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Perspectives

Graph r-hued colorings—A survey

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ABSTRACT

A (k, r)**-coloring** of a graph G is a proper k-vertex coloring of G such that the neighbors of each vertex of degree d will receive at least $\min\{d, r\}$ different colors. The r**-hued chromatic number**, denoted by $\chi_r(G)$, is the smallest integer k for which a graph G has a (k, r)-coloring. This article is intended to survey the recent developments on the studies related to this r-hued colorings. Emphases are on the r-hued colorings of planar graphs, graph families with forbidden minors, and sparse graphs, as well as on the comparison between the r-hued chromatic number and the chromatic number of a graph, and the sensitivity studies of the r-hued chromatic number. It also surveys other related results on r-hued colorings and list r-hued colorings.

Contents

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1. Introduction

Graph coloring has been one of the most fascinating and motivating problems in graph theory. Many of the research topics and methods have been developed by in the long journey of turning the four color conjecture into the four color theorem. As commented by Tutte [131], "The Four Colour Theorem is the tip of the iceberg, the thin end of the wedge and the first cuckoo of spring". As of today, there have been many variations of graph coloring problems under intensive studies. Our objectives are to survey results and developments on the r-hued colorings of graphs.

Unless otherwise specified, graphs considered in this survey are simple undirected finite graphs. Terms and notation not defined in the survey will follow those in [18]. As in [18], $\delta(G)$, $\Delta(G)$, $\kappa(G)$, $\kappa'(G)$ and $\chi(G)$ denote the minimum degree, the maximum degree, the connectivity, the edge-connectivity, and the chromatic number of a graph G, respectively. Throughout the survey, a cycle on n vertices is denoted by C_n and often referred as an n-cycle. The **girth** of G, denoted by $g(G)$, is the smallest n such that G has an n-cycle. For graph H and G, we write $H \subseteq G$ to mean that H is a subgraph of G. The **maximum average degree** of a graph G is defined as

$$mad(G) = \max\left\{ \frac{\sum_{v\in V(H)} d_H(v)}{|V(H)|} \ :\ H \text{ is a subgraph of } G \right\}. \tag{1}$$

For a vertex $v \in V(G)$, the **neighborhood** of v in G is $N_G(v) = \{u \in V(G) : u \text{ is adjacent to } v \text{ in } G\}$. Vertices in $N_G(v)$ are called **neighbors of** v.

We use $\mathbb{N}$ to denote the set of all positive integers, and often use elements in $\mathbb{N}$ as colors. Let $2^{\mathbb{N}}$ denote the collection of subsets of $\mathbb{N}$. For $k \in \mathbb{N}$, define $\overline{k} = \{1, 2, \dots, k\}$. The concept of r-hued coloring was first initiated in the dissertation of Bruce Montgomery [110] and in the paper [89] for the special case when $r = 2$. In this early stage, a 2-hued coloring is called a **dynamic coloring**. The first research paper on r-hued coloring for generic values of r appeared in [87], where the r-hued coloring was called a **conditional coloring**. In later studies, the terminology has not been unified, but most people are using either r-hued colorings or r-dynamic colorings.

We present the formal definition of r-hued colorings of graphs. For an integer $k > 0$, if $c : V(G) \to \overline{k}$ is a mapping, and if $S \subseteq V(G)$, then define $c(S) = \{c(u) : u \in S\}$.

Definition 1.1. Let k and r be positive integers. A (k, r)-**coloring** of a graph G is a mapping $c : V(G) \to \overline{k}$ satisfying both the proper coloring condition (C1) and the r-hued coloring condition (C2), as follows:
(C1) if $u, v \in V(G)$ are adjacent in G, then $c(u) \neq c(v)$;
(C2) for any $v \in V(G)$, $|c(N_G(v))| \geq \min\{|N_G(v)|, r\}$.

Following [18], a mapping $c : V(G) \to \overline{k}$ satisfying (C1) only is a proper k-coloring of G, and $\chi(G)$, the chromatic number of G, is the smallest integer k such that G has a proper k-coloring. Likewise, the r-**hued chromatic number** of G, denoted by $\chi_r(G)$, is the smallest k such that G has a (k, r)-coloring. It follows from the definitions of $\chi(G)$ and $\chi_r(G)$ that $\chi(G) = \chi_1(G)$, and so $\chi_r(G)$ can be viewed as a generalization of the classical graph coloring. Since any (k, i)-coloring of G is also a (k, j)-coloring of G for any integers $i > j > 0$, it has been firstly observed in [87] that

$$\chi(G) = \chi_1(G) \leq \chi_2(G) \leq \cdots \leq \chi_{r-1}(G) \leq \chi_r(G) \leq \cdots \leq \chi_{\Delta(G)}(G) = \chi_{\Delta(G)+1}(G) = \cdots . \tag{2}$$

There is also a list coloring version of r-hued colorings. We follow [18] to define list colorings of graphs. A list of a graph G is an assignment $L : V(G) \to 2^{\mathbb{N}}$ that assigns every $v \in V(G)$ a list $L(v)$ of colors available at v. An L-**coloring** is a mapping $c : V(G) \to \mathbb{N}$ satisfying the proper coloring condition (C1) in Definition 1.1, and the following list coloring condition:
(C3) $c(v) \in L(v)$, for every $v \in V(G)$.

If G has an L-coloring, then G is L-**colorable**. For an integer $k \in \mathbb{N}$, a list L of a graph G is a k-**list** if $|L(v)| = k$ for any $v \in V(G)$. A graph G is k-**list-colorable** if for any k-list L of G, G is L-colorable. The smallest integer k such that G is k-list-colorable is called the **list chromatic number** of G, and is denoted by $\chi_L(G)$.

The list 2-hued coloring of graphs is first introduced in [4], under the name of list dynamic coloring of graphs. In [30], the list r-hued coloring of graphs for generic values of r is formally introduced. Let $r \in \mathbb{N}$ be an integer. For a given assignment $L : V(G) \to 2^{\mathbb{N}}$ in a graph G, an (L, r)-**coloring** c is a mapping $c : V(G) \to \mathbb{N}$ satisfying both (C1) and (C2) in Definition 1.1, as well as the list coloring condition (C3). The **list r-hued chromatic number**, denoted by $\chi_{L,r}(G)$, is the smallest integer k such that for any k-list L of G, G has an (L, r)-coloring. By definition,

$$\chi_L(G) = \chi_{L,1}(G) \leq \cdots \leq \chi_{L,r-1}(G) \leq \chi_{L,r}(G) \leq \cdots \leq \chi_{L,\Delta(G)}(G) = \chi_{L,\Delta(G)+1}(G) = \cdots . \tag{3}$$

It also follows from the definitions that for every graph G and any positive integer r, one always has $\chi_r(G) \leq \chi_{L,r}(G)$.

The inequalities in (2) and (3) indicate that r-hued colorings and list r-hued colorings are closely related to the 2-distance colorings of graphs, introduced by Kramer and Kramer [78,79], and the list 2-distance k-coloring of graphs, respectively. For a graph G, a **2-distance k-coloring** of G is a mapping $c : V(G) \to \overline{k}$ satisfying (C1) and the following:
(C4) If $u, v \in V(G)$ are of distance 2 in G, then $c(u) \neq c(v)$.
The paper [80] is a very resourceful survey on 2-distance colorings of graphs.

The 2-**distance chromatic number** of G is the smallest integer k such that G has a 2-distance k-coloring. The **list 2-distance k-coloring** and **list** 2-**distance chromatic number** of G can be similarly defined. It follows from the definitions that for a graph G with maximum degree $\Delta(G)$, the 2-distance chromatic number of G equals $\chi_{\Delta(G)}(G)$ and the list 2-distance chromatic number of G equals $\chi_{L,\Delta(G)}(G)$.

Another way to view 2-distance coloring of graphs is to consider the coloring of the powers of a graph. The **square of a graph** G, denoted by G^2, has $V(G^2) = V(G)$, where $uv \in E(G^2)$ if and only if the distance between u and v in G is at most 2. By definition, $\chi(G^2) = \chi_{\Delta(G)}(G)$ and $\chi_L(G^2) = \chi_{L,\Delta(G)}(G)$. For any integer $\ell > 0$, the concept of the square of a graph has been extended to the **ℓth power of a graph** G, denoted by G^ℓ, which has $V(G^\ell) = V(G)$, where two vertices u and v are adjacent in G^ℓ if and only if the distance between u and v in G is at most ℓ. There have been intensive studies in 2-distance coloring of a graph G and in coloring the powers of a graph G, as seen in [1,11,15,16,19–22,25,26,28,33,35–37,43–45,58,59,61,62,65,76–79,95,96,104,107,109,120,138–140,144], among others.

Another motivation of r-hued colorings of graphs arises from the communication problem in the deployment of multi-agent systems in optimal reconfigurations of electric power networks. An agent is a computer system that is capable of autonomous action in this environment in order to meet its designed objectives. Autonomy means that the components in an environment function are solely under their own control. Using multiagent systems in optimal reconfigurations of electric power networks in conjunction with graph algorithms has been widely applied in recent years, as seen in [34,51,52,111,130], among others. In a graph modeling of an multiagent system deployed in an optimal reconfiguration process of power networks, a graph model is formed to mimic the communicating and working relationship of the agents, in which agents are modeled as vertices and two vertices are adjacent if the corresponding agents need to communicate at work. In order to perform an optimal reconfiguration, each agent needs to acquire a certain but different kinds of information from its neighboring agent sensors, and use the information to make its own decisions and to take actions. In the modeling, agents are communicating in a wireless way. Each agent can receive signals with different frequencies but can only have one emission frequency. It is required that neighboring agents must have different emission frequencies and, for a fixed integer $r > 0$, every agent needs to receive at least r kinds of different information from its neighboring agents, which are distinguished by their emission frequencies. This amounts to requiring that every agent must have neighbors using at least r different emission frequencies. To assign emission frequencies to the agents, it is natural to consider it as a graph r-hued coloring problem with emission frequencies being colors.

2. Basic properties

It is observed in [87] that for any graph G, the following always holds:

$$\min\{r, \Delta(G)\} + 1 \leq \chi_r(G) \leq \chi_{L,r}(G) \leq |V(G)|. \tag{4}$$

Thus, graphs reaching either inequalities in (4) would be of interest. Theorem 2.1 shows that all trees reach the lower bound and all complete graphs reach the upper bound in (4).

Theorem 2.1 (*Lai, Lin, Montgomery, Shui and Fan [87]*)**.** *Each of the following holds:*
(i) $\chi_r(K_n) = n$.
(ii) Let $n \geq 3$ be an integer and C_n be the cycle of order n. If $r \geq 2$, then

$$\chi_r(C_n) = \begin{cases} 5, & \text{if } n = 5; \\ 3, & \text{if } n \equiv 0 \pmod 3; \\ 4, & \text{otherwise.} \end{cases}$$

(iii) If G is a tree, then $\chi_r(G) = \min\{r, \Delta(G)\} + 1$.
(iv) Suppose that $m \geq n \geq 2$, then $\chi_r(K_{m,n}) = \min\{2r, n+m, r+n\}$.
(v) If $k \geq r+1$, then $\chi_r(K_{i_1,\ldots,i_k}) = k$.

The list coloring version of Theorem 2.1(ii) is straightforwardly extended by Akbari et al. in [4] when $r = 2$.

Theorem 2.2 (*Akbari, Ghanbari and Jahanbekam [4]*)**.** *Let $n \geq 3$ be an integer and C_n be the cycle of order n. Then*

$$\chi_{L,2}(C_n) = \begin{cases} 5, & \text{if } n = 5; \\ 3, & \text{if } n \equiv 0 \pmod 3; \\ 4, & \text{otherwise.} \end{cases}$$

Additional studies on conditions for a graph with the property $\chi_r(G) = |V(G)|$ are conducted in [87].

