Asymptotically sharpening the
s-Hamiltonian index bound

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For a non-negative integer $s \leq |V(G)| - 3$, a graph G is s-Hamiltonian if the removal of any $k \leq s$ vertices results in a Hamiltonian graph. Given a connected simple graph G that is not isomorphic to a path, a cycle, or a $K_{1,3}$, let $\delta(G)$ denote the minimum degree of G, let $h_s(G)$ denote the smallest integer i such that the iterated line graph $L^i(G)$ is s-Hamiltonian, and let $\ell(G)$ denote the length of the longest non-closed path P in which all internal vertices have degree 2 such that P is not both of length 2 and in a K_3. For a simple graph G, we establish better upper bounds for $h_s(G)$ as follows.

$$h_s(G) \leq \begin{cases} \ell(G) + 1, & \text{if } \delta(G) \leq 2 \text{ and } s = 0; \\ \widetilde{d}(G) + 2 + \lceil \lg(s+1) \rceil, & \text{if } \delta(G) \leq 2 \text{ and } s \geq 1; \\ 2 + \left\lceil \lg \dfrac{s+1}{\delta(G) - 2} \right\rceil, & \text{if } 3 \leq \delta(G) \leq s + 2; \\ 2, & \text{otherwise,} \end{cases}$$

where $\widetilde{d}(G)$ is the smallest integer i such that $\delta(L^i(G)) \geq 3$. Consequently, when $s \geq 6$, this new upper bound for the s-hamiltonian index implies that $h_s(G) = o(\ell(G) + s + 1)$ as $s \to \infty$. This sharpens the result, $h_s(G) \leq \ell(G) + s + 1$, obtained by Zhang et al. in [Discrete Math., 308 (2008) 4779-4785].

Keywords: s-Hamiltonian; (s, t)-supereulerian; Collapsible graphs; k-triangular; Line graph stable properties

1 Introduction

Finite loopless graphs permitting parallel edges are considered with undefined terms being referenced to [5]. As in [5], a simple graph is one that is loopless and without parallel edges, and the minimum degree of a graph G is denoted by $\delta(G)$. For a subset $X \subseteq V(G)$ or $E(G)$, let $G[X]$ denote the subgraph induced by X, and let $G - X = G[V(G) - X]$ or $G[E(G) - X]$, respectively. When $X = \{x\}$, we write $G - x$ for $G - \{x\}$. Throughout this paper, if $X \subseteq E(G)$, then, for notational convenience, we often use X to denote both the edge subset of $E(G)$ and $G[X]$. We also use $\lg x$ as an alternative notation for $\log_2 x$, and set $[m, n] = \{m, m+1, \ldots, n\}$ for two integers m, n with $m \leq n$.

A graph is considered *Hamiltonian* if it has a spanning cycle. For a non-negative integer $s \leq |V(G)| - 3$, a graph is called *s-Hamiltonian* if the removal of any $k \leq s$ vertices results in a Hamiltonian graph. A subgraph H of G is *dominating* if $G - V(H)$ is edgeless.

Following [4,6], a graph is *supereulerian* if it has a spanning closed trail. Harary and Nash-Williams [13] characterized Hamiltonian line graphs as follows, which implies that the line graph of every supereulerian graph is Hamiltonian.

Theorem 1.1 (Harary and Nash-Williams, Proposition 8 of [13]). *Let G be a graph with at least three edges. Then $L(G)$ is Hamiltonian if and only if G has a dominating closed trail.*

The *line graph* of a graph G, denoted $L(G)$, is a simple graph with $E(G)$ being its vertex set, where two vertices in $L(G)$ are adjacent whenever the corresponding edges in G are adjacent. A *claw-free* graph is one that does not have an induced subgraph isomorphic to $K_{1,3}$. Beineke [2] and Robertson (Page 74 of [12]) showed that line graphs are claw-free graphs. For a positive integer i, we define $L^0(G) = G$, and the ith iterated line graph of G, denoted $L^i(G)$, is defined recursively as $L^i(G) = L(L^{i-1}(G))$.

