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A Note on the Connectivity of Binary Matroids

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Abstract

In [J. Combinatorial Theory, Ser. B, 28 (1980), 305-359], Seymour introduced the binary matroid 3-sums and proved that if a 3-connected binary matroid M is a 3-sum of matroids M_1 and M_2, then each of M_1 and M_2 is isomorphic to a proper minor of M. For a 3-connected binary matroid M expressed as a 3-sum of M_1 and M_2, we show that in general, both M_1 and M_2 are 2-connected, and if M_1 and M_2 are simple matroids, then both M_1 and M_2 are also 3-connected.

1 Introduction

We consider finite binary matroids in this note. Undefined terms and notations can be found in [1]. Thus we use r_M, cl_M, $\mathscr{I}(M)$, $\mathscr{B}(M)$ and $\mathscr{C}(M)$ to denote the rank function, the closure operator, the collections of independent sets, bases and circuits of a matroid M, respectively. If $X \subseteq E$, then M/X and $M|X$ denotes the matroid contractions, matroid restrictions, respectively. Define $M - X = M|(E - X)$. A **cycle** of a matroid is a disjoint union of circuits of M, and we use $\mathscr{C}_0(M)$ to denote the collection of all cycles of M. For sets X and Y, the **symmetric difference** of X and Y is defined as $X \triangle Y = (X \cup Y) - (X \cap Y)$. It is known (for example, Theorem 9.1.2 of [1],) that $\mathscr{C}_0(M)$ with the symmetric difference is a vector space over $GF(2)$, the 2-element field.

Let $M_1 = (E_1, \mathscr{I}_1)$ and $M_2 = (E_2, \mathscr{I}_2)$ be binary matroids and $E = E_1 \triangle E_2$. Seymour in [2] showed that there is a matroid $M_1 \triangle M_2$ with ground set E with $\mathscr{C}_0(M) = \mathscr{C}_0(M_1) \triangle \mathscr{C}_0(M_2)$. Three special cases of $M_1 \triangle M_2$ are introduced by Seymour ([2] and [3]) as follows.
(S1) If $E_1 \cap E_2 = \emptyset$ and $|E_1|, |E_2| < |E_1 \triangle E_2|$, $M_1 \triangle M_2$ is a **1-sum** of M_1 and M_2, denoted by $M_1 \oplus M_2$.
(S2) If $|E_1 \cap E_2| = 1$ and $E_1 \cap E_2 = \{p\}$, say, and p is not a loop or coloop of M_1 or M_2, and $|E_1|, |E_2| < |E_1 \triangle E_2|$,

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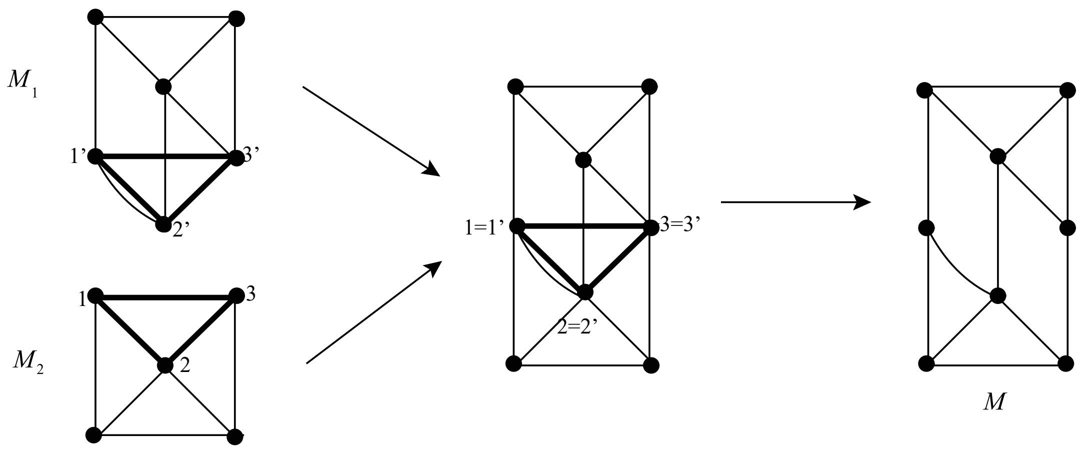

Fig. 1 M is a 3-sum of M_1 and M_2.