Theorem 2.3 (*Lai, Lin, Montgomery, Shui and Fan [87]*)**.** *Let G be a connected graph and $|V(G)| = n$. Each of the following holds.*
(i) For any $r \geq 2$, $\chi_r(G) = |V(G)|$ if and only if any two nonadjacent vertices of G are adjacent to a vertex of degree at most r.
(ii) Let $r > 0$ be an integer. If $\chi_r(G) = |V(G)|$, then either $G = K_n$ or $n \leq r^2+1$. Furthermore, if G is incomplete with $n = r^2+1$, then G must be r-regular.

In [89], the problem whether the difference between $\chi_2(G) - \chi(G)$ can be bounded was raised. Following [56], for a graph G, the **subdivision graph** of G, denoted by $S(G)$, is the graph obtained from G by subdividing each of its edges exactly once (i.e. by replacing each edge by a path of length two). The new vertices are called the **middle vertices** of $S(G)$, and the other vertices are called the **original vertices**. It is shown in [89] that in any 2-hued coloring of $S(K_n)$, all the original vertices must be colored with mutually distinct colors, and since $S(K_n)$ is bipartite, with the set of all middle vertices and the set of all original vertices being the vertex bipartition, $\lim_{n\to\infty} \chi_2(S(K_n)) - \chi(S(K_n)) = \infty$. So even among bipartite graphs G, the gap between $\chi_2(G) - \chi(G)$ is unbounded.

Determining $\chi_r(G)$ or $\chi_{L,r}(G)$ has always been the main objective in the study. Akbari et al. in [4] conjectured that for any graph G, one always has $\chi_{L,2}(G) = \max\{\chi_L(G), \chi_2(G)\}$. This conjecture is disproved by Esperet in [50], who proved the following.

Theorem 2.4 (*Esperet [50]*)**.** *Each of the following holds.*
(i) There exists a planar bipartite graph G with $\chi_L(G) = \chi_2(G) = 3$ and $\chi_{L,2}(G) = 4$.
(ii) For any integer $k \geq 5$, there exists a bipartite graph G_k with $\chi_L(G_k) = \chi_2(G_k) = 3$ and $\chi_{L,2}(G_k) \geq k$.

In [50], Esperet utilized the subdivision graph $S(G)$. He considered a (possibly improper) coloring of the edges of G such that the set of edges incident to any vertex of degree more than one contains at least two distinct colors. Define $ch_2^*(G)$ as the smallest integer k such that if every edge of G is given a list of k colors, G has such a coloring with the additional property that every edge is assigned a color from its list. In [50], Esperet proved the following.

Theorem 2.5 (*Esperet [50]*)**.** *For any graph G, $\chi_{L,2}(S(G)) \leq \max\{\chi_L(G), ch_2^*(G) + 2\}$. In particular, $\chi_L(G) \leq \chi_{L,2}(S(G)) \leq \max\{5, \chi_L(G)\}$.*

There are other elementary studies in [64,117], investigating the r-hued chromatic number of paths and cycles, and repeating results are stated in Theorems 2.1–2.3.

2.1. Brooks-type theorems and generic upper bounds

Brooks' Theorem [24], see also Theorem 14.4 of [18] on graph colorings states that a connected graph G satisfies $\chi(G) \leq \Delta(G) + 1$, where equality holds if and only if G is a complete graph or an odd cycle. Upper bounds of the r-hued chromatic number for generic graphs have also been intensively studied. The following is the first to appear in Montgomery's dissertation [110] as well as in [89].

Theorem 2.6 (*Lai, Montgomery and Poon [89]*)**.** *Let G be a connected graph.*
(i) If $G \neq C_5$ and $\Delta(G) \leq 3$, then $\chi_2(G) \leq 4$.
(ii) If $\Delta(G) \geq 4$, then $\chi_2(G) \leq \Delta(G) + 1$.

Theorem 2.6 is straightforwardly extended to its list coloring version in [4] later.

Theorem 2.7 (*Akbari, Ghanbari and Jahanbekam [4]*)**.** *Let G be a connected graph.*
(i) If $G \neq C_5$ and $\Delta(G) \leq 3$, then $\chi_{L,2}(G) \leq 4$.
(ii) If $\Delta(G) \geq 4$, then $\chi_{L,2}(G) \leq \Delta(G) + 1$.

The graphs reaching the upper bounds in either Theorem 2.6 or Theorem 2.7 seem to be difficult to be determined. Efforts in this direction have been made. Introduced by Hoffman and Singleton in [63], a **Moore graph** is a regular graph of degree d and diameter k, for some positive integers d and k, whose number of vertices equals the upper bound

$$1 + d\sum_{i=0}^{k-1}(d-1)^i.$$

For a connected graph G, Lai et al. [87] proved that if $\Delta(G) \leq r$ for any integer $r \geq 2$, then $\chi_r(G) \leq \Delta(G) + r^2 - r + 1$. Ding et al. [42] found that the best upper bound can be achieved by a Moore graph when $r = \Delta(G)$. Lin and Wang [97,98] improved the results as follows, which seems to be the only result of this kind that has the extremal graphs determined.

Theorem 2.8 (*Lin [97], Lin and Wang [98]*)**.** *Let G be a connected graph. $\chi_r(G) \leq r\Delta(G) + 1$, where the equality holds if and only if $r = \Delta(G)$ and G is a Moore graph.*

Karpov [70] also proved the following Brooks' Theorem type results.

Theorem 2.9 (*Karpov [70]*). *Let G be a connected graph that does not have a vertex of degree 2. If* $\Delta(G) \geq 8$, *then* $\chi_2(G) \leq \Delta(G)$.

Theorem 2.10 (*Karpov [70]*). *Let* $d \geq 8$ *be an integer. Each of the following holds.*
(i) If $G = K_{d+1}$ *or G is a subdivision of a* K_{d+1}, *then* $\chi_2(G) = d + 1$.
(ii) If $\Delta(G) \leq d$, $G \neq K_{d+1}$ *and G is not a subdivision of* K_{d+1}, *then* $\chi_2(G) \leq d$.

2.2. The r-hued chromatic number of certain graph families

Let k and n be positive integers with $n > 2k$. We use $\mathbb{Z}_n$ to denote the set of integers modulo n as well as the corresponding additive cyclic group. As in [18], the **generalized Petersen graph** $P_{k,n}$ is the simple graph with vertices $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$, and edges $x_i x_{i+1}, y_i y_{i+k}, x_i y_i$ where $i \in \mathbb{Z}_n$. The graph $P_{2,5}$ is known as the **Petersen graph**.

Li, Fan and Zhong [90] determined the best lower bound for the 3-hued chromatic number of the generalized Petersen graphs. As every $P_{k,n}$ is a 3-regular graph, by (4) and Theorem 2.8, it is known that $4 \leq \chi_3(P_{k,n}) \leq 10$. By Theorem 2.8, and as $P_{2,5}$ is a Moore graph, the upper bound can be achieved by $\chi_3(P_{2,5}) = 10$. Cranston and Kim [37] proved that except for $P_{2,5}$, the best upper bound for $\chi_3(P_{k,n})$ is 8.

Theorem 2.11. *For any positive integers* k, n *with* $n > 2k$, *each of the following holds.*
(i) (Li, Fan and Zhong [90]) $\chi_3(P_{k,n}) \geq 4$ *and the bound is best possible;*
(ii) (Cranston and Kim [37]) If $k \neq 2$ *or* $n \neq 5$, *then* $\chi_3(P_{k,n}) \leq 8$ *and the bound is best possible.*

Following [18], a **Halin graph** is a graph $H := T \cup C$, where T is a plane tree on at least four vertices in which no vertex has degree two, and C is a cycle connecting the leaves of T in the cyclic order determined by the embedding of T. Thus, a Halin graph G is formed by an external cycle C and a tree T bounded inside the cycle. Liu and Zhang in [101] first introduced pseudo-Halin graphs, as a generalization of Halin graphs. A 2-connected planar graph G with $\delta(G) \geq 3$ is a **pseudo-Halin** graph if for some face f_0 of G, $G - E(f_0)$ is a tree. Meng et al. [106] studied the 2-hued chromatic number of all Pseudo-Halin graphs.

Theorem 2.12 (*Meng, Miao, Su and Li [106]*). *Let G be a pseudo-Halin graph. Then* $\chi_2(G) \leq 4$. *Moreover, this bound is best possible.*

Motivated by a list coloring theorem in [8], Akbari et al. in [5] proved the following result by constructing a (4, 2)-coloring for graphs satisfying the conditions in the theorem below.

Theorem 2.13 (*Akbari, Ghanbari, Jahanbekam and Jamaali [5]*). *Let G be a connected graph and* $\ell \geq 3$ *be an integer.*
(i) If the length of every cycle is divisible by 3, then $\chi_2(G) \leq 3$.
(ii) If G is not isomorphic to C_5 *and the length of every cycle of G is divisible by* ℓ, *then* $\chi_2(G) \leq 4$.

Yang et al. [139] studied r-hued chromatic numbers of powers of graphs, including paths, trees and cycles.

Theorem 2.14 (*Yang, Deng and Shao [139]*). *Let* k, ℓ, m, n *be positive integers with* $n \geq 3$. *Each of the following holds.*
(i)

$$\chi(P_n^2) = \chi_2(P_n^2) = 3.$$

$$\chi_3(P_n^2) = \begin{cases} 3, & \text{if } n = 3; \\ 4, & \text{otherwise.} \end{cases}$$

and for $r \geq 4$,

$$\chi_r(P_n^2) = \begin{cases} 3, & \text{if } n = 3; \\ 4, & \text{if } n = 4; \\ 5, & \text{otherwise.} \end{cases}$$

(ii) If $3 \leq \ell \leq n - 1$, *then* $\chi(P_n^\ell) = \chi_2(P_n^\ell) = \ell + 1$.
(iii) If T is a tree with $|V(T)| \geq 5$ *and* $T \neq P_n$, *then*

$$\chi_r(T^2) \leq \begin{cases} \Delta(T) + 1, & \text{if } r \leq \Delta(T); \\ \Delta(T) + 2, & \text{if } r = \Delta(T) + 1; \\ \min\{2\Delta(T) + 1, r\}, & \text{otherwise.} \end{cases}$$

(iv) If $m \geq 3$, *then*

$$\chi(C_m^2) = \chi_2(C_m^2) = \begin{cases} 3, & \text{if } m \equiv 0 \pmod 3; \\ 5, & \text{if } m = 5; \\ 4, & \text{otherwise.} \end{cases}$$

$$\chi_3(C_m^2) = \begin{cases} 3, & \text{if } m = 3; \\ 4, & \text{if } m \notin \{3, 5, 6, 7, 11\}; \\ 5, & \text{otherwise.} \end{cases}$$

and for $r \geq 4$,

$$\chi_r(C_m^2) = \begin{cases} m, & \text{if } 3 \leq m \leq 9; \\ 5, & \text{if } m \equiv 0 \pmod 5; \\ 7, & \text{if } m \in \{13, 14, 19\}; \\ 6, & \text{otherwise}. \end{cases}$$

(v)[1] *Let t be an integer. If $m > \ell + 1 \geq 4$ with $m = k(\ell + 1) + t$ and $0 \leq t \leq \ell$, then for $1 \leq r \leq 2$,*

$$\chi_r(C_m^\ell) = \begin{cases} \ell + 1 + \lfloor \frac{t}{k} \rfloor, & \text{if } t \equiv 0 \pmod k; \\ \ell + 2 + \lfloor \frac{t}{k} \rfloor, & \text{otherwise}. \end{cases}$$

The r-hued chromatic numbers of a number of special graphs were obtained by using elementary and straightforward coloring constructions in [39,115–117,132]. Reddy and Iyer [115–117] studied a few classes of graphs, including the windmill graphs, line graphs of windmill graphs, middle graphs of friendship graphs, middle graphs of a cycle, line graphs of friendship graphs, middle graphs of complete k-partite graphs, middle graphs of a bipartite graph, squares of cycles, among others. Dafik et al. [39] focused on the prism graphs, three-cyclical ladder graphs and circulant graphs, while Vivin et al. [132] studied 2-hued chromatic numbers of the middle graphs, total graphs, central graphs of a cycle, a path or an n-sunlet graph, respectively. They also proved a list of results on the 2-hued chromatic number of the Mycielskian graphs of a cycle or a path, and line graphs of an n-sunlet graph. It is of interest to discover more general results on those graphs.