Let J_1 and J_2 be two graphs obtained from $K_{1,3}$ via identifying two and three vertices of degree 1, respectively. Let $K_{1,3}^+ = \{J_1, J_2, K_{1,3}\}$. Since the line graph of a cycle remains unchanged, in general, we assume that graphs are not isomorphic to paths, cycles or any members in $K_{1,3}^+$. For this reason, we define

$$\mathcal{G} = \{G : G \text{ is connected and is not isomorphic to a path, or a cycle, or a member in } K_{1,3}^+\}.$$

Chartrand in [9] introduced and studied the Hamiltonian index of a graph, and initiated the study of indices of graphical properties. More generally, we have the following definition.

Definition 1.1 (Definition 5.8 of [17]). *For a property $\mathcal{P}$, the $\mathcal{P}$-index of $G \in \mathcal{G}$ is defined by*

$$\mathcal{P}(G) = \begin{cases} \min\{i : L^i(G) \text{ has property } \mathcal{P}\}, & \text{if one such integer } i \text{ exists;} \\ \infty, & \text{otherwise.} \end{cases}$$

A property $\mathcal{P}$ is **line graph stable** if $L(G)$ has $\mathcal{P}$ whenever G has $\mathcal{P}$. Chartrand [9] showed that for every graph $G \in \mathcal{G}$, the Hamiltonian index exists as a finite number, and the characterization of Hamiltonian line graphs (Theorem 1.1) by Harary and Nash-Williams implies that being Hamiltonian is line graph stable. Z. Ryjáček et al. [25] indicated that determining the value of the Hamiltonian index is difficult. Clark and Wormald [11] showed that for all graphs in $\mathcal{G}$, other Hamiltonian-like indices also exist as finite numbers; and in [17], it is shown that these Hamiltonian-like properties are also line graph stable. Let $h(G), h_s(G)$ and $s(G)$ be the *Hamiltonian index*, *s-Hamiltonian index* and *supereulerian index* of $G \in \mathcal{G}$, respectively. By definitions, $h(G) = h_0(G)$.

Let $P = v_0 e_1 v_1 e_2 \cdots v_{s-1} e_s v_s$ be a path of a graph G where each $e_i \in E(G)$ and each $v_i \in V(G)$. Then P is called a (v_0, v_l)-path or an (e_1, e_s)-path of G. A path P of G is *divalent* if every internal vertex of P has degree 2 in G. For two non-negative integers s and t, a divalent path P of G is a *divalent (s,t)-path* if the two end vertices of P have degrees s and t, respectively. A non-closed divalent path P is considered *proper* if P is not both of length 2 and in a K_3. As in [16, 26], for a graph $G \in \mathcal{G}$, define

$$\ell(G) = \max\{m : G \text{ has a length } m \text{ proper divalent path}\}. \tag{1}$$

Theorem 1.2. *Let $G \in \mathcal{G}$ be a simple graph. Each of the following holds.*
(i) (Corollary 6 of [16]) $s(G) \leq \ell(G)$.
(ii) (Corollary 6 of [16]) $h(G) \leq s(G) + 1 \leq \ell(G) + 1$.
(iii) (Zhang et al., Theorem 1.1 of [26]) $h_s(G) \leq \ell(G) + s + 1$.

Several natural questions arise here. Can we improve the upper bounds above? Can we generalize Theorem 1.2(i) in the way as Theorem 1.2(iii) extends Theorem 1.2(ii)? As a generalization of supereulerian graphs, given two non-negative integers s and t, it is defined in [21] that a graph G is (s,t)-*supereulerian* if for any disjoint sets $X, Y \subset E(G)$

with $|X| \leq s$ and $|Y| \leq t$, $G - Y$ contains a spanning closed trail that traverses all edges in X. Former studies on (s, t)-supereulerian graphs can be found in [19–21], among others. Let $i_{s,t}(G)$ denote the (s, t)-*supereulerian index* of a graph $G \in \mathcal{G}$. Thus, $i_{0,0}(G) = s(G)$. By the characterization of Hamiltonian line graphs (Theorem 1.1), the line graph of every $(0, s)$-supereulerian graph is s-Hamiltonian, and then we obtain the following observation.