$M_1 \triangle M_2$ is a **2-sum** of M_1 and M_2, denoted by $M_1 \oplus_2 M_2$.
(S3) If $|E_1 \cap E_2| = 3$ and $E_1 \cap E_2 = Z$, and Z is a circuit of M_1 and M_2, and Z includes no cocircuit of either M_1 or M_2, and $|E_1|, |E_2| < |E_1 \triangle E_2|$, $M_1 \triangle M_2$ is a **3-sum** of M_1 and M_2, denoted by $M_1 \oplus_3 M_2$.

The following is known for matroid 2-sums.

Lemma 1. *(Proposition 7.1.22 of [1]) A 2-sum $M = M_1 \oplus_2 M_2$ is connected if and only if both M_1 and M_2 are connected.*

The similar conclusion may not be made for 3-sums of binary matroids. An example is presented in Figure 1, which is a slight modification of Figure 9.3 of [1]. In Figure 1, it is shown that while a binary matroid $M = M_1 \oplus_3 M_2$ is 3-connected, one of the summand M_1, as it contains a 2-circuit, is not 3-connected. Therefore, it is of interests to determine natural conditions that would assure both M_1 and M_2 are 3-connected when a binary matroid $M = M_1 \oplus_3 M_2$ is 3-connected. The purpose of this research is to investigate such conditions. The main result of this note is the following.

Theorem 2. *Let M be a simple 3-connected binary matroid and $M = M_1 \oplus_3 M_2$ is a 3-sum of two matroids. Each of the followings holds.*
(i) Both M_1 and M_2 are connected matroids.
(ii) If both M_1 and M_2 are simple, then each of M_1 and M_2 is 3-connected.

We present some preliminaries in Section 2 and prove the main result in Section 3.

2 Priliminaries

Following [1], the **connectivity function** $\lambda_M(X)$ of a matroid $M = (E, \mathscr{I})$ satisfies $\lambda_M(X) = r(X) + r(E - X) - r(M) = r_M(X) + r_{M^*}(X) - |X|$ for any subset $X \subseteq E(M)$. A partition $(X, E - X)$ of E is a k**-separation** of M if both $\lambda(X) < k$ and $\min\{|X|, |E - X|\} \geq k$. A subset $X \subset E$ is a **separator** if $(X, E - X)$ is a 1-separation. A matroid M is n**-connected** if M does not has a k-separation for any integer $k \leq n$.

Lemma 3. *(Proposition 2.1.11 of [1]) If C is a circuit and C^* is a cocircuit of the matroid M, then $|C \cap C^*| \neq 1$.*

Lemma 4. *(Lemmas 1.4.3 and 1.4.6 of [1]) Suppose $X \subseteq E$ and $x \in E$.*
(i) If $X \in \mathscr{I}(M)$ and $X \cup x \notin \mathscr{I}(M)$, then $x \in cl(X)$.
(ii) If $X \subseteq Y \subseteq E$, then $cl(X) \subseteq cl(Y)$.

Lemma 5. *Let $M = (E, \mathscr{I})$ be a matroid and $X \subset E$ be a proper nonempty subset of E. The followings are equivalent:*
(i) There exists a base B of M, such that $B \cap X \in \mathscr{B}(M|X)$ and $B - X \in \mathscr{B}(M - X)$.
(ii) For any two bases B_1, B_2 of M, $|B_1 \cap X| = |B_2 \cap X|$. (Equivalently, $|B_1 - X| = |B_2 - X|$.)
(iii) X is a separator of matroid M.

Proof. (i) $\Rightarrow$ (ii). Suppose there exists a base B of M such that $B \cap X \in \mathscr{B}(M|X)$ and $B - X \in \mathscr{B}(M - X)$. For any $B_1, B_2 \in \mathscr{B}(M)$, and for $i \in \{1,2\}$, as $B_i \cap X \in \mathscr{I}(M|X)$ and $B_i - X \in \mathscr{B}(M - X)$, it follows that

$$|B_i \cap X| \leq |B \cap X| \text{ and } |B_i - X| \leq |B - X|. \tag{1}$$

Thus

$$|B_i| = |B_i \cap X| + |B_i - X| \leq |B \cap X| + |B - X| = |B|.$$

As $|B_i| = |B|$, we conclude that $|B_1 \cap X| = |B \cap X| = |B_2 \cap X|$.

(ii) $\Rightarrow$ (iii). Assume that (ii) holds. Let $B_X \in \mathscr{B}(M|X)$ and $B_{E-X} \in \mathscr{B}(M - X)$. Augment B_X to a base $B_1 \in \mathscr{B}(M)$ and augment B_{E-X} to a base $B_2 \in \mathscr{B}(M)$. By (ii), $\lambda_M(X) = r_M(X) + r_M(E - X) - r(M) = |B_X| + |B_{E-X}| - |B_1| = 0$. As $X \notin \{\emptyset, E\}$, we have $\min\{|X|, |E - X|\} \geq 1$. By definition, X is a separator.