2.3. Complexity and algorithms

It is known ([GT4] of [54]) that determining if a graph is 3-colorable is an NP-complete problem. Thus, it is natural to consider the complexity for r-hued colorings. Li and Zhou [94] and Li et al. [92] proved that the same could also be said for generic r-hued colorings. In these papers, it is shown that (3,2)-colorability remains NP-complete when restricted to planar bipartite graphs with maximum degree at most 3 and arbitrarily high girth. This differs considerably from the well-known result that classical 3-colorability is polynomially solvable for graphs with maximum degree at most 3.

Theorem 2.15 (*Li and Zhou [94]*)**.** *For $K_{1,3}$-free graphs G with $\Delta(G) = 3$, $(3, 2)$-colorability is NP-complete.*

Theorem 2.16 (*Li, Yao, Zhou and Broersma [92]*)**.** *Each of the following holds.*
(i) For every fixed integers k and r with $2 \leq r < k$, (k, r)-colorability is NP-complete.
(ii) The problem $(3, 2)$-colorability remains NP-complete for planar bipartite graphs with maximum degree at most 3 and arbitrarily high girth.
(iii) Within the class of graphs with $\Delta(G) = 3$, it is NP-hard to determine whether $\chi_2(G) = 3$ or $\chi_2(G) = 4$.
(iv) The problem $(3, 2)$-colorability remains NP-complete when restricted to hamiltonian graphs with $\Delta(G) \leq 6$.
(v) The problem $(3, 2)$-colorability is NP-complete for planar hamiltonian graphs.

Nevertheless, heuristic and bionic algorithms for r-hued colorings have been investigated in [99], in which such algorithms for generalized Petersen graphs and other family of graphs are presented.

3. Planar graphs and graphs with forbidden minors

The four color problem has been one of the most fascinating problems in graph theory. Even today, after the four color problem has become the Four-Color Theorem, the interests on planar graph coloring problems continue to draw the attention of many researchers. Determining best possible upper bounds of the r-hued chromatic number of planar graphs is undoubtedly one of the most focused problems.

3.1. Planar graphs

Without turning to the Four-Color Theorem, Chen et al. [30] applied the Lebesgue distributions [113] to prove that for a connected planar graph G, $\chi_{L,2}(G) \leq 5$ if $\Delta(G) \leq 4$; $\chi_{L,2}(G) \leq 6$; $\chi_2(G) \leq 5$ and in which the bound of $\chi_2(G)$ is sharp as $\chi_2(C_5) = 5$. They also conjectured that C_5 is the only planar graph G with $\chi_2(G) = 5$. With a smart application of the Four-Color Theorem, this conjecture was proved by Kim et al. in [72], and they also improved the results by Chen et al. as follows.

Theorem 3.1 (*Kim, Lee and Park [72]*)**.** *Let G be a connected planar graph. Each of the following holds.*
(i) If $G \neq C_5$, then $\chi_2(G) \leq 4$.
(ii) $\chi_{L,2}(G) \leq 5$.

[1] The results from [139] are summarized in (v) by authors.

By (4), if $r \geq 2$, then $\chi_r(G) = 2$ for a connected graph G if and only if $G = K_2$. Theorem 3.1 indicates that if G is a connected planar graph other than C_5, then $\chi_2(G) \leq 4$. Therefore, characterizing planar graphs G with $\chi_2(G) = 3$ and $\chi_2(G) = 4$ would be of interest. In general, for any integer $r \geq 2$, the problem of characterizing all graphs G with $\chi_r(G) = 3$ remains to be investigated. However, as indicated in Theorems 2.15 and 2.16, such problems will be very difficult. Thus, for some common and well-attended graph families $\mathcal{F}$, characterizing all graphs G in $\mathcal{F}$ with $\chi_r(G) = 3$ seems quite interesting. For values $r \geq 2$, it is of interest to determine the smallest integers $p(r)$ and $p_L(r)$ such that for any graph G in $\mathcal{F}$, $\chi_r(G) \leq p(r)$ and $\chi_{L,r}(G) \leq p_L(r)$. In particular, when $\mathcal{F}$ denotes the family of all planar graphs, there have been many researches towards this end, as seen in the following.

Theorem 3.2. *Let G be a connected planar graph. Each of the following holds.*
(i) (Qi, Li and Li [114]) $\chi_3(G) \leq 12$.
(ii) (Thomassen [129]) If G is a cubic graph, then $\chi_3(G) \leq 7$. Moreover, The upper bound 7 cannot be replaced by 6.
(iii) (Loeb, Mahoney, Reiniger and Wise [102]) $\chi_3(G) \leq \chi_{L,3}(G) \leq 10$.
(iv) (Asayama, Kawasaki, Kim, Nakamoto and Ozeki [12]) If G is a planar triangulation, then $\chi_3(G) \leq 5$. The bound is best possible.

For a cubic graph G, utilizing $\chi_3(G) = \chi(G^2)$, Thomassen's proof of Theorem 3.2(ii) is based on a decomposition method: color the vertices of the planar cubic graph by two colors, red and blue, such that the induced subgraph of the square-graph by blue vertices is 3-colorable, and the induced subgraph of the square-graph by red vertices is planar and hence by the Four-Color Theorem, is 4-colorable.

As a common extension of Theorem 3.2 (iii) and (iv), Gu et al. [55] considered planar near-triangulations. A **planar near-triangulation** is a planar graph with a plane embedding in which all bounded faces are 3-cycles.

Theorem 3.3 (*Gu, Kim, Ma and Shi [55]*). *If G is a planar near-triangulation, then $\chi_{L,3}(G) \leq 6$.*

A graph G is called **subcubic** if $\Delta(G) \leq 3$. The following conjecture is proposed.

Conjecture 3.4 (*Dvořák, Škrekovski and Tancer [46]*). *Let G be a planar subcubic graph. If G is triangle-free, then $\chi_{\Delta}(G) \leq 6$.*

It is indicated in [46] that the upper bound in Conjecture 3.4 is tight, as shown by the example in Fig. 1.

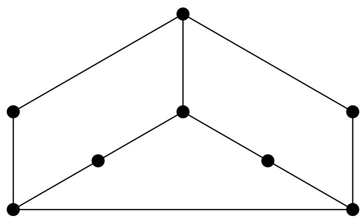

Fig. 1. An example for Conjecture 3.4.

For generic values of r, a number of results have also been obtained. Some results on bounded girth conditions can be found in Table 1 on sparse graphs.

Theorem 3.5 (*Song and Lai [123]*). *Let $r \geq 8$ be an integer, and G be a planar graph. Then $\chi_r(G) \leq 2r + 16$.*

As $\chi_{\Delta(G)}(G) = \chi(G^2)$, a number of researches on r-hued colorings are motivated by the coloring of the square graph. There have been lots of researches on the study of $\chi_{\Delta(G)}(G)$ for a planar graph G. In 2003, Wang and Lih [134] proposed a conjecture that for every integer $k \geq 5$, there exists an integer Δ_k such that for any planar graph G, if $g(G) \geq k$ and $\Delta(G) \geq \Delta_k$, then $\chi_{\Delta(G)}(G) \leq \Delta(G) + 1$, which is known to be false for $k = 5, 6$, and is solved with $k \geq 7$. Some efforts on this conjecture are summarized in the Table 2. The following theorem concludes some results not summarized in the table. Dvořák et al. [45], Borodin and Ivanova [20,21], and Yancey [138] made further advances towards Wang and Lih's conjecture.

Theorem 3.6. *Let G be a planar graph. Each of the following holds.*
(i) (Dvořák, Král, Nejedlý and Škrekovski [45]) If $g(G) \geq 6$ and $\Delta(G) \geq 8821$, then $\chi_{\Delta(G)}(G) \leq \Delta(G) + 2$.
(ii) (Borodin and Ivanova [20]) If $g(G) \geq 6$ and $\Delta(G) \geq 18$, then $\chi_{\Delta(G)}(G) \leq \Delta(G) + 2$.
(iii) (Borodin and Ivanova [21]) If $g(G) \geq 6$ and $\Delta(G) \geq 36$, then $\chi_{L,\Delta(G)}(G) \leq \Delta(G) + 2$.
(iv) (Yancey [138]) If $g(G) \geq 5$ and $\Delta(G) \geq 63,500$, then $\chi_{L,\Delta(G)}(G) \leq \Delta(G) + 6$.
(v) (Borodin, Glebov, Ivanova, Neutroeva and Tashkinov [19]) For each integer $D \geq 2$, there exists a planar graph G_D with girth 6, $\Delta(G_D) = D$ and $\chi_D(G_D) \geq D + 2$.

As suggested by Theorem 3.6(i), Dvořák et al. [45] presented a revised version of Wang and Lih's conjecture.

Table 1
This table summarizes the results on r-hued chromatic numbers of the sparse graphs with bounded girths. As an example, line 8 and column $r+1$ is interpreted as "for a graph G with girth at least 8 and $r \geq 9$, we have $\chi_r(G) \leq r+1$, and this statement is verified for planar graphs only".

$g(G)$	χ_r					
	$r+1$	$r+2$	$r+3$	$r+4$	$r+5$	$r+10$
3	–	$r=2$ [72][a]	$r=2$ [30,72][d]	$r=2$ [73][d,e]		
4	–					
5					$r \geq 15$ [143,146][d,e]	all r [143][d,e]
6					$r \geq 3$ [74,124][c,d]	
7		$r=2$ [73][d,e]		$r=3$ [74][e]		
8	$r \geq 9$ [86][d]	$r=2$ [32,73][b,e]				
9	$r \geq 8$ [145]		$r=3$ [74]			
10	$r \geq 6$ [145]					
11						
12	$r \geq 5$ [145]		$r=3$ [32]			
13						
14		$r=3$ [32]				

[a]Results are verified for $G \neq C_5$.
[b]Results are verified for G that has no C_5-component.
[c]The bound is best possible for $r=3$.
[d]Results are verified for planar graphs only.
[e]Results are verified for list hued chromatic numbers as well.

Table 2
This table summarizes the results on Δ-hued chromatic numbers of the sparse graphs with bounded girths. As an example, line 8 and column $\Delta+1$ is interpreted as "for a graph G with girth at least 8 and $\Delta(G) \geq 9$, we have $\chi_r(G) \leq \Delta(G)+1$, and this statement is verified for planar graphs only".

$g(G)$	χ_Δ						
	$\Delta+1$	$\Delta+2$	$\Delta+3$	$\Delta+4$	$\Delta+5$	$\Delta+6$	$\Delta+8$
3	–			$\Delta=3$ [58,129][a]			
4	–						
5	–	$\Delta \geq 10^7$ [15][a,b]	$\Delta \geq 339$ [44][a]			$\Delta \geq 12$ [26][a,b]	All Δ [43,147][a]
6	–	$\Delta \geq 17$ [17][a]	$\Delta \geq 9$ [26][a,b]		$4 \leq \Delta \leq 7$ [28][a]	$\Delta=5$ [25]	
7	$\Delta \geq 16$ [65][a,b]			$\Delta=5$ [25] $\Delta \leq 4$ [36][a,b]			
8	$\Delta \geq 9$ [86][a]		$\Delta=5$ [25]				
9	$\Delta \geq 8$ [16]	$\Delta=5$ [25]	$\Delta=4$ [36,37][a,b]				
10	$\Delta \geq 6$ [16,65]	$\Delta=3$ [46][b]					
11	$\Delta \geq 6$ [65][a,b]	$\Delta=4$ [36][a,b]					
12	$\Delta \geq 5$ [16] $\Delta=5$ [22][a,b]	$\Delta=3$ [22] [a,b]					
13	$\Delta=5$ [65][a,b]						
14	$\Delta \geq 4$ [16]		$\Delta=3$ [46][b]				

[a]Results are verified for planar graphs only.
[b]Results are verified for list hued chromatic numbers as well.

Conjecture 3.7. *There exists an integer k such that if G is a planar graph with $g(G) \geq 5$ and $\Delta(G) \geq k$, then $\chi_{\Delta(G)}(G) \leq \Delta(G)+2$.*

This conjecture is recently proved by Bonamy et al. [15].