Observation 1. *Let $G \in \mathcal{G}$. Then $h_s(G) \leq i_{0,s} + 1$. In particular, $h(G) \leq s(G) + 1$.*

To present the main results, an additional notation would be needed. Since $G \in \mathcal{G}$, it is observed that (for example, Theorem 18 of [10]) there exists an integer $i > 0$ such that $\delta(L^i(G)) \geq 3$. Define

$$\widetilde{d}(G) = \min\{i : \delta(L^i(G)) \geq 3\}. \tag{2}$$

Our main results can now be stated as follows.

Theorem 1.3. *Let $G \in \mathcal{G}$ be a simple graph with $\delta = \delta(G)$ and $\widetilde{d} = \widetilde{d}(G)$. Then, given two non-negative integers s and t,*

$$i_{s,t}(G) \leq \begin{cases} \ell(G), & \text{if } \delta \leq 2 \text{ and } s = t = 0; \\ \widetilde{d} + 1 + \lceil \lg(s + t + 1) \rceil, & \text{if } \delta \leq 2 \text{ and } s + t \geq 1; \\ 1 + \left\lceil \lg \dfrac{s + t + 1}{\delta - 2} \right\rceil, & \text{if } 3 \leq \delta \leq s + t + 2; \\ 1, & \text{otherwise.} \end{cases} \tag{3}$$

Using Observation 1, Theorem 1.3 implies Corollary 1.4 below.

Corollary 1.4. *Let $G \in \mathcal{G}$ be a simple graph with $\delta = \delta(G)$ and $\widetilde{d} = \widetilde{d}(G)$. Then, given a non-negative integer $s \leq |V(G)| - 3$,*

$$h_s(G) \leq \begin{cases} \ell(G) + 1, & \text{if } \delta \leq 2 \text{ and } s = 0; \\ \widetilde{d} + 2 + \lceil \lg(s + 1) \rceil, & \text{if } \delta \leq 2 \text{ and } s \geq 1; \\ 2 + \left\lceil \lg \dfrac{s + 1}{\delta - 2} \right\rceil, & \text{if } 3 \leq \delta \leq s + 2; \\ 2, & \text{otherwise.} \end{cases} \tag{4}$$

Given a simple graph $G \in \mathcal{G}$ with $\ell = \ell(G)$ and $\widetilde{d} = \widetilde{d}(G)$. By the formula to compute $\widetilde{d}$ to be presented in Section 3.1, we have $\widetilde{d} \leq \ell + 2$. When $s \geq 6$, as $\lceil \lg(s+1) \rceil + 2 \leq s - 1$, we have $\widetilde{d} + 2 + \lceil \lg(s+1) \rceil \leq \ell + 1 + s$. Moreover, since $\lceil \lg(s+1) \rceil = o(s)$ as $s \to \infty$, it follows that $\widetilde{d} + 2 + \lceil \lg(s+1) \rceil = o(\ell + s + 1)$ as $s \to \infty$. Similarly, when $s \geq 1$ and $n \geq 1$, we have $\lceil \lg \frac{s+1}{n} \rceil \leq s$ and $\lceil \lg \frac{s+1}{n} \rceil = o(s)$ as $s \to \infty$. It means that $2 + \lceil \lg \frac{s+1}{n} \rceil \leq s + 2$ and $2 + \lceil \lg \frac{s+1}{n} \rceil = o(s + 2)$ as $s \to \infty$. Hence, when $s \geq 6$, the upper bounds above sharpen the result of Theorem 1.2(iii).

In the next section, we present preliminaries and tools that will be used in our discussions. In Section 3, we shall show some important lemmas, including a corrected formula to compute $\widetilde{d}(G)$, which are very helpful to prove the main result, Theorem 1.3, in Section 4.

2 Preliminaries

For a vertex $v \in V(G)$, we denote $N_G(v)$ to be the set of all neighbors of vertex v in a graph G, and denote $E_G(v)$ to be the set of all edges incident with v in G. The degree of vertex v is denoted by $d_G(v) = |E_G(v)|$. For integer $i \geq 0$, let $D_i(G)$ be the set of all vertices of degree i in G, and let $O(G)$ be the set of all odd degree vertices in G. For an edge $e \in E(G)$, let $E_G(e)$ be the set of all edges adjacent with e in G and $d_G(e) = |E_G(e)|$. For notational convenience, if $v \in V(G)$ and $e \in E(G)$, we write $d(v)$ and $d(e)$ for $d_G(v)$ and $d_G(e) = d_{L(G)}(e)$, respectively, when G or $L(G)$ is understood from context.