(iii) $\Rightarrow$ (i). Let X be a separator of M. Then $r_M(X) + r_M(E - X) = r(M)$. Pick a $B_X \in \mathscr{B}(M|X)$ and augment B_X to a $B \in \mathscr{B}(M)$. Then as $B - X \in \mathscr{I}(M - X)$ and $|B - X| = |B| - |B \cap X| = r(M) - r_M(X) = r_M(E - X)$, it follows that $B \cap X = B_X \in \mathscr{B}(M|X)$ and $B - X \in \mathscr{B}(M - X)$. ■

3 Proof of theorem 2

Let M be a simple 3-connected binary matroid and $M = M_1 \oplus_3 M_2$. Then by Lemma 1, both M_1 and M_2 are 2-connected matroid. Assume now that for some $i \in \{1,2\}$, M_i is simple. We are to show that M_i is 3-connected.

Proof. Assume that M_i is a simple matroid. Let $Z = E(M_1) \cap E(M_2) = \{z_1, z_2, z_3\}$. By the definition of a binary 3-sum, $Z \in \mathscr{C}(M_1) \cap \mathscr{C}(M_2)$. For some $i \in \{1,2\}$, if $X \subseteq E(M_i)$, then we denote $Y = E(M_i) - X$ throughout the proof. If $X \cap Z = \emptyset$, then direct computing yields

$$\begin{aligned} \lambda_{M_i}(X) &= r_{M_i}(X) + r_{M_i^*}(X) - |X| = r_{M_i - Z}(X) + r_{M_i^* - Z}(X) - |X| \\ &= r_{M-(E(M_{3-i})-Z)}(X) + r_{M^*-(E(M_{3-i})-Z)}(X) - |X| \\ &= r_M(X) + r_{M^*}(X) - |X| = \lambda_M(X). \end{aligned} \tag{2}$$

We argue by contradiction to show (i) and assume, for some $i \in \{1,2\}$, that M_i has a separator X. So $\min\{|X|, |Y|\} \geq 1$ and $\lambda_{M_i}(X) = 0$.

Suppose first that $X \cap Z = \emptyset$. By (2) and $\lambda_{M_i}(X) = 0$, it follows that X is a separator of M, contrary to the assumption that M is connected. Hence we must have $X \cap Z \neq \emptyset$. Similarly, we also have $Y \cap Z \neq \emptyset$.

By symmetry, we may assume that $\{z_1, z_2\} \subseteq X$ and $z_3 \in Y$. Augment the independent set $\{z_1, z_2\}$ to be a base $B_X \in \mathscr{B}(M_i|X)$, and then augment B_X to be a base $B \in \mathscr{B}(M_i)$. As Z is a circuit, $z_3 \notin B$, and so Z must be the fundamental circuit in $B \cup z_3$. It follows that $B' = (B - z_1) \cup z_3 \in \mathscr{B}(M_i)$ and $|B \cap X| = |B' \cap X| + 1$. By Lemma 5, X can not be a separator of M_i, contrary to the assumption. This proves Theorem 2(i).

To show (ii), we assume that both M_1 and M_2 are simple, to show that for each $i \in \{1,2\}$, and for arbitrarily chosen proper nonempty subset $X \subset E(M_i)$ with $Y = E(M_i) - X$, satisfying $\min\{|X|, |Y|\} \geq 2$, we always have $\lambda_{M_i}(X) \geq 2$.

Claim 1. If $X \cap Z = \emptyset$, then $\lambda_{M_i}(X) \geq 2$.

By contradiction, assume that $X \cap Z = \emptyset$ and $\lambda_{M_i} < 2$. By (2), we have $\lambda_M(X) = \lambda_{M_i}(X) < 2$, and $|E(M) - X| \geq |Y - Z| + |E(M_{3-i}) - Z| \geq 2$. It follows that $(X, E(M) - X)$ is a 2-separation of M, contrary to the assumption that M is 3-connected. This proves Claim 1.