Theorem 3.8 (*Bonamy, Cranston and Postle [15]*)**.** *There exists an integer $k = 1,730^2+1 = 2,992,901$ such that if G is a planar graph with $g(G) \geq 5$ and $\Delta(G) \geq k$, then $\chi_{L,\Delta(G)}(G) \leq \Delta(G)+2$.*

While Theorem 3.8 almost closes the chapter of Wang and Lih's conjecture and Conjecture 3.7, it would be interesting to know for each integer $r > 0$, the optimal values of $g(r)$, $D(r)$ and $h(r)$ such that if G is a planar graph with $g(G) \geq g(r)$ and $\Delta(G) \geq D(r)$, then $\chi_r(G) \leq h(r)$. The same can also be considered for the list r-hued colorings of planar graphs. Some related results can be found in Table 1.

3.2. Graphs with forbidden minors

Let K be a graph without isolated vertices. A graph G contains K as a **minor** (or a **topological minor**, respectively) if K is the contraction of a subgraph of G (or if G contains a subgraph isomorphic to a subdivision of K, respectively). The Kuratowski's Theorem [85] has the following form.

Theorem 3.9 (*Kuratowski [85], Wagner [133], Harary and Tutte [57]*)**.** *A graph G is planar if and only if G has no minor isomorphic to a K_5 or a $K_{3,3}$.*

Thus, graphs that do not have a K_5 minor constitute a graph family properly containing all planar graphs. Utilizing the result in Theorem 3.1, (therefore, using the Four-Color Theorem), and the structural characterization of graphs without a K_5 minor by Wagner [133], Theorem 3.1 has been extended by Kim et al. in [71].

Theorem 3.10 (*Kim, Lee and Oum [71]*)**.** *If G is a connected graph without a K_5 minor and if $G \neq C_5$, then $\chi_2(G) \leq 4$.*

For a generic clique minor and a topological clique minor, Kim et al. in [71] obtained the following results.

Theorem 3.11 (*Kim, Lee and Oum [71]*)**.** *For any integer $t \geq 2$, the following hold.*
(i) If G contains no K_t topological minor, then $\chi_2(G) \leq 10t^2 + 2$.
(ii) If G contains no K_t minor, then $\chi_2(G) \leq \lfloor 64t\sqrt{\log_2 t} \rfloor + 3$.

3.3. Wegner's conjecture

Wegner [135] posed the following conjecture, originally stated for 2-distance colorings, which has drawn the attention of many researchers.

Conjecture 3.12 (*Wegner [135]*)**.** *If G is a planar graph, then*

$$\chi_{\Delta(G)}(G) = \begin{cases} \Delta(G) + 5, & \text{if } 4 \leq \Delta(G) \leq 7; \\ \lfloor 3\Delta(G)/2 \rfloor + 1, & \text{if } \Delta(G) \geq 8. \end{cases}$$

There are several studies towards Conjecture 3.12.

Theorem 3.13. *Let G be a planar graph with $\Delta = \Delta(G)$. Then each of the following holds.*
(i) (Zhu and Bu [144]) If $\Delta \leq 5$, then $\chi_\Delta(G) \leq 20$.
(ii) (Zhu and Bu [144]) If $\Delta \geq 6$, then $\chi_\Delta(G) \leq 5\Delta - 7$.
(iii) (Molloy and Salavatipour [109]) If $\Delta \geq 241$, then $\chi_\Delta(G) \leq \lceil 5\Delta/3 \rceil + 25$.
(iv) (Molloy and Salavatipour [109]) $\chi_\Delta(G) \leq \lceil 5\Delta/3 \rceil + 78$.
(v) (Havet, Heuvel, McDiarmid and Reed [59]) If $\Delta \geq 8$, then $\chi_\Delta(G) \leq 3\Delta(1 + o(1))/2$.
(vi) (Zhu, Chen, Miao and Lv [147]) If $\Delta \geq 26$ and G does not have a 4-cycle, then $\chi_\Delta(G) \leq \lfloor 3\Delta/2 \rfloor + 1$.

Lih et al. [96] proved that Conjecture 3.12 holds for graphs without K_4-minors. Define

$$K(r) = \begin{cases} r + 3, & \text{if } 2 \leq r \leq 3; \\ \lfloor 3r/2 \rfloor + 1, & \text{if } r \geq 4. \end{cases}$$

The function $K(r)$ is used in the following theorems.

Theorem 3.14 (*Lih, Wang and Zhu [96]*)**.** *Let G be a K_4-minor free graph. Then*

$$\chi_{\Delta(G)}(G) \leq K(\Delta(G)).$$

Utilizing a structural theorem in [96], a number of results have been obtained. In 2006, Theorem 3.14 was generalized to list Δ-hued coloring by Hetherington and Woodall [61]. In 2014, Song et al. [122] studied the general r and extended Theorem 3.14 from $r = \Delta(G)$ to arbitrary values of r. In another paper, Kostochka et al. [76] sharpened the bound in Theorem 3.14.

Theorem 3.15. *Let G be a K_4-minor free graph with $\Delta = \Delta(G)$, and $r \geq 2$ be an integer. Each of the following holds.*
(i) (Hetherington and Woodall [61]) $\chi_{L,\Delta}(G) \leq K(\Delta)$.
(ii) (Song, Fan, Chen, Sun and Lai [122]) $\chi_r(G) \leq K(r)$ and $\chi_{L,r}(G) \leq K(r) + 1$.
(iii) (Kostochka, Özkahya and Woodall [76]) If $\Delta \geq 6$ and G^2 does not contain a complete subgraph on $\lfloor 3\Delta/2 \rfloor + 1$ vertices, then $\chi_\Delta(G) \leq \lfloor 3\Delta/2 \rfloor$.

Chen et al. extended Theorem 3.14 and Theorem 3.15(ii) in [31]. Let H be a graph. If J is a graph obtained from H by a (possibly empty) sequence of edge subdivisions, then we call J an H-**subdivision**. Thus by definition, if $\Delta(H) \leq 3$, then G contains an H-minor if and only if G contains an H-subdivision. For an integer $n \geq 4$, define $K_4(n)$ to be the collection of all non-isomorphic K_4-subdivision on n vertices. For a collection $\mathcal{K}$ of graphs, define

$$EX(\mathcal{K}) = \{G : G \text{ does not have a minor isomorphic to a member in } \mathcal{K}\}.$$

By definition, for each $n \geq 4$, we have

$$EX(\{K_4\}) \subseteq \cdots \subseteq EX(K_4(n)) \subseteq EX(K_4(n+1)) \subseteq \cdots . \tag{5}$$

and for each fixed integer $n \geq 4$, $EX(K_4(n))$ contains all graphs with order less than n. Hence $\bigcup_{n=4}^{\infty} EX(K_4(n))$ contains all graphs. The following is a result on the graphs in $EX(K_4(7))$.

Theorem 3.16 (*Chen, Fan, Lai, Song and Xu [31]*). *Let $r \geq 2$ be an integer. If $G \in EX(K_4(7))$ and $G \ncong K_6$, then $\chi_r(G) \leq K(r)$.*

Recently, Wei et al. [136] investigate the r-hued list coloring version of Theorem 3.16 and obtained the following.

Theorem 3.17 (*Wei, Liu, Xiong and Lai [136]*). *Let $r \geq 2$ be an integer. If $G \in EX(K_4(7))$ and $G \ncong K_6$, then $\chi_{L,r}(G) \leq K(r)+1$.*

Motivated by Theorem 3.15, the following conjecture is presented.

Conjecture 3.18 (*Song, Fan, Chen, Sun and Lai [122]*). *Let r be a positive integer. If G is a planar graph, then $\chi_r(G) \leq f(r)$, where*

$$f(r) = \begin{cases} r+3, & \text{if } 1 \leq r \leq 2; \\ r+5, & \text{if } 3 \leq r \leq 7; \\ \lfloor 3r/2 \rfloor + 1, & \text{if } r \geq 8. \end{cases}$$

We believe there is also a list r-hued coloring version of Conjecture 3.18, which is left for us to pursue further. Note that Conjecture 3.18 is valid for $r \in \{1, 2\}$, as when $r = 1$, this is equivalent to the Four-Color Theorem, and when $r = 2$, this has been proved by Chen et al. in [30] and in Theorem 3.1. For other values of r, the conjecture is open.

For $r \geq 8$, the following example indicates that the upper bound in Conjecture 3.18 could not be relaxed. The graph in Fig. 2 was first introduced in [135].

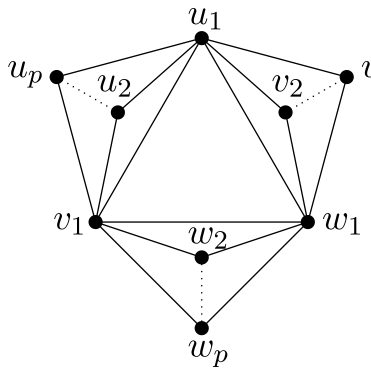

Fig. 2. $G(p, 1)$ with $r = 2p$, $mad(G(p, 1)) = 4 - \frac{2}{p}$ and $\chi_r(G(p, 1)) = \frac{3r}{2}$.

Song, Lai and Wu [125] showed that there exists an infinite sequence of fractional number q_r with $3 \leq q_r < 4$ and $\lim_{r\to\infty} q_r = 4$, such that for any even integer $r > 0$, there exists a planar graph G satisfying that $mad(G) \leq q_r$, $\Delta(G) \geq r$ and $\chi_r(G) \geq \frac{3r}{2}$. Such a graph can be constructed as follows.

Let $s \geq 1$ and $p \geq 2$ be integers. For $i = 1, \ldots, s$, let J_i be a graph with

$$V(J_i) = \{u_1^i, v_1^i, w_1^i, u_2^i, u_3^i, \ldots, u_p^i, v_2^i, v_3^i, \ldots, v_p^i, w_2^i, w_3^i, \ldots, w_p^i\},$$

and

$$E(J_i) = \{u_1^i v_1^i, v_1^i w_1^i, w_1^i u_1^i\} \cup \{u_1^i u_j^i, u_1^i v_j^i, v_1^i u_j^i, v_1^i w_j^i, w_1^i w_j^i, w_1^i v_j^i : 2 \leq j \leq p\}.$$

Obtain a graph $G(p, s)$ from the disjoint union of $J_1, J_2, \ldots, J_s$ by identifying $w_p^1, w_p^2, \ldots, w_p^s$ into one vertex w_p.

The graphs in Fig. 2 and Fig. 3 are examples for $s = 1$ and $s = 2$, respectively. The following observations justify the conclusions of the graph stated by Song et al. in [125].

Proposition 3.19. *Let $s \geq 1$ and $p \geq 2$ be integers. The graph $G(p, s)$ satisfies each of the followings:*
(i) $\Delta(G(p, s)) = \max\{2p, 2s\}$.
(ii) $4 - \frac{2}{p} \leq mad(G(p, s)) < 4$.
(iii) If $r = 2p$, then $\chi_r(G(p, s)) \geq \frac{3r}{2}$.