2.1 Iterated Line Graphs

For a subset $X \subseteq E(G)$, let $L^0(X) = X$ and $L^i(X) = L^i(G)[L^{i-1}(X)]$ for each integer $i \geq 1$. Moreover, for a subset $Y \subseteq E(L^i(G))$, by the definition of the iterated line graphs, there exists a unique $Z \subseteq E(L^{i-j}(G))$ for each $j \in [0, i]$ such that $L^j(Z) = Y$, denoted $Z = L^{-j}(Y)$. Thus, for two integers i, j and an edge subset $X \subseteq E(G)$, $L^i(L^j(X)) = L^{i+j}(X)$.

Lemma 2.1. *Given an integer $i \geq 0$ and a graph G. If P is a divalent (s, t)-path in $L^i(G)$ of length r that is not in a K_3, then for each $j \in [0, i]$, $L^{-j}(P)$ is a divalent (s, t)-path in $L^{i-j}(G)$ of length $r + j$.*

Proof: Assume that j_0 is the smallest number such that $L^{-j_0}(P)$ is not a divalent (s, t)-path of length $r + j_0$ where $0 < j_0 \leq i$. Let $Q = L^{-j_0+1}(P)$. Thus, Q is a divalent (s, t)-path in $L^{i-(j_0-1)}(G)$ of length $r + j_0 - 1$. First, we claim that Q is not in a K_3. If Q is in a K_3, then $P = L^{j_0-1}(Q)$ is in a K_3 since the line graph of a K_3 is still a K_3, which contradicts the assumption of P being not in a K_3.

Now, set $J = L^{i-j_0}(G)$, and then $L(J) = L^{i-(j_0-1)}(G)$. Let Q be a (u, v)-path of $L(J)$, where $u \in D_s(L(J))$ and $v \in D_t(L(J))$. As Q is not in a K_3 and the definition of divalent paths, $L^{-j_0}(P) = L^{-1}(Q)$ is a divalent (u, v)-path in J, where $\{u, v\} \subset E(J)$. Let $L^{-j_0}(P)$ be a (x, y)-path where $\{x, y\} \subset V(J)$. Since $d(x) = d(u) - 2 + 2 = s$ and $d(y) = d(v) - 2 + 2 = t$, $L^{-j_0}(P)$ is a divalent (s, t)-path of length $r + j_0$, which contradicts our choice of j_0. □

2.2 Collapsible Graphs

In [7], Catlin defined collapsible graphs as a useful tool to study supereulerian graphs. A graph G is *collapsible* if for every subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a subgraph Γ_R such that $O(\Gamma_R) = R$ and $G - E(\Gamma_R)$ is connected. By definition, all complete graphs K_n except K_2 are collapsible. As shown in Proposition 1 of [18], a graph G is collapsible if and only if for every subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph L_R with $O(L_R) = R$. As $L_\emptyset$ is a spanning eulerian subgraph, every collapsible graph is supereulerian. Collapsible graphs have been considered to be very useful to study eulerian subgraphs via the graph contraction. For an edge subset $X \subseteq E(G)$, the *contraction* G/X is obtained from G by identifying the two ends of each edge in X and deleting the resulting loops. If H is a subgraph of G, then we write G/H for $G/E(H)$. The following theorem summarizes some useful properties of collapsible graphs for our proofs.

Theorem 2.2. *Let G be a graph and H be a subgraph of G.*
(i) (Catlin, Theorem 3 of [8]) If each edge of a connected graph G is in a cycle of length 2 or 3, then G is collapsible.
(ii) (Catlin, Corollary of Theorem 3 of [7]) If H is collapsible, then G is collapsible if and only if G/H is collapsible.

3 The k-Triangular Index

A cycle of length 3 is often called a triangle. Following [3], for an integer $k > 0$, a graph G is *k-triangular* if every edge lies in at least k distinct triangles in G; a graph G is *triangular* if G is 1-triangular. Let $\mathcal{T}_k$ denote the family of all k-triangular graphs. Thus, $\delta(G) \geq k+1$ if $G \in \mathcal{T}_k$.

Triangular graphs are often considered as models for some kinds of cellular networks ([14]) and for certain social networks ([22]), as well as mechanisms to study network stabilities and to classify spam websites ([1]). In addition to its applications in the hamiltonicity of line graphs ([3]), triangular graphs are also related to design theory. In 1984, Moon in [23] introduced the Johnson graphs $J(n, s)$, named after Selmer M. Johnson for

the closely related Johnson scheme. The vertex set of $J(n,s)$ is all s-element subsets of an n-element set, where two vertices are adjacent whenever the intersection of the corresponding two subsets contains exactly $s-1$ elements. For example, $J(n,1)$ is isomorphic to K_n. By definitions, for any integers $n \geq 3$ and s with $n > s$, $J(n,s)$ is $(n-2)$-triangular. Therefore, it is of interests to investigate k-triangular graphs for a generic value of k.

For an integer $k > 0$, define $t_k(G)$ to be the *k-triangular index* of $G \in \mathcal{G}$, that is, the smallest integer m such that $L^m(G) \in \mathcal{T}_k$. The triangular index $t_1(G)$ is first investigated by Zhang et al.

Theorem 3.1. *Let $G \in \mathcal{G}$ be a simple graph. Each of the following holds.*
(i) (Zhang et al., Proposition 2.3 (i) of [27]) Being triangular is line graph stable.
(ii) (Zhang et al., Lemma 3.2 (iii) of [26]) $t_1(G) \leq \ell(G)$.

One of the purposes of this section is to determine, for any positive integer k, the best possible bounds for $t_k(G)$ and to investigate whether being k-triangular is line graph stable.

3.1 A Formula to Compute $\widetilde{d}(G)$

Recall that $\widetilde{d}(G) = \min\{i : \delta(L^i(G)) \geq 3\}$, which is defined in (2). Define

$$\ell_1(G) = \max\{|E(P)| : P \text{ is a divalent } (1,3)\text{-path of } G\},$$
$$\ell_2(G) = \max\{|E(P)| : P \text{ is a divalent } (1,t)\text{-path of } G, \text{ where } t \geq 4\}, \quad (5)$$
$$\ell_3(G) = \max\{|E(P)| : P \text{ is a divalent } (s,t)\text{-path of } G, \text{ where } s,t \geq 3\},$$

and

$$\ell_0(G) = \max\{\ell_1(G) + 1, \ell_2(G), \ell_3(G) - 1\}.$$

In [15], it is claimed that "It is easy to see $\widetilde{d}(G) = \ell_0(G)$." However, there exists an infinite family of graphs each of which shows that this claim might be incorrect. Let $\mathcal{B} = \{T : T$ is a tree with $V(T) = D_1(T) \cup D_3(T)\}$. For each $G \in \mathcal{B}$, we have $\ell_1(G) = \ell_3(G) = 1$ and $\ell_2(G) = 0$. Direct computation indicates that $\widetilde{d}(G) = 3 > \ell_0(G)$. See Figure 1 for an illustration.
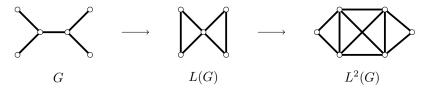

$$G \qquad\qquad L(G) \qquad\qquad L^2(G)$$

Fig. 1: A member $G \in \mathcal{B}$ and its iterated line graphs.

Thus what would be the correct formula to compute $\widetilde{d}(G)$ becomes a question to be answered. Before presenting our answer to it, we need some notation. Let $U = \{v \in V(G) : |N_G(v)| = 1\}$ and $F = \bigcup_{v \in U} E_G(v)$.

Lemma 3.2. *Let $G \in \mathcal{G}$ be a graph with $\delta(G) \leq 2$, $\widetilde{d} = \widetilde{d}(G)$ and $\ell_0 = \ell_0(G)$. The formula below computes $\widetilde{d}$:*

$$\widetilde{d} = \begin{cases} \max\{\ell_0, 3\}, & \textit{if } |E_G(v) \cap F| = 2 \textit{ for some } v \in D_3(G); \\ \ell_0, & \textit{otherwise.} \end{cases} \quad (6)$$

Proof: Let m be the right-hand side of (6) and let $\ell_i = \ell_i(G)$ for each $i \in \{1,2,3\}$. Then $m \leq \widetilde{d}$ by definitions of $\widetilde{d}$ and line graphs. Now, it suffices to show that $\delta(L^m(G)) \geq 3$. We assume that $\delta(L^m(G)) \leq 2$ to seek a contradiction.

If $\delta(L^m(G)) = 1$, then $L^m(G)$ has a divalent $(1,t)$-path of length r where $t \geq 3$. By Lemma 2.1, G has a divalent $(1,t)$-path of length $r + m$. If $t = 3$, then $m + 1 \leq m + r \leq$

$\ell_1 \leq m - 1$, a contradiction; if $t > 3$, then $m + 1 \leq m + r \leq \ell_2 \leq m$, which is also a contradiction.

Then, $\delta(L^m(G)) = 2$. Pick $u \in D_2(L^m(G))$. If u is not in any triangles of $L^m(G)$, then u is in a divalent (s', t')-path of length $r' \geq 2$ in $L^m(G)$ that is not in a K_3, where $s' \geq 3$ and $t' \geq 3$. It follows that G has a divalent (s', t')-path of length $r' + m$ by Lemma 2.1, which shows that $2 + m \leq r' + m \leq \ell_3 \leq m + 1$, a contradiction. Thus, $u \in V(H)$ where $H \cong K_3$ is a subgraph of $L^m(G)$. By the definition of line graphs, $L^{-1}(H)$ is isomorphic to one member of $\{K_3, K_{1,3}, J_1, J_2\}$. Let $u = xy \in E(L^{-1}(H))$.

When $L^{-1}(H) \cong K_{1,3}$, as $d(u) = 2$, we have $\ell_1(L^{m-1}(G)) \geq 1$. By Lemma 2.1, $\ell_1 \geq 1 + (m - 1) = m \geq \ell_1 + 1$, a contradiction.

When $L^{-1}(H) \cong J_1$ or J_2, as there is no parallel edges in line graphs, $m = 1$. If $L^{-1}(H) \cong J_2$, then $G \cong J_2$ as $d(u) = 2$, contradicting the definition of $\mathcal{G}$. Then, $L^{-1}(H) \cong J_1$. If $u = xy$ is one of the parallel edges of J_1, then one of end vertices of u, say x, of degree 3 in G satisfies $|E_G(x) \cap F| = 2$, which implies $m \geq 3$ by (6). It is a contradiction with $m = 1$.

When $L^{-1}(H) \cong K_3$, we have $d(x) = d(y) = 2$ and $\ell_3 \geq 3$ as $d(u) = 2$. If $m = 1$, as $\ell_3 \geq 3$, then $1 = m \geq \ell_3 - 1 \geq 2$, a contradiction. So, $m \geq 2$. Note that $L^{-2}(H)$ is isomorphic to one member of $\{K_3, K_{1,3}, J_1, J_2\}$. If $L^{-2}(H) \cong K_3$ or J_2, then $L^{m-2}(H) \cong G \cong K_3$ or J_2, respectively, as $d(x) = d(y) = 2$. It contradicts $G \in \mathcal{G}$. Now, $L^{-2}(H)$ is isomorphic to one member of $\{K_{1,3}, J_1\}$. Since $d(x) = d(y) = 2$ as well as line graphs are claw-free and contain no parallel edges, it shows that $m = 2$. As $d(x) = d(y) = 2$, $\{x, y\} \subseteq F$ and there is a common end vertex of edges x and y of degree three, which shows $m \geq 3$ by (6). It contradicts the fact we got before that $m = 2$. $\qquad\square$

3.2 The k-Triangular Index

Before establishing the bounds for $t_k(G)$, we need some lemmas.

Theorem 3.3 (Niepel, Knor and Šoltés, Lemma 1(1) of [24]). *Let G be a simple graph with $\delta(G) \geq 3$. Then, $\delta(L^i(G)) \geq 2^i(\delta(G) - 2) + 2$ for each integer $i \geq 0$.*

By the definition of line graphs, if G is a regular graph, then for each integer $i \geq 0$, we always have $\delta(L^i(G)) = 2^i(\delta(G) - 2) + 2$, and so the lower bound in Theorem 3.3 is best possible in this sense.

Lemma 3.4. *Let $G \in \mathcal{G}$ be a simple graph with $\delta = \delta(G)$. Each of the following holds for each integer $i > 0$.*
(i) If $\delta \geq 3$, then $L^i(G)$ is $(2^{i-1}(\delta - 2))$-triangular.
(ii) If $\delta \leq 2$, then $L^{\widetilde{d}+i}(G)$ is $(2^{i-1}(\delta_0 - 2))$-triangular where $\delta_0 = \delta(L^{\widetilde{d}(G)}(G))$. In particular, $L^{\widetilde{d}+i}(G)$ is 2^{i-1}-triangular.

Proof: Let $e_1 e_2 \in E(L(G))$ be an arbitrary edge in $L(G)$. Then there exists a vertex $u \in V(G)$ such that $\{e_1, e_2\} \subset E_G(u)$. Suppose $\delta \geq 3$. In general, as $L(G)[E_G(u)] \cong K_{d(u)}$, the edge $e_1 e_2$ lies in at least $d(u) - 2 \geq \delta - 2 \geq 1$ distinct triangles. It means that $L(G)$ is $(\delta - 2)$-triangular. By Theorem 3.3, for each integer $i > 0$, $\delta(L^{i-1}(G)) \geq 2^{i-1}(\delta - 2) + 2 \geq 3$. It follows that $L^i(G)$ is $(2^{i-1}(\delta - 2))$-triangular and (i) is proved.

To show (ii), as $\delta_0 \geq 3$, it follows by (i) that $L^{\widetilde{d}+i}(G) = L^i(L^{\widetilde{d}}(G))$ is $(2^{i-1}(\delta_0 - 2))$-triangular. $\qquad\square$

Theorem 3.5. *Let $k \geq 2$ be an integer and $G \in \mathcal{G}$ be a simple graph with $\delta = \delta(G)$ and $\widetilde{d} = \widetilde{d}(G)$. Each of the following holds.*
(i) Being k-triangular is line graph stable.

(ii)

$$t_k(G) \leq \begin{cases} \widetilde{d} + 1 + \lceil \lg k \rceil, & \text{if } \delta \leq 2; \\ 1 + \left\lceil \lg \dfrac{k}{\delta - 2} \right\rceil, & \text{if } 3 \leq \delta \leq k + 1; \\ 1, & \text{otherwise.} \end{cases} \tag{7}$$

Moreover, the equality holds for sufficiently large k when $\delta \leq k + 1$.

Proof: (i) Suppose $G \in \mathcal{G}$ is a simple k-triangular graph for given $k \geq 2$. Then $\delta(G) \geq k + 1 \geq 3$. Pick an edge $e_1 e_2 \in E(L(G))$. To show that $L(G) \in \mathcal{T}_k$, it is enough to prove that $e_1 e_2$ lies in at least k distinct triangles in $L(G)$. Let x be the common vertex of e_1 and e_2 in G, and X be the set of all edges adjacent with both edges e_1 and e_2, that is, $X = E_G(e_1) \cap E_G(e_2)$. If $d(x) \geq k + 2$, then $|X| \geq k$. It means that $e_1 e_2$ lies in at least k distinct triangles in $L(G)$. Now, we consider that $d(x) = k + 1$. Since $G \in \mathcal{T}_k$ is a simple graph, $G[N_G(x)]$ is a complete graph and then $e_1 e_2$ lies in at least k distinct triangles in $L(G)$.

(ii) Let $t = t_k(G)$. First, we consider the situation when $\delta \leq 2$. As $k \geq 2$, by the definition of $\widetilde{d}$, we have $t \geq \widetilde{d}$. If $t < \widetilde{d} + 2$, then $t < \widetilde{d} + 1 + \lceil \lg k \rceil$ as $k \geq 2$. Assume next that k is so large that $t \geq \widetilde{d} + 2$. As $L^t(G) \in \mathcal{T}_k$ but $L^{t-1}(G) \notin \mathcal{T}_k$, by Lemma 3.4(ii), $2^{t - \widetilde{d} - 2} < k \leq 2^{t - \widetilde{d} - 1}$. Then algebraic manipulation leads to $t - \widetilde{d} - 2 < \lg k \leq t - \widetilde{d} - 1$, which means that $\lceil \lg k \rceil = t - \widetilde{d} - 1$. Hence we conclude that $t = \widetilde{d} + 1 + \lceil \lg k \rceil$.

Now, we suppose that $\delta \geq 3$. If $\delta \geq k + 2$, then $L(G) \in \mathcal{T}_{\delta - 2}$ by Lemma 3.4(i), which implies that $L(G) \in \mathcal{T}_k$ and then $t \leq 1$.

If $\delta \leq k + 1$ and $t \geq 2$, then, by Lemma 3.4(i), for each integer $i > 0$, $L^i(G)$ is $(2^{i-1}(\delta - 2))$-triangular. So $2^{t-2}(\delta - 2) < k \leq 2^{t-1}(\delta - 2)$ by the definition of $t = t_k(G)$. It follows that $t = 1 + \left\lceil \lg \frac{k}{\delta - 2} \right\rceil$. Then, $t \leq 1 + \left\lceil \lg \frac{k}{\delta - 2} \right\rceil$ when $3 \leq \delta \leq k + 1$. $\qquad \square$

4   Proof of Theorem 1.3

An *elementary subdivision* of a graph G at an edge $e = uv$ is a graph $G(e)$ obtained from $G - e$ by adding a new vertex v_e and two new edges uv_e and $v_e v$. For a subset $X \subseteq E(G)$, we define $G(X)$ to be the graph obtained from G by elementarily subdividing every edge of X.

Lemma 4.1. *For an integer $k > 1$, if $G \in \mathcal{G}$ is a k-triangular simple graph and $X \subset E(G)$ with $|X| = s$ where $1 \leq s < k$, then $G - X \in \mathcal{T}_{k-s}$.*

Proof: Pick $e \in E(G - X)$. Since $G \in \mathcal{T}_k$, edge e lies in at least k distinct triangles in G, say $C_1^e, C_2^e, \ldots, C_k^e$. As $E(C_i^e \cap C_j^e) = \{e\}$ for each $\{i, j\} \subseteq [1, k]$ and $|X| = s < k$, there exist $k - s$ such triangles $C_{i'}^e$ where $i' \in [1, k]$ such that $E(C_{i'}^e) \cap X = \emptyset$. It follows that $G - X \in \mathcal{T}_{k-s}$. $\qquad \square$

Lemma 4.2. *Given two non-negative integers s and t. If $G \in \mathcal{G}$ is a $(s + t + 1)$-triangular simple graph, then G is (s, t)-supereulerian.*

Proof: For any $X, Y \subset E(G)$ with $X \cap Y = \emptyset$, $|X| = s_1 \leq s$ and $|Y| \leq t$. Then $|X \cup Y| \leq s + t$. Let $H = G - (X \cup Y)$. By Lemma 4.1, $H \in \mathcal{T}_1$. It follows that H is collapsible by Theorem 2.2(i). Let $X = \{x_1, x_2, \ldots, x_{s_1}\}$. Then $V(G(X)) = V(G) \cup \{v_{x_1}, v_{x_2}, \ldots, v_{x_{s_1}}\}$. Note that $G(X) - Y - \{v_{x_1}, v_{x_2}, \ldots, v_{x_{s_1}}\} = H$ is collapsible. Since every edge of $(G(X) - Y)/H$ lies in a cycle of length 2, which implies that $(G(X) - Y)/H$ is collapsible by Theorem 2.2(i). It indicates that $G(X) - Y$ is collapsible by Theorem 2.2(ii) as H is collapsible. Then $G(X) - Y$ is supereulerian, which means that $G(X) - Y$ has a spanning eulerian subgraph J. Note that $d_{G(X) - Y}(v_{x_i}) = 2$ for each $i \in [1, s_1]$.

Then subgraph J contains all edges incident with some v_{x_i}, which means that $G - Y$ has a spanning eulerian subgraph J' containing X, and so G is (s, t)-supereulerian. □

Proof of Theorem 1.3: Combine Theorem 3.1(ii), Theorem 3.5(ii) and Lemma 4.2, and then we complete the proof of it. □

Acknowledgements

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