyset$. Let t_{i_1} be a minimum integer and t_{i_2} be a maximum integer such that $(a_{t_{i_1}}, c_i), (c_i, a_{t_{i_2}}) \in A(J(D))$. Thus c_i uniquely determines the pair of integers $\{t_{i_1}, t_{i_2}\}$. Moreover, the choices of t_{i_1} and t_{i_2} imply that $(a_1, c_i), (a_2, c_i), \ldots, (a_{t_{i_1}-1}, c_i) \notin A(J(D))$ and $(c_i, a_{t_{i_2}+1}), (c_i, a_{t_{i_2}+2}), \ldots, (c_i, a_t) \notin A(J(D))$. By Corollary 2.3 (ii), we have

$$\text{for any two distinct vertices } a_{\ell_1}, a_{\ell_2} \in \{a_1, \ldots, a_{t_{i_1}-1}\}, (a_{\ell_1}, a_{\ell_2}), (a_{\ell_2}, a_{\ell_1}) \in A(J(D)), \quad (3.4)$$

and by Corollary 2.3 (i), we have

$$\text{for any two distinct vertices } a_{j_1}, a_{j_2} \in \{a_{t_{i_2}+1}, \ldots, a_t\}, (a_{j_1}, a_{j_2}), (a_{j_2}, a_{j_1}) \in A(J(D)). \quad (3.5)$$

By (3.2), we can obtain that $t_{i_1} \neq t_{i_2}$. Next, we consider two subcases in the following.

Subcase 2.1. There exists a vertex $c_i \in V(J(D)) - V(T)$ such that $t_{i_1} < t_{i_2}$.

If there exists a vertex $a_j \in \{a_{t_{i_1}+1}, a_{t_{i_1}+2}, \ldots, a_{t_{i_2}-1}\}$ such that $(a_j, c_i) \in A(J(D))$, then by (3.2), we have $(c_i, a_j) \notin A(J(D))$. Hence $a_j \neq a_{t_{i_2}}$, and so let $a_{t_{i_1}} := a_j$. If there exists a vertex $a_{j'} \in \{a_{t_{i_1}+1}, a_{t_{i_1}+2}, \ldots, a_{t_{i_2}-1}\}$ such that $(c_i, a_{j'}) \in A(J(D))$, then by (3.2), we have $(a_{j'}, c_i) \notin A(J(D))$. Hence $a_{j'} \neq a_{t_{i_1}}$, and so let $a_{t_{i_2}} := a_{j'}$. Thus, for any $a_k \in \{a_{t_{i_1}+1}, a_{t_{i_1}+2}, \ldots, a_{t_{i_2}-1}\}$, $(c_i, a_k), (a_k, c_i) \notin A(J(D))$, by (3.2), we can obtain that $(c_i, a_{t_{i_1}}), (a_{t_{i_2}}, c_i) \notin A(J(D))$. By Corollary 2.3 (i), we have for any two distinct vertices $a_{k_1}, a_{k_2} \in \{a_{t_{i_1}}, a_{t_{i_1}+1}, \ldots, a_{t_{i_2}-1}\}, (a_{k_1}, a_{k_2})$, $(a_{k_2}, a_{k_1}) \in A(J(D))$. Hence

$$\text{for any vertex } a_k \in \{a_{t_{i_1}+1}, a_{t_{i_1}+1}, \ldots, a_{t_{i_2}-1}\}, (a_{t_{i_1}}, a_k), (a_k, a_{t_{i_1}}) \in A(J(D)). \quad (3.6)$$

Let $A = V(T) - (V(T[a_1, a_{t_{i_1}}]) \cup V(T[a_{t_{i_2}}, a_t]))$, hence $A \subseteq \{a_{t_{i_1}+1}, a_{t_{i_1}+1}, \ldots, a_{t_{i_2}-1}\}$. Thus, by (3.6), we have

$$T'' = T[a_1, a_{t_{i_1}}](a_{t_{i_1}}, c_i)(c_i, a_{t_{i_2}})T[a_{t_{i_2}}, a_t] \cup \bigcup_{a \in A} \{(a, a_{t_{i_1}}), (a_{t_{i_1}}, a)\}$$

is an (a, b)-trail of $J(D)$ with $|V(T'')| > |V(T)|$, contrary to (3.1).