Proof. We shall use the notation in Fig. 3 to facilitate our arguments. Direct computation yields Proposition 3.19(i). Let $m, n \geq 1$ be two integers. Straightforward algebraic manipulations lead to that if $\frac{m}{n} < 4$, then

$$\frac{m}{n} < \frac{m+4}{n+1} < 4. \tag{6}$$

Let $G = G(p, s)$ and H be a subgraph of G with the maximum average degree among all subgraphs of G. Thus H is an induced subgraph of G with $|E(H)| > 0$. Hence we may assume that $E(H) \cap E(J_1) \neq \emptyset$. By inspection, (for example, using Fig. 2), we note that $\frac{2|E(H)|}{|V(H)|} < 4$. By symmetry and as H is an induced subgraph, we may assume that either $w_1^1 u_1^1 \in E(H)$ or $w_1^1 v_2^1 \in E(H)$. Suppose that $w_1^1 u_1^1 \in E(H)$, by (6), adding a vertex of degree 2 will strictly increase the average degree to H, then H is not of the maximum average degree. Hence we must have $w_1^1 v_1^1 \in E(H)$. Again by (6), all the vertices of degree 2 in J_1 must also be in $V(H)$, which implies that $J_1 \subseteq H$, and so $w_p \in V(H)$. We now consider the case that $w_1^1 v_2^1 \in E(H)$. By (6), we must have $w_1^1 u_1^1 \in E(H)$ since H is of the maximum average degree. Similar to the prior case, $w_1^1 v_1^1$ must be contained in $E(H)$ and consequently, all the vertices of degree 2 in J_1 must also be in $V(H)$. Therefore, $J_1 \subseteq H$, and thus $w_p \in V(H)$. Likewise, if for some i with $1 < i \leq s$, we have $E(H) \cap E(J_i) \neq \emptyset$, then we must also have $J_i \subseteq H$. Hence we assume that $H \cong G(p, t)$ for some t with $1 \leq t \leq s$ such that $H \cap J_i = J_i$ for $1 \leq i \leq t$ and $H \cap J_i = \{w_p\}$ for $t < i \leq s$ (see an example in Fig. 3 when $t = 2$). The average degree of H is

$$\frac{2t(6p-3)}{t(3p-1)+1},$$

which is an increasing function in t. As $\frac{2|E(H)|}{|V(H)|}$ is maximized among all subgraphs of G, we must have $t = s$ and so $H = G(p, s)$.

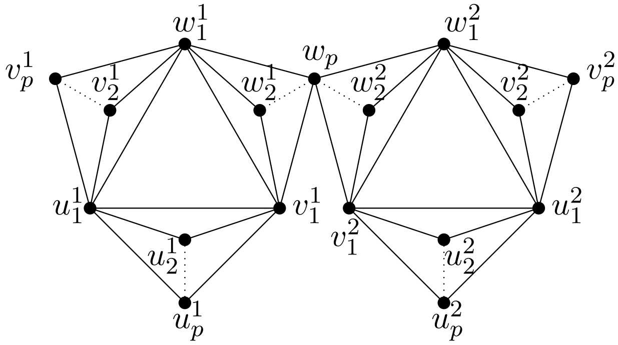

Fig. 3. An example of $G(p, 2)$.

As $p \geq 2$ and $s \geq 1$, with $q_s = \frac{2s(6p-3)}{s(3p-1)+1}$, Proposition 3.19(ii) follows from the fact that

$$3 \leq 4 - \frac{2}{p} \leq q_s = \frac{2s(6p-3)}{s(3p-1)+1} \leq \frac{2s(6p-3)}{s(3p-1)} = \frac{2(6p-3)}{(3p-1)} < \frac{2(6p-2)}{3p-1} = 4.$$

It remains to justify Proposition 3.19(iii). Let $r = 2p$. Suppose that $G(p, s)$ has a (k, r)-coloring $c : V(G(p, s)) \to \overline{k} = \{1, 2, \ldots, k\}$. Let $G = G(p, s)$. Since $N_G(u_1^1) = \{v_1^1, w_1^1, u_2^1, u_3^1, \ldots, u_p^1, v_2^1, v_3^1, \ldots, v_p^1\}$, it follows by $r = 2p$ that $|c(N_G(u_1^1))| = 2p$. Similarly, $|c(N_G(v_1^1))| = |c(N_G(w_1^1))| = 2p$. It follows that $|c(V(J_1))| = |V(J_1)| = 3p$, and so $k \geq |c(V(J_1))| = 3p = \frac{3r}{2}$. ■

For large values of r, Zhu et al. in [148], and Bu and Wang in [27] studied the r-hued coloring problem of planar graphs without cycles of fixed length.

Theorem 3.20. *Let G be a planar graph and r be an integer.*
(i) (Zhu, Gu, Sheng and Lv [148]) If $r \geq 3$ and G does not have cycles of length from 4 to 9, then $\chi_r(G) \leq r + 5$.
(ii) (Bu and Wang [27]) If G does not have 4,5-cycles and not have 3-cycle that intersects with i-cycles, $i = \{3, 6, 7\}$, then $\chi_r(G) \leq r + 5$, where $r \geq 3$.
(iii) (Bu and Wang [27]) If $r \geq 13$ and G does not have 3, 4, 8-cycles, then $\chi_r(G) \leq r + 5$.

3.4. Graphs on surfaces

As planar graphs are graphs that can be embedded on the sphere, it is natural to follow the foot steps of Heawood [60] to consider the coloring problem of graphs that can be embedded in other 2-manifolds. Following the definition in [137], the genus of a surface obtained by adding handles to a sphere is the number of handles added; we use S_γ for the surface of genus γ. The **genus** of a graph G, denoted $\gamma(G)$, is the minimum γ such that G embeds on S_γ.

For graphs embedded on surfaces with higher genera, some results have been obtained. Using Lebesgue edge-distributions, Chen et al. [30] proved a 2-hued version of Heawood's Theorem. With discharge arguments, Loeb et al. [102] obtained an upper bound for any integer $r \geq 2$. Using an approach similar to that in [59], Amini et al. [11] proved a more general result for graphs embedded in any surface.

Theorem 3.21. *Let G be a connected graph embedded in the surface S with genus γ.*
(i) (Amini, Esperet and van den Heuvel [11]) For any real number $\epsilon > 0$, there exists a constant $c(S, \epsilon)$ such that for any $c \geq c(S, \epsilon)$, if $\Delta(G) \leq c$, then $\chi_{L,\Delta(G)} \leq (\frac{3}{2} + \epsilon)c$.
(ii) (Chen, Fan, Lai, Song and Sun [30]) If G is a graph, then $\chi_{L,2}(G) \leq \frac{1}{2}(7 + \sqrt{1 + 48\gamma(G)})$.
(iii) (Loeb, Mahoney, Reiniger and Wise [102]) If $\gamma \leq 2$, then $\chi_r(G) \leq (r+1)(\gamma+5)+3$. If $\gamma \geq 3$, then $\chi_r(G) \leq (r+1)(2\gamma+2)+3$.
(iv) (Loeb, Mahoney, Reiniger and Wise [102]) If S is a torus, then $\chi_3(G) \leq \chi_{L,3}(G) \leq 10$, and this bound is sharp for toroidal graphs.

3.5. Outer planar graphs and k-planar graphs

A graph G is an **outerplanar** graph if G has a plane embedding in such a way that every vertex is on the exterior cycle. Based on the structures of outerplanar graphs, Lih and Wang [95] studied $\chi_{\Delta(G)}(G)$ for an outerplanar graph G. Later, Lih and Wang's results have been improved by Agnarsson and Halldórsson [1] and Hetherington and Woodall [62].

Theorem 3.22. *Let G be an outerplanar graph with $\Delta = \Delta(G)$. Then each of the following holds.*
(i) (Lih and Wang [95]) If $\Delta \geq 3$, then $\chi_\Delta(G) \leq \Delta + 2$.
(ii) (Lih and Wang [95]) If $\Delta \geq 7$, then $\chi_\Delta(G) \leq \Delta + 1$.
(iii) (Agnarsson and Halldórsson [1]) If $\Delta = 6$, then $\chi_\Delta(G) = 7$.

A **maximal outerplanar** graph is an outerplanar graph with maximum possible edges for a given number of vertices. Luo [104] proved the following.

Theorem 3.23 (*Luo [104]*)**.** *If G is a maximal outerplanar graph with $\Delta = \Delta(G)$, then $\Delta + 1 \leq \chi_\Delta(G) \leq \Delta + 2$.*

As an analogue to Kuratowski's Theorem, it is known in [29] that outerplanar graphs are precisely the graphs that do not have a minor isomorphic to K_4 or to $K_{2,3}$.

Theorem 3.24 (*Hetherington and Woodall [62]*)**.** *Let G be a $K_{2,3}$-minor-free graph with $\Delta = \Delta(G)$. Then each of the following holds.*
(i) If $\Delta \geq 3$, then $\Delta + 1 \leq \chi_\Delta(G) \leq \chi_{L,\Delta}(G) \leq \Delta + 2$, and these inequalities are sharp for $3 \leq \Delta \leq 5$, even for outerplanar graphs.
(ii) If $\Delta \geq 6$, then $\chi_\Delta(G) = \chi_{L,\Delta}(G) = \Delta + 1$.

Utilizing the structural results in [20], a related result is proved by Civan et al. [35].

Theorem 3.25 (*Civan, Deniz and Yetim [35]*)**.** *Let $\ell \geq 1$ be an integer and G be a K_4-minor-free graph. If G contains no subgraph isomorphic to $K_{2,\ell}$, then $\chi_{\Delta(G)}(G) \leq \Delta(G) + \ell$.*

Outerplanar graphs form a special class of planar graphs, while k-planar graphs are generalizations of planar graphs. For a given integer $k \geq 0$, a graph G is k-**planar** if G can be drawn in the plane so that each edge is crossed at most k times. Therefore, by definition, planar graphs are 0-planar. An **outer**-k-**planar** graph is a graph admitting a drawing in the plane for which all vertices belong to the outer face of the drawing and there are at most k crossings on each edge.

The r-hued colorings of outer-1-planar graphs and 1-planar graphs are studied in [93,141], in which Li and Zhang proved the following.

Theorem 3.26. *Let G be a graph.*
(i) (Li and Zhang [93]) If G is outer-1-planar, then $\chi_{L,3}(G) \leq 6$.
(ii) (Zhang and Li [141]) If G is 1-planar, then $\chi_{L,2}(G) \leq 11$.

Similarly to Conjecture 3.18, it is of interest to seek, for given nonnegative integers k and r, the smallest integers $h(k,r)$ and $h_L(k,r)$ such that for any k-planar graph G, $\chi_r(G) \leq h(k,r)$ and $\chi_{L,r}(G) \leq h_L(k,r)$. In this sense, Conjecture 3.18 is aiming at determining $h(0,r)$. In [141], Zhang and Li proposed to study $h(1,2)$ and $h_L(1,2)$. As it is known that K_6 is 1-planar and K_7 is 2-planar but not 1-planar, 6 is a lower bound of $h(1,r)$ and $h_L(1,r)$ and 7 is a lower bound of $h(2,r)$ and $h_L(2,r)$. The values and the behavior of $h(k,r)$ and $h_L(k,r)$ are far from being understood.

4. Sparse graphs

One of the hot research topics is to determine the r-hued chromatic number $\chi_r(G)$ and the list r-hued chromatic number $\chi_{L,r}(G)$, for a graph G with $mad(G)$ bounded by a small constant. The methods used to study such problems are mostly focused on the discharging method. While this study is of interest of its own, it is often motivated by Wegner's Conjecture (Conjecture 3.12). The observation below, following from Euler's formula, is commonly observed for a planar graph.

Observation 4.1. *Let G be a planar graph. Then $(mad(G) - 2)(g(G) - 2) < 4$.*

Observation 4.1 describes, for a planar graph G, a relationship between the girth $g(G)$ and the maximum average degree $mad(G)$, as partially illustrated in Table 3.

As shown in (4), for any graph G, one has $\chi_{L,r}(G) \geq \chi_r(G) \geq \min\{r, \Delta(G)\} + 1$. Many studies in the r-hued colorings of sparse graphs are focused on results suggesting that the r-hued chromatic number or the list r-hued chromatic number are bounded by $r + c_0$ for some small constant c_0. Many of such studies have also been surveyed in Table 1 for generic r-hued colorings and in Table 2 for Δ-hued colorings.

Yancey [138] and Bonamy et al. [16] studied the upper bounds for the 2-distance chromatic number and the list 2-distance chromatic number for graphs with maximum average degree bounded by small rational numbers, and with or without sufficiently large maximum degrees.

Table 3
The relationship between the girth $g(G)$ and the maximum average degree $mad(G)$.