By Claim 1 and by symmetry, we may assume that both $X \cap Z \neq \emptyset$ and $Y \cap Z \neq \emptyset$. Thus we may again assume that $\{z_1, z_2\} \subseteq X$ and $z_3 \in Y$. Augment the independent set $\{z_1, z_2\}$ to be a base $B_X \in \mathscr{B}(M_i|X)$, and then augment B_X to be a base $B \in \mathscr{B}(M_i)$. As $z_3 \notin B$, Z must be the fundamental circuit in $B \cup z_3$, and so $B' = (B - z_1) \cup z_3 \in \mathscr{B}(M_i)$ with

$$|B \cap X| = |B' \cap X| + 1. \tag{3}$$

Claim 2. $r_{M_i}(Y) \geq 2$ and $r_{M_i^*}(Y) \geq 2$.

Let $y \in Y - z_3$ be an element. By Theorem 2(i), $\{y\}$ is not a separator of M_i, and so $1 \leq \lambda_{M_i}(y) = r_{M_i}(y) + r_{M_i^*}(y) - |y| \leq 1 + 1 - 1 = 1$, forcing $\lambda_{M_i}(y) = r_{M_i}(y) = r_{M_i^*}(y) = 1$. Since Z is a circuit of M_i, it follows by Lemma 3 that $\{z_3, y\}$ is coindependent in M_i. Since M_i is simple matroid, $\{z_3, y\}$ is independent in M_i. This proves that $r_{M_i}(Y) \geq |\{z_3, y\}| \geq 2$ and $r_{M_i^*}(Y) \geq |\{z_3, y\}| \geq 2$, and so Claim 2 holds.

If $|Y| = 2$, then by Claim 2, we have $\lambda_{M_i}(X) = \lambda_{M_i}(Y) = r_{M_i}(Y) + r_{M_i^*}(Y) - |Y| = 2 + 2 - 2 = 2$, a contradiction. Therefore, throughout the rest of the arguments, we assume that $|Y| \geq 3$.

Claim 3. $B' \cap Y \notin \mathscr{B}(M_i|Y)$.

By contradiction, assume that $B' \cap Y \in \mathscr{B}(M_i|Y)$. Let $Y' = Y - z_3$ and $X' = E(M_i) - Y'$. Then $X' = X \cup z_3$, $|Y'| \geq 2$ and $Y' \subseteq E(M_i) - Z$. By rank function properties, we have

$$r_{M_i}(Y') \leq r_{M_i}(Y) \leq r_{M_i}(Y') + 1 \text{ and } r_{M_i}(Y) - 1 \leq r_{M_i}(Y') \leq r_{M_i}(Y). \tag{4}$$

Since $Z \in \mathscr{C}(M_i)$ and $Z - X = \{z_3\}$, it follows that $r_{M_i}(X \cup \{z_3\}) = r_{M_i}(X) = |B_X| = |B \cap X|$. This, together with (4) and $\lambda_{M_i}(Y') = r_{M_i}(X') + r_{M_i}(Y') - r(M_i)$, implies that

$$r_{M_i}(X) + r_{M_i}(Y) - 1 - r(M_i) \leq \lambda_{M_i}(Y') \leq r_{M_i}(X) + r_{M_i}(Y) - r(M_i). \tag{5}$$

As $r_{M_i}(X) = |B \cap X|$, $r_{M_i}(Y) = |B' \cap Y|$ and $r(M_i) = |B| = |B'|$, and by (3), we have $r_{M_i}(X) + r_{M_i}(Y) - r(M_i) = |B \cap X| + |B' \cap Y| - |B'| = |B \cap X| - |B' \cap X| = 1$. This, together with (5), implies that $0 \leq \lambda_{M_i}(Y') \leq 1$. On the other hand, by $Y' \subseteq E(M_i) - Z$ and by Claim 1, we have $\lambda_{M_i}(Y') \geq 2$. It is a contradiction. This proves Claim 3.

By Claim 3, and as $|Y| \geq 3$. there exists an element $y' \in Y - B'$, such that the fundamental circuit $C_{M_i}(y', B')$ in $B' \cup y'$ contains elements not in Y, and so $C_{M_i}(y', B') \cap X \neq \emptyset$. Let $x \in C_{M_i}(y, B') \cap X$. Then $B'' = (B' - x) \cup y \in \mathscr{B}(M_i)$, and $|B'' \cap X| = |B' \cap X| - 1 = |B \cap X| - 2$. It follows that

$$\begin{aligned} r(M_i) &= |B''| = |B'' \cap X| + |B'' \cap Y| = |B \cap X| - 2 + |B'' \cap Y| \\ &= r_{M_i}(X) - 2 + |B'' \cap Y| \leq r_{M_i}(X) - 2 + r_{M_i}(Y). \end{aligned}$$

Thus $\lambda_{M_i}(X) \geq 2$. It follows that M_i does not have any 2-separations and so M_i is 3-connected. This completes the proof. ■

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