Subcase 2.2. For any vertex $c_i \in V(J(D)) - V(T)$, $t_{i_1} > t_{i_2}$.

By (3.2), we can obtain that

$$(c_i, a_{t_{i_1}}), (a_{t_{i_2}}, c_i) \notin A(J(D)). \quad (3.7)$$

If $\{a_1, a_2, \ldots, a_{t_{i_2}-1}\} \neq \emptyset$, then there exists a vertex $a \in \{a_1, a_2, \ldots, a_{t_{i_2}-1}\}$ such that $a \neq a_{t_{i_2}}$. Since $t_{i_1} > t_{i_2}$, and t_{i_1} is a minimum integer and t_{i_2} is a maximum integer such that $(a_{t_{i_1}}, c_i), (c_i, a_{t_{i_2}}) \in A(J(D))$, we can obtain that $a_1 \neq a_{t_{i_1}}$, $a_t \neq a_{t_{i_2}}$, $a \neq a_{t_{i_1}}$ and $(c_i, a_t), (a, c_i) \notin A(J(D))$. By (3.7) and Corollary 2.3 (iii), we have $(a_{t_{i_2}}, a_t), (a, a_{t_{i_1}}), (a_{t_{i_2}}, a_{t_{i_1}}) \in A(J(D))$. Since $a \neq a_{t_{i_2}}$, we have $(a_{t_{i_2}}, a_t) \neq (a, a_{t_{i_1}})$. By (3.4), for any $a_k \in A'$, we have $(a_k, a_{t_{i_2}})$, $(a_{t_{i_2}}, a_k) \in A(J(D))$, where $A' = \{a_2, a_3, \ldots, a_{t_{i_1}-1}\} - \{a_{t_{i_2}}\}$. And by (3.5), for any

$a_{k'} \in A''$, we have $(a_{k'}, a_{t_{i_1}}), (a_{t_{i_1}}, a_{k'}) \in A(J(D))$, where $A'' = \{a_{t_{i_1}+1}, a_{t_{i_1}+2}, \ldots, a_{t-1}\}$.

Next, we consider vertices a_1 and a. If $a_1 = a$, then

$$T'' = (a_1, a_{t_{i_1}})(a_{t_{i_1}}, c_i)(c_i, a_{t_{i_2}})(a_{t_{i_2}}, a_t) \cup \bigcup_{a_k \in A'} \{(a_k, a_{t_{i_2}}), (a_{t_{i_2}}, a_k)\} \cup \bigcup_{a_{k'} \in A''} \{(a_{k'}, a_{t_{i_1}}), (a_{t_{i_1}}, a_{k'})\}$$

is an (a, b)-trail of $J(D)$ with $|V(T'')| > |V(T)|$, contrary to (3.1). Hence we assume that $a_1 \neq a$. By (3.4), $(a_1, a) \in A(J(D))$, and so $(a_1, a) \neq (a, a_{t_{i_1}})$. If $(a_1, a) \neq (a_{t_{i_2}}, a_t)$, then

$$T'' = (a_1, a)(a, a_{t_{i_1}})(a_{t_{i_1}}, c_i)(c_i, a_{t_{i_2}})(a_{t_{i_2}}, a_t) \cup \bigcup_{a_k \in A'} \{(a_k, a_{t_{i_2}}), (a_{t_{i_2}}, a_k)\} \cup \bigcup_{a_{k'} \in A''} \{(a_{k'}, a_{t_{i_1}}), (a_{t_{i_1}}, a_{k'})\}$$

is an (a, b)-trail of $J(D)$ with $|V(T'')| > |V(T)|$, contrary to (3.1). If $(a_1, a) = (a_{t_{i_2}}, a_t)$, then $a_1 = a_{t_{i_2}}$ and $a = a_t$. Since $a \neq a_{t_{i_1}}$, we have $(a_1, a_t) \neq (a_{t_{i_2}}, a_{t_{i_1}})$. Thus