$g(G) \geq$	4	5	6	7	8	9	10	11	12	13	14
$mad(G) <$	4	$\frac{10}{3}$	3	$\frac{14}{5}$	$\frac{8}{3}$	$\frac{18}{7}$	$\frac{5}{2}$	$\frac{22}{9}$	$\frac{12}{5}$	$\frac{26}{11}$	$\frac{7}{3}$

Theorem 4.2 (*Yancey [138]*). *Let c be a fixed number such that $c \geq 3$. If G is a graph such that $mad(G) < 4 - \frac{4}{c+1} - \epsilon$ for some $\frac{4}{c(c+1)} > \epsilon > 0$, then $\chi_{L,\Delta(G)}(G) \leq \max\{\Delta(G) + c, 16c^2\epsilon^{-2}\}$.*

Theorem 4.3 (*Bonamy, Lévêque and Pinlou [16]*). *There exists a function f such that for $\epsilon > 0$, every graph with $mad(G) < 14/5 - \epsilon$ and $\Delta(G) \geq f(\epsilon)$ satisfies $\chi_{\Delta(G)}(G) \leq \Delta(G) + 1$.*

Theorem 4.4 (*Bonamy, Lévêque and Pinlou [16]*). *For any sufficiently small real number $\epsilon > 0$, there exists an integer $h(\epsilon)$ such that every graph G with $mad(G) < 4 - \epsilon$ satisfies $\chi_{L,\Delta(G)}(G) \leq \Delta(G) + h(\epsilon)$.*

By Theorem 2.1(iii), all forests are $(r+1, r)$-colorable. As forests are of maximum average degree at most 2, this observation, together with Theorems 4.3 and 4.4, and with Conjectures 3.12 and 3.18, leads to the following problems, which are proposed in [125].

Problem 4.5. For any real number $x > 0$, is there a smallest integer $f(x)$ such that, when $r \geq f(x)$, every graph G with $mad(G) < x$ satisfies $\chi_r(G) \leq r + 1$?

Problem 4.6. Determine the set $\mathcal{X}$ of positive real numbers such that $x \in \mathcal{X}$ if and only if there exists an integer $h(x)$, for every graph G with $mad(G) < x$, we have $\chi_r(G) \leq r + h(x)$ for all sufficiently large r.

The example in Proposition 3.19 indicates that $\sup\{x \in \mathcal{X}\} \leq 4$ in Problem 4.6. The following example suggests that in Problem 4.5, $f(x)$ does not exist for any $x \geq 3$.

Example 4.7. There exists an infinite sequence of fractional number q_r with $\frac{7}{3} \leq q_r < 3$ and $\lim_{r\to\infty} q_r = 3$, such that for any integer $r \geq 3$, there exists a graph G satisfying that $mad(G) \leq q_r$, $\Delta(G) \geq r$ and $\chi_r(G) \geq r+2$. Such graphs can be constructed as follows. Let $s \geq 1$ and $t \geq 1$ be integers. For $i = 1, \ldots, s$, let J_i be a graph with

$$V(J_i) = \{w_1^i, w_2^i, w_3^i, w_4^i, x_1^i, x_2^i, \ldots, x_t^i, y_1^i, y_2^i, \ldots, y_t^i\},$$

and

$$E(J_i) = \{w_1^i w_3^i, w_2^i w_3^i, w_1^i w_4^i, w_2^i w_4^i\} \cup \{w_1^i x_j^i, x_j^i y_j^i, y_j^i w_2^i : 1 \leq j \leq t\}.$$

Obtain a graph $G(s,t)$ from the disjoint union of $J_1, J_2, \ldots, J_s$ by identifying $w_1^1, w_1^2, \ldots, w_1^s$ into one vertex w_1. The graph in Fig. 4 is an example for $s = 2$. Then we have the following observations which justify the conclusions stated in this example.
(i) $\Delta(G(s,t)) = s(t+2)$;
(ii) $\frac{7}{3} \leq mad(G(s,t)) = \frac{2s(3t+4)}{s(2t+3)+1} < 3$;
(iii) If $r = t+2$, then $\chi_r(G(s,t)) \geq r+2$.

Proof. Direct computation yields Example 4.7(i). In the following, we utilize the notation of Fig. 4 in our arguments. Let $m, n \geq 1$ be positive integers. If $\frac{m}{n} < 3$, then direct computation yields that

$$\frac{m}{n} < \frac{m+6}{n+2} < 3. \tag{7}$$

Let $G = G(s,t)$ and H be a subgraph of G with the maximum average degree among all subgraphs of G. Then H is an induced subgraph with $|E(H)| > 0$, and so we may assume that $E(H) \cap E(J_1) \neq \emptyset$. By inspection, $\frac{2|E(H)|}{|V(H)|} < 3$ and $w_1, w_2^1 \in V(H)$. As H is a subgraph of G with the maximum average degree, we must have $w_3^1, w_4^1 \in V(H)$. By (7), for all $1 \leq j \leq t$, we must have $x_j^1, y_j^1 \in V(H)$. Thus $J_1 \subseteq H$. Likewise, if for some i with $1 < i \leq s$, we have $E(H) \cap E(J_i) \neq \emptyset$, then we must have $J_i \subseteq H$. Therefore, we may assume that for k with $1 \leq k \leq s$, we have $H \cong G(k,t)$ such that $H \cap J_i = J_i$ for $1 \leq i \leq k$ and $H \cap J_i = \{w_1\}$ for $k < i \leq s$ (see an example in Fig. 4 when $k = 2$). It follows that the average degree of H is

$$\frac{2k(3t+4)}{k(2t+3)+1},$$

which is an increasing function in k. By the maximality of $\frac{2|E(H)|}{|V(H)|}$, we must have $k = s$ and so $H = G(s,t)$.

As $s \geq 1$ and $t \geq 1$, Example 4.7(ii) follows from the fact that

$$\frac{7}{3} \leq \frac{2(3t+4)}{2t+4} \leq \frac{2s(3t+4)}{s(2t+3)+1} \leq \frac{2s(3t+4)}{s(2t+3)} = \frac{2(3t+4)}{(2t+3)} < \frac{6t+9}{2t+3} = 3.$$

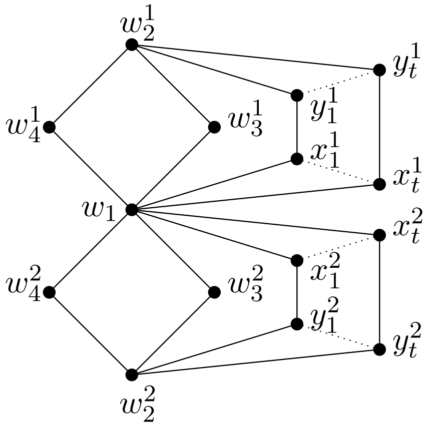

Fig. 4. An example of $G(2, t)$.

To justify Example 4.7(iii), we assume, by contradiction, that $G(s, t)$ has an $(r+1, r)$-coloring $c : V(G(s, t)) \to \overline{r+1} = \{1, 2, \dots, r, r+1\}$. Let $G = G(s, t)$. Since $N_G(w_2^1) = \{w_3^1, w_4^1, y_1^1, y_2^1, \dots, y_t^1\}$, it follows by $r = t+2$ that $c(N_G(w_2^1) \cup \{w_2^1\}) = \overline{r+1}$. Since $|c(N_G(w_3^1))| = 2$, we have $c(w_1) \notin \{c(w_2^1), c(w_3^1), c(w_4^1)\}$. For each j with $1 \le j \le t$, as $|c(N_G(x_j^1))| = 2$, we have $c(w_1) \neq c(y_j^1)$. It follows that $c(w_1) \notin c(N_G(w_2^1) \cup \{w_2^1\}) = \overline{r+1}$, contrary to the assumption that c is an $(r+1, r)$-coloring. This proves Example 4.7(iii). ■

Motivated by Theorems 4.3 and 4.4, Song et al. [125] investigated Problems 4.5 and 4.6 and obtained the following result.

Theorem 4.8 (*Song, Lai and Wu [125]*).
(i) For any fraction $q < 14/5$, there exists an integer $R = R(q)$ such that for each $r \ge R$, every graph G with maximum average degree q is list $(r+1, r)$-colorable.
(ii) For any sufficiently small real number $\epsilon > 0$, there exists an integer $h = h(\epsilon)$ such that every graph G with maximum average degree $4 - \epsilon$ satisfies $\chi_{L,r}(G) \le r + h(\epsilon)$.

5. The comparison between χ_r and χ

As $\chi_1(G) = \chi(G)$ is the classical chromatic number, it is naturally of interest to study the difference between $\chi_r(G)$ and $\chi(G)$, and how this difference behaves. Such studies were first initiated in [87,110]. The following example arises from the ideas in [87,110].

Example 5.1. For an integer $n > r \ge 2$, let $\binom{\overline{n}}{r}$ be the set of all r-subsets of $\overline{n}$. We construct a bipartite graph $G(n, r)$ with vertex bipartition (X, Y) with $X = \overline{n}$ and $Y = \binom{\overline{n}}{r}$, where for any $x \in X$ and $y \in Y$, $xy \in E(G(n, r))$ if and only if $x \in y$. When $r = 2$, this is a subdivision of the complete graph K_n, denoted by SK_n, is formed from K_n by subdividing every edge of K_n exactly once. As $G(n, r)$ is bipartite, $\chi(G(n, r)) = 2$. However, as every vertex $y \in Y$ has degree r in $G(n, r)$, any (k, r)-coloring of $G(n, r)$ must color the vertices in X with n different colors. It follows that for any fixed r, $\lim_{n\to\infty} \chi_r(G(n, r)) - \chi(G(n, r)) = \infty$. Thus the gap between $\chi_r(G)$ and $\chi(G)$, among all graphs, can be arbitrarily large.

Nevertheless, the next result, extending an idea from [9], indicates that, in an Erdös and Rényi random graph model [48,49], one observes that almost all graphs G satisfy $\chi_r(G) = \chi(G)$. It has been observed in Lemma 3.1 of [87], as well as by Sun and Ma [127], that if any vertex of degree greater than one is in a triangle, then $\chi_2(G) = \chi(G)$. This observation can be extended and utilized to show the following.

Proposition 5.2. *Let $\mathcal{G}_p(n)$ be the probability space of all labeled simple graph on n vertices with each edge occurring independently with probability p, where $0 < p < 1$ is a constant. Then in $\mathcal{G}_p(n)$, almost all graphs G satisfying $\chi_r(G) = \chi(G)$.*

Proof. Note that if G is a graph in which every vertex lies in a subgraph of G isomorphic to K_{r+1}, then any proper coloring of G is an r-hued coloring of G. Therefore, it suffices to show that almost all graphs G have the property that every vertex lies in a subgraph of G isomorphic to K_{r+1}. Define $A(r)$ to be the event that every vertex lies in a subgraph of G isomorphic to K_{r+1}. By definition, it suffices to show that

$$\lim_{n\to\infty} Prob(A(r)) = 1.$$

Let $r > 0$ be an integer and p be a real number with $0 < p < 1$. Assume that $n \geq r + 1$. Pick any $G \in \mathcal{G}_p(n)$. For each vertex $v \in V(G)$, partition $V(G - v)$ into parts $V_1, V_2, \ldots, V_s, V_{s+1}$, where

$$s = \lfloor \frac{n-1}{r} \rfloor, \ |V_i| = r \text{ for } 1 \leq i \leq s, \ \text{and } V_{s+1} = V(G) - (\{v\} \cup (\cup_{i=1}^{s} V_i)).$$

For each fixed v and i with $1 \leq i \leq s$, define an event in $\mathcal{G}_p(n)$ by

$$A_{v,i} := \{G \in \mathcal{G}_p(n) : G[V_i \cup \{v\}] \cong K_{r+1}\}.$$

By the definition of $\mathcal{G}_p(n)$, we have

$$Prob(A_{v,i}) = p^{\binom{r+1}{2}}.$$

Define

$$B_v = \cap_{i=1}^{s} \overline{A_{v,i}}.$$

Then B_v occurs if and only if none of the subgraphs $G[V_i \cup \{v\}]$ is isomorphic to K_{r+1}, for any i with $1 \leq i \leq s$. By the independence, and as the V_i's are disjoint,

$$Prob(B_v) = \prod_{i=1}^{s} \left(1 - Prob(A_{v,i})\right) = \left(1 - p^{\binom{r+1}{2}}\right)^s.$$

Call a vertex $v \in V(G)$ **bad** if $G \in B_v$, or if B_v occurs. (This is because when B_v occurs, a proper coloring of G may not have r-different colors appearing in $N_G(v)$.) Thus $\cup_{v \in V(G)} B_v$ means there might be at least one bad vertex in G. Let $B(r) = \overline{\cup_{v \in V(G)} B_v}$. Then $B(r)$ represents the event that none of the vertices is bad. This implies that when $G \in B(r)$, every vertex of G lies in a K_{r+1} of G. Hence $B(r) \subseteq A(r)$.

$$\begin{aligned} 1 \geq Prob(A(r)) \geq Prob(B(r)) &= 1 - Prob(\cup_{v \in V(G)} B_v) \\ &\geq 1 - \sum_{v \in V(G)} Prob(B_v) \\ &= 1 - n\left(1 - p^{\binom{r+1}{2}}\right)^s \end{aligned}$$

As $s \approx \frac{n}{r}$ when n is sufficiently large, from Calculus, we have, for each fixed r,

$$\lim_{n \to \infty} n\left(1 - p^{\binom{r+1}{2}}\right)^{\frac{n}{r}} = 0,$$

we conclude that

$$\lim_{n \to \infty} Prob(A(r)) = 1.$$

This completes the proof of the proposition. ■

5.1. Bounding the difference $\chi_r(G) - \chi(G)$ in general graphs

Efforts have been made to investigate the difference between $\chi_r(G)$ and $\chi(G)$. The concept of normal graphs is proposed in [87,91,110]. For positive integers r and s, a graph G is (r, s)-**normal** if $\chi_r(G) - \chi(G) \leq s$. For $r \geq 2$, an $(r, 0)$-normal graph is called an r-**normal graph** in [87,110], and when $r = 2$, it is simply called a **normal graph** in [87,110]. We say that a set of vertices are **independent** (or, alternatively, **stable**) if there is no edge between these vertices. In a graph G, **the independent number** (also called the **stability number**), $\alpha(G)$, is the size of a largest independent set of G. A **dominating set** of a graph G is a set $T \subseteq V(G)$ such that every vertex not in T is joined to at least one vertex of T. The **domination number**, $\gamma(G)$, is the number of vertices in a smallest dominating set of G. A set $T \subseteq V(G)$ is called a **total dominating set** in G if, for every vertex $v \in V(G)$, there is at least one vertex $u \in T$ adjacent to v. The **total domination number**, $\gamma_t(G)$, is the number of vertices in a smallest total dominating set of G. The set $T \subset V(G)$ is called a **double total dominating** set if T and its complement $V(G) \setminus T$ are both total dominating sets.

The following summarizes the studies on $(r, 0)$-normal graphs.

Theorem 5.3 (*Lai, Lin, Montgomery, Shui and Fan [87]*). *Let G be a connected graph and $|V(G)| = n$.*
(i) A graph G is $(r, 0)$-normal for all $r \geq 2$ if and only if G is a complete graph or an odd cycle of length a multiple of three.
(ii) If $\delta(G) \geq \lfloor (r-1)n/r \rfloor + 1$, then G is $(r, 0)$-normal. The lower bound on $\delta(G)$ is best possible.

Theorem 5.4. *Let G be a graph with $|V(G)| = n$ and let $L(G)$ denote the line graph of G.*
(i) (Sun and Ma [127]) If G is simple, $\alpha(G) = 2$ and $\Delta(G) \leq n - 5$, then G is $(2, 0)$-normal.
(ii) (Sun and Ma [127]) For any $u, v \in V(G)$, and $uv \in E(G)$, if $d(v) + d(u) > n$, then G is $(2, 0)$-normal.
(iii) (Sun and Ma [127]) Let G be connected and $\{v_1, v_2, \ldots, v_r\} \subset V(G)$ be any vertex set of G. If $\sum_{i=1}^{r} d(v_i) > n(r-1)$, then G is $(r, 0)$-normal.

Theorem 5.5 (*Liu and Sun [100]*). *Let G be a graph with $|V(G)| = n$. Then G is a $(3, 0)$-normal graph if one of the following holds.*
(i) For any $x, y \in V(G)$ with $xy \in E(G)$, $d(x) + d(y) \geq n + 2$, and G does not contain an even cycle without a chord as an induced subgraph.
(ii) $\Delta(G) \leq \lceil \frac{n-3\alpha(G)}{\alpha(G)-1} - 1 \rceil$.

5.2. Bounding the difference $\chi_r(G) - \chi(G)$ in regular graphs

In the case of regular graphs, Montgomery [110] conjectured the following:

Conjecture 5.6 (*Montgomery [110]*). *If G is a regular graph, then G is $(2, 2)$-normal.*

Ahadi et al. [3] generalized the conjecture as below.

Conjecture 5.7 (*Ahadi, Akbari, Dehghan and Ghanbari [3]*). *For a graph G with no isolated vertex, G is $(2, \lceil \frac{\Delta(G)}{\delta(G)} \rceil + 1)$-normal.*

Conjecture 5.8 (*Ahadi, Akbari, Dehghan and Ghanbari [3]*). *Let G be a regular graph and $\chi(G) \geq 4$, then G is $(2, 0)$-normal.*

However, Conjecture 5.7 and Conjecture 5.8 were disproved by Alishahi in [10] by constructing two counterexamples. Conjecture 5.6 was also settled negatively by Jahanbekam, Kim, O and West in [66]. Nevertheless, there are many results showing that the difference of $\chi_2(G) - \chi(G)$ can be bounded by functions involving parameters of the graph G. Thus for generic values of $r \geq 2$, determining sharp upper bounds for the differences $\chi_r(G) - \chi(G)$ and $\chi_{L,r}(G) - \chi(G)$ in terms of other characteristics of the graph becomes an interesting problem. Authors in [7] define a graph G to be **strongly k-regular** if there are parameters k, λ and μ such that G is k-regular, every adjacent pair of vertices has λ common neighbors, and every nonadjacent pair of vertices has μ common neighbors.

Theorem 5.9. *Let G be a graph.*
(i) (Akbari, Ghanbari and Jahanbekam [5]) If G is bipartite and regular, then G is $(2, 2)$-normal.
(ii) (Akbari, Ghanbari and Jahanbekam [7]) If G is strongly regular and $G \notin \{C_4, C_5, K_{k,k}\}$, then G is $(2, 1)$-normal.
(iii) (Alishahi [9]) If G is regular, then $\chi_2(G) \leq 2\chi(G)$.
(iv) (Dehghan and Ahadi [41]) If G is regular, then G is $(2, \lceil \alpha(G)/2 \rceil + 1)$-normal.

Theorem 5.10 (*Dehghan and Ahadi [41]*). *Let G be a graph, $\alpha'(G)$ and $\omega(G)$ be the matching number and the clique number of G.*
(i) If G is a regular graph, then G is $(2, 2\lfloor log_2(\alpha(G)) \rfloor + 3)$-normal.
(ii) In general, if G is a graph, then G is $(2, 2 + min\{\alpha'(G), \frac{\alpha(G)+\omega(G)}{2}\})$-normal.

Utilizing the 2-colorability (the Property B) of a hypergraph and the Lovász Local Lemma [47,105], Alishahi proved the following results on the comparing $\chi_2(G)$ and $\chi(G)$ in [9].

Theorem 5.11 (*Alishahi [9]*). *Let G be a k-regular graph. Each of the following holds.*
(i) G is $(2, \lfloor 14.06 \ln k \rfloor + 1)$-normal.
(ii) Let ϵ be a positive constant. For any large enough k, $\chi_2(G) \leq \chi_{L,2}(G) \leq \lceil (1 + \epsilon)\chi_L(G) \rceil$.

In the same paper, Alishahi advanced the study and obtained the following interesting results, upper bounding the 2-hued chromatic number of a graph G by the chromatic numbers of two subgraphs of G, induced by a total dominating set and its complement, respectively.

Theorem 5.12 (*Alishahi [9]*). *Let G be a graph.*
(i) If $e(\Delta(G)^2 - \Delta(G) + 2) \leq 2^{\delta(G)}$ and there exists a total dominating set $T \subseteq V(G)$, then $\chi_2(G) \leq \chi(G[V \setminus T]) + 2\chi(G[T])$.
(ii) If G has a double total dominating set $T \subset V(G)$, then $\chi_2(G) \leq \chi(G[V \setminus T]) + \chi(G[T])$.

Additional and arguably stronger results were later obtained by Alishahi in [10], Taherkhani in [128], and Jahanbekam et al. in [66].

Theorem 5.13. *Let G be a graph.*
(i) (Alishahi [10]) If $\chi(G) \geq 4$ and $e(\Delta(G)^2 - \Delta(G) + 2) \leq 2^{\delta(G)}$, then G is $(2, \alpha(G^2))$-normal.
(ii) (Alishahi [10]) If G is a k-regular graph with diameter 2, $k \geq 4$ and $e(\Delta(G)^2 - \Delta(G) + 1) \leq 2^{\delta(G)}$, then G is $(2, 1)$-normal.
(iii) (Alishahi [10]) If G is k-regular with no induced C_4, then for any $k \geq 35$, G is $(2, 2\lceil 4 \ln k + 1 \rceil)$-normal.
(iv) (Taherkhani [128]) If G is k-regular with $k \geq 3$, then G is $(2, \lceil 5.437 \ln k + 2.721 \rceil)$-normal.

Theorem 5.14 (*Jahanbekam, Kim, O and West [66]*). *Let G be a graph. Each of the following holds.*
(i) If G is k-regular with $k \geq (3 + x)r \ln r$, where $x - \frac{2 \ln \ln r}{\ln r}$ is a small positive constant, then $\chi_r(G) \leq r\chi(G)$.
(ii) If G has diameter 2, then G is $(2, 2)$-normal. The equality holds only for complete bipartite graphs and C_5.
(iii) If G has diameter 3, then $\chi_2(G) \leq 3\chi(G)$. This bound is sharp.

The next example shown in [66] indicates that for infinitely many r, there is an r-regular graph G such that $\chi_r(G) > r^{1.37744}\chi(G)$.

Example 5.15 (*Jahanbekam, Kim, O and West [66]*)**.** Let $\binom{\overline{n}}{t}$ be the set of all t-element subsets of a set $\overline{n}$. Let $K(n,t)$ be the Kneser graph with

$$V(K(n,t)) = \binom{\overline{n}}{t},$$

where two vertices in $v, w \in V(K(n,t))$ are adjacent if and only if $v \cap w = \emptyset$. Thus every $v \in V(K(n,t))$ is adjacent to $\binom{n-t}{t}$ other vertices.

Let $n = 3t-1$, $r = \binom{n-t}{t}$, and $G = K(3t-1,t)$. Then G is r-regular with diameter 2. This forces $\chi_r(G) = |V(G)| = \binom{3t-1}{t}$. But by [13,103], $\chi(G) = n-2t+2 = t+1$.

Applying Stirling's Formula to get

$$\frac{\chi_r(G)}{r\chi(G)} \approx \frac{\frac{2}{3}\binom{3t}{t}}{\frac{(t+1)}{2}\binom{2t}{t}} \approx \frac{1}{t}\sqrt{\frac{4}{3}}\left(\frac{27}{16}\right)^t.$$

Set this ratio to be r^x, where $r \approx \frac{4^t}{2\sqrt{\pi t}}$, and take logarithms on both sides to get $t\lg(\frac{27}{16}) = (1+o(1))tx\lg(4)$, leading to $x = \frac{\lg(27)-4}{2} > 0.37744$. Thus, $\chi_r(G) > r^{1.37744}\chi(G)$.

Ahadi et al. took an interesting approach in [3]. They defined for a graph G,

$$k^*(G) = \begin{cases} 2, & \text{if } \chi(G) = 2; \\ 1, & \text{if } \chi(G) \in \{3,4,5\}; \\ 0, & \text{otherwise.} \end{cases}$$

and proved the theorem below indicating that the parameter k^* is a good descriptor in such studies.

Theorem 5.16 (*Ahadi, Akbari, Dehghan and Ghanbari [3]*)**.** *Let G be a graph with $|V(G)| = n$ and $k^*(G) = k^*$. Each of the following holds.*
(i) G is $(2, \gamma(G)+k^*)$-normal.
(ii) If G is non-bipartite, then G is $(2, \max\{0, \alpha(G)-\delta(G)+1+k^*\})$-normal.
(iii) If G is non-bipartite, then G is $(2, n-\alpha(G)+1+k^*)$-normal.

In 2017, Bowler et al. disproved Conjecture 5.6 in [23] by showing that the bound of $\chi_2(G) \le 2\chi(G)$ is sharp. They ingeniously constructed regular graphs G with $\chi(G) = n$ and $\chi_2(G) = 2n$ for each integer $n \ge 2$. They also proved the following result.

Theorem 5.17 (*Bowler, Erde, Lehner, Merker, Pitz and Stavropoulos [23]*)**.** *For all natural numbers $r, n, \delta \ge 2$, there exists a k-regular graph G with $k > \delta$, $\chi(G) = n$ and $\chi_r(G) = r\chi(G)$.*

5.3. Bounding the difference $\chi_r(G) - \chi(G)$ in graphs with forbidden induced subgraphs

There are always some attempts to extend the study of the vertex r-hued coloring to its edge-coloring version. An initial idea to start is to define the r-hued chromatic index by $\chi_r(L(G))$, where $L(G)$ denotes the line graph of G. As it is well known that line graphs do not have an induced $K_{1,3}$ as a subgraph [14,56,118], this motivates the investigation of r-hued coloring in graphs with a forbidden induced subgraph.

Let H be a graph. A graph G is H-**free** if G does not have an induced subgraph isomorphic to H. A number of results have been advanced in this direction for $K_{1,k}$-free graphs, with $k \ge 3$.

Theorem 5.18 (*Lai, Lin, Montgomery, Shui and Fan [87]*)**.** *Let G be a $K_{1,3}$-free graph. Each of the following holds.*
(i) G is $(2,2)$-normal.
(ii) If G is connected, then $\chi_2(G) = \chi(G)+2$ if and only if G is a cycle of length 5 or of even length not a multiple of 3.

Theorem 5.19 (*Gao, Sun, Song and Lai [53]*)**.** *Let G be a graph. A vertex $u \in V(G)$ in a graph G is called the* **unique middle vertex** *if $d_G(u) = 2$, $N_G(u) = \{v_1, v_2\}$ with $d_G(v_1) \ge 3$ and $d_G(v_2) \ge 3$. Each of the following holds.*
(i) If G is $K_{1,3}$-free with $\chi(G) \ge 4$, and there is no unique middle vertex in G, then G is $(2,0)$-normal.
(ii) If G be $K_{1,4}$-free, then $\chi_2(G) \le 2\chi(G)$.
(iii) For a positive integer $n \ge 2$, if G does not have a subgraph isomorphic to a cycle of length $2n$, then $\chi_2(G) \le 2\chi(G)$.

Theorem 5.20 (*Li and Lai [91]*)**.** *Let G be a $K_{1,3}$-free graph. Then $\chi_3(G) \le \chi_{L,3}(G) \le \max\{\chi(G)+3, 7\}$. These bounds are best possibles.*

In [3], Ahadi et al. showed that if G is a P_4-free graph, then G is $(2,2)$-normal. This has been generalized to all values of r, and partially extended to P_5-free graphs in [88].

Theorem 5.21 (*Lai, Lv and Xu [88]*). *Let G be a connected graph. Each of the following holds.*
(i) If G is P_4-free, then G is $(r, 2(r-1))$-normal. Furthermore, a P_4-free graph G satisfies $\chi_r(G) - \chi(G) = 2(r-1)$ if and only if $G = K_{s,t}$ with $\min\{s, t\} \geq r$.
(ii) If G is a bipartite P_5-free graph, then $\chi_r(G) \leq r\chi(G)$.
(iii) If G is P_5-free, then $\chi_2(G) \leq 2\chi(G)$.

An infinite family of graphs described in the example below indicates that the upper bounds in Theorem 5.21 would be best possible. We include the discussions on these examples in [88] for completeness.

Example 5.22 (*Lai, Lv and Xu [88]*). Let $k \geq 2$ and $r \geq 1$ be integers. For positive integers $n_1, n_2, \ldots, n_k, (n_i \geq r, i = 1, 2, \ldots, k)$, let $K = K_{n_1,n_2,\ldots,n_k}$ denote a complete k-partite graph such that the k partite vertex sets are $V_1, V_2, \ldots, V_k$ with $|V_i| = n_i$ for $1 \leq i \leq k$. Let $U = \{u_1, u_2, \ldots, u_k\}$ be a set of vertices with $U \cap V(K) = \emptyset$; and let $n = \sum_{i=1}^{k} n_i + k$. Obtain a graph $G = G(n, k, r)$ from K and U by joining u_i to every vertex in V_i but not to any other vertices, for each i with $1 \leq i \leq k$. Thus, $n = |V(K)| + |U| = |V(G)|$. Let $\mathcal{F}$ be the collection of all graphs $G(n, k, r)$ for some values n, k, r with $n \geq k \geq r \geq 1$.

Proposition 5.23 indicates that every graph $G \in \mathcal{F}$ satisfies $\chi_r(G) = r\chi(G)$.

Proposition 5.23 (*Lai, Lv and Xu [88]*). *For any graph $G \in \mathcal{F}$, each of the following holds.*
(i) $\chi(G) = \omega(G) = k$.
(ii) $\chi_r(G) = rk$.
(iii) G is P_5-free.

Proof. Let $G \in \mathcal{F}$. Then for some integers n, k and r, we have $G = G(n, k, r)$. We shall use the same notations above. For each i with $1 \leq i \leq k$, fix a vertex $w_i \in V_i$; and let $W = \{w_1, w_2, \ldots, w_k\}$. Since K is a complete k-partite graph, $G[W]$ is isomorphic to K_k.

(i) By definition of G, $G[W]$ is a k-clique of G and so $\chi(G) \geq \omega(G) = k$. Let $c : V(G) \to [k]$ be so defined that $c(V_i) = i$ and $c(u_i) = i + 1 \pmod k$. Since K is a k-partite graph, each V_i is a stable set; since $N_G(u_i) = V_i$, it follows that c is a proper k-coloring of G. This proves (i).

(ii) Suppose that $\ell = \chi_r(G)$ and let $c : V(G) \to [\ell]$ be a (k, r)-coloring of G. Since $G[W]$ is isomorphic to K_k, we may assume that for each i with $1 \leq i \leq k$, $c(w_i) = i$.

Fix an i with $1 \leq i \leq k$. Since $n_i \geq r$ and $N_G(u_i) = V_i$, there must be a vertex subset $Z_i \subseteq V_i$ such that $|c(Z_i)| = |Z_i| = r$. Randomly pick a vertex $z_i \in Z_i$, and let $Z = \{z_1, z_2, \ldots, z_k\}$. As K is a complete k-partite graph, $G[Z]$ is isomorphic to K_k and so $|c(Z)| = k$. It follows that $\ell \geq |c(\cup_{i=1}^{k} Z_i)| = rk$.

To justify (ii), it suffices to present a (rk, r)-coloring of G. Construct a mapping $c : V(G) \to [rk]$ as follows. For $1 \leq i \leq k$, define $c(V_i) = \{(i-1)r + 1, (i-1)r + 2, \ldots, (i-1)r + r\}$ and $c(u_i) = (i-1)r + r + 1$. As K is a complete k-partite graph with $k \geq r$, the restriction of c to $V(K)$ is a (rk, r)-coloring. Since $N_G(u_i) = V_i$, and since $|c(V_i)| = r$, it follows that c is indeed a (rk, r)-coloring. This proves that $\ell = \chi_r(G) \leq rk$, and so completes the proof of (ii).

(iii) Let $P = x_1x_2x_3 \ldots x_t$ be a longest induced path in G. Since K is a complete k-partite graph, and since P is induced, we must have $|V(P) \cap V(K)| \leq 3$ and $|V(P) \cap V(K)| = 3$ if and only if $V(P) \cap V(K) = \{x_{i-1}, x_i, x_{i+1}\}$ for some i with $1 < i < 5$ such that x_{i-1} and x_{i+1} are in the same partite set of K. If x_{i-1} and x_{i+1} are both in a V_j, then we must have $t = 3$ and $P = x_{i-1}x_ix_{i+1}$ since $N(u_j) = V_j$. If $|V(P) \cap V(K)| = 2$, then as P is a longest induced path, $V(P) \cap V(K) = \{x_{i-1}, x_i\}$. We may assume, without lot of generality, that $x_{i-1} \in V_1$ and $x_i \in V_2$. It follows that $P = u_1x_{i-1}x_iu_2$. Hence in any case, $|V(P)| \leq 4$ and so G must be P_5-free. ■

Example 5.22 and Proposition 5.23 lead to the following Problem.

Problem 5.24 (*Lai, Lv and Xu [88]*). For integers $k > 0$, $r \geq 2$ and $t \geq 4$, determine a best possible function $f(k, r, t)$ such that for every connected P_t-free graph G with $\chi(G) = k$, we have $\chi_r(G) \leq f(k, r, t)$. More specifically, is there a best possible value $c = c(r, t)$ such that for every connected P_t-free graph G, we have $\chi_r(G) \leq c(r, t)\chi(G)$? In particular, does this equation $c(r, 5) = 5$ hold?

6. Products of graphs

There have been different kinds of products of graphs, as well as other graphical operations forming new graphs from some input graphs. It is of interest to study the relationship between the chromatic number of the inputting graphs and that of the resulting graph, as seen in the survey of Klavžar [75].

6.1. Cartesian product

For two graphs G and H, the **Cartesian product** of G and H, denoted by $G\square H$, has vertex set $V(G)\times V(H)$, where (u_1,v_1) and (u_2,v_2) are adjacent if and only if either $u_1=u_2$ and $v_1v_2\in E(H)$, or $v_1=v_2$ and $u_1u_2\in E(G)$. The problem of determining the r-hued chromatic number of the Cartesian product of graphs has attracted lots of attention. Akbari et al. in [6] started studying r-hued colorings of Cartesian products of paths and cycles. Since the maximum degree is 4 for these graphs, $\chi_r(G)=\chi_4(G)$ when $r\geq 4$. Jahanbekam et al. [66], O [112] and Kang et al. [69] proved the cases when $3\leq r\leq 4$.

Theorem 6.1 (Akbari, Ghanbari and Jahanbekam [6]). *For natural numbers m and n, each of the following holds.*
(i) If $m,n\geq 2$, then $\chi_2(P_m\square P_n)=4$.
(ii) If $m\geq 3$, then

$$\chi_2(C_m\square P_n)=\begin{cases}\chi_2(C_m), & \text{if } n=1;\\ 3, & \text{if } m\equiv 0 \pmod 3;\\ 4, & \text{otherwise.}\end{cases}$$

(iii) If $m,n\geq 3$, then

$$\chi_2(C_m\square C_n)=\begin{cases}3, & \text{if } mn\equiv 0 \pmod 3;\\ 4, & \text{otherwise.}\end{cases}$$

Theorem 6.2. *For natural numbers m and n, $m,n\geq 2$, each of the following holds.*
(i) (Jahanbekam, Kim, O and West [66])

$$\chi_3(P_m\square P_n)=\begin{cases}4, & \text{if } \min\{m,n\}=2;\\ 4, & \text{if } m \text{ and } n \text{ are both even;}\\ 5, & \text{if } m,n \text{ are not both even and } mn\not\equiv 2 \pmod 4.\end{cases}$$

$$\chi_4(P_m\square P_n)=\begin{cases}4, & \text{if } \min\{m,n\}=2;\\ 5, & \text{otherwise.}\end{cases}$$

(ii) (Kang, Müller and West [69]) If $mn\equiv 2 \pmod 4$, then $\chi_3(P_m\square P_n)=5$.

Theorem 6.3. *For natural numbers $m,n\geq 3$, each of the following holds.*
(i) (Jahanbekam, Kim, O and West [66], and O [112]) For $m\leq n \pmod 