$$T'' = (a_1, a_t)(a_1, a_{t_{i_1}})(a_{t_{i_1}}, c_i)(c_i, a_1) \cup \bigcup_{a_k \in A'} \{(a_k, a_{t_{i_2}}), (a_{t_{i_2}}, a_k)\} \cup \bigcup_{a_{k'} \in A''} \{(a_{k'}, a_{t_{i_1}}), (a_{t_{i_1}}, a_{k'})\}$$

is an (a, b)-trail of $J(D)$ with $|V(T'')| > |V(T)|$, contrary to (3.1).

Likewise, if $\{a_{t_{i_1}+1}, a_{t_{i_1}+2}, \ldots, a_t\} \neq \emptyset$, then a contradiction will be obtained similarly.

Hence assume that $t_{i_2} = 1$ and $t_{i_1} = t$. If $A''' = \{a_{t_{i_2}+1}, a_{t_{i_2}+2}, \ldots, a_{t_{i_1}-1}\} \neq \emptyset$, then let $a' \in \{a_{t_{i_2}+1}, a_{t_{i_2}+2}, \ldots, a_{t_{i_1}-1}\}$. As t_{i_1} is a minimum integer and t_{i_2} is a maximum integer such that $(a_{t_{i_1}}, c_i), (c_i, a_{t_{i_2}}) \in A(J(D))$, we can obtain that $a' \neq a_{t_{i_1}}, a' \neq a_{t_{i_2}}$ and $(a', c_i), (c_i, a') \notin A(J(D))$. By (3.2), we have $(c_i, a_{t_{i_1}}), (a_{t_{i_2}}, c_i) \notin A(J(D))$. By Corollary 2.3 (iii), we have $(a_{t_{i_2}}, a'), (a', a_{t_{i_1}}), (a_{t_{i_2}}, a_{t_{i_1}}) \in A(J(D))$. By (3.4), for any $a_k \in A'''$, $(a_{t_{i_2}}, a_k), (a_k, a_{t_{i_2}}) \in A(J(D))$. Thus

$$T'' = (a_1, a_t)(a_t, c_i)(c_i, a_1)(a_1, a')(a', a_t) \cup \bigcup_{a_k \in A''' - \{a'\}} \{(a_k, a_{t_{i_2}}), (a_{t_{i_2}}, a_k)\}$$

is an (a, b)-trail of $J(D)$ with $|V(T'')| > |V(T)|$, contrary to (3.1).

Hence we may assume that $t = 2$, $a_{t_{i_2}} = a_1$ and $a_{t_{i_1}} = a_2$. Thus

$$V(J(D)) = \{a_1, a_2, c_1, c_2, \ldots, c_\ell\}, \tag{3.8}$$

and

$$\text{for any } c_i \in V(J(D)) - V(T), [a_1, c_i]_{J(D)} = \{(c_i, a_1)\} \text{ and } [a_2, c_i]_{J(D)} = \{(a_2, c_i)\}. \tag{3.9}$$

By (3.9), for any two distinct vertices $c_i, c_j \in V(J(D)) - V(T)$, we have $(a_1, c_i), (a_1, c_j) \notin A(J(D))$. By Corollary 2.3 (i), $(c_i, c_j), (c_j, c_i) \in A(J(D))$. Thus

$$J(D)[V(J(D)) - V(T)] \text{ is a complete digraph.} \tag{3.10}$$

Since $(a_1, a_2) \in A(J(D))$, by (3.8)–(3.10), we can obtain that $J(D) \cong J(D(\ell))$. By Proposition

3.2 (iv), $D \cong D(\ell)$, and so $D \in \mathcal{D}$, a contradiction. Hence $J(D)$ is strongly trail-connected. $\square$

Theorem 3.4 *Every strongly connected jump digraph $J(D)$ of a directed multigraph D is weakly trail-connected.*

Proof Let $\mathcal{D}$ be a directed multigraph family as defined in Definition 3.1. By Theorem 3.3, if $D \notin \mathcal{D}$, then $J(D)$ is strongly trail-connected, and so $J(D)$ is weakly trail-connected; if $D \in \mathcal{D}$, by Proposition 3.2 (ii), $J(D)$ is weakly trail-connected. Hence every strongly connected jump digraph $J(D)$ of directed multigraph D is weakly trail-connected. $\square$

Theorem 3.5 *Every weakly connected jump digraph $J(D)$ of a directed multigraph D has a spanning trail.*

Proof If $J(D)$ is weakly connected but is not strongly connected, then by Theorem 2.7, $D \in \mathcal{H}_1$ and $D \notin \mathcal{H}_2$. So $B^+ \neq \emptyset$ and $B^- \neq \emptyset$ and, $t_1 \neq 0$ or $t_2 \neq 0$ or $t_3 \neq 1$. Hence $\sum_{i=1}^{t_1} \ell_{x_i} \neq 0$ or $\sum_{j=1}^{t_2} \ell'_{y_j} \neq 0$ or, $\sum_{k=1}^{t_3} \ell_{z_k} \neq 1$ and $\sum_{k=1}^{t_3} \ell'_{z_k} \neq 1$. If $t_3 = 0$, since $B^+ \neq \emptyset$ and $B^- \neq \emptyset$, by Proposition 2.5 (i), we can obtain that $\sum_{i=1}^{t_1} \ell_{x_i} \geq 1$ and $\sum_{j=1}^{t_2} \ell_{y_j} \geq 1$. By Proposition 2.5 (v), we have

$$
\begin{aligned}
|(B^+, B^-)_{J(D)}| &= \Big(\sum_{i=1}^{t_1} \ell_{x_i} + \sum_{k=1}^{t_3} \ell_{z_k}\Big)\Big(\sum_{j=1}^{t_2} \ell'_{y_j} + \sum_{k=1}^{t_3} \ell'_{z_k}\Big) - \sum_{k=1}^{t_3} \ell_{z_k}\ell'_{z_k} \\
&= \Big(\sum_{i=1}^{t_1} \ell_{x_i}\Big)\Big(\sum_{j=1}^{t_2} \ell'_{y_j}\Big) \geq 1.
\end{aligned}
$$

If $t_3 \geq 2$, then $(\sum_{k=1}^{t_3} \ell_{z_k})(\sum_{k=1}^{t_3} \ell'_{z_k}) > \sum_{k=1}^{t_3} \ell_{z_k}\ell'_{z_k}$. By Proposition 2.5 (v),

$$
\begin{aligned}
|(B^+, B^-)_{J(D)}| &= \Big(\sum_{i=1}^{t_1} \ell_{x_i} + \sum_{k=1}^{t_3} \ell_{z_k}\Big)\Big(\sum_{j=1}^{t_2} \ell'_{y_j} + \sum_{k=1}^{t_3} \ell'_{z_k}\Big) - \sum_{k=1}^{t_3} \ell_{z_k}\ell'_{z_k} \\
&\geq \Big(\sum_{k=1}^{t_3} \ell_{z_k}\Big)\Big(\sum_{k=1}^{t_3} \ell'_{z_k}\Big) - \sum_{k=1}^{t_3} \ell_{z_k}\ell'_{z_k} \geq 1.
\end{aligned}
$$

Thus, $(B^+, B^-)_{J(D)} \neq \emptyset$. Let $a \in B^+$ and $b \in B^-$ such that $(a, b) \in A(J(D))$. By Proposition 2.5 (iii), $J(D)[B^+]$ and $J(D)[B^-]$ are complete digraphs, by (1.2), $J(D)[B^+]$ has a spanning closed trail T_1 and $J(D)[B^-]$ has a spanning closed trail T_2. Thus

$$T_1 \cup (a, b) \cup T_2$$

is a spanning trail of $J(D)$. $\square$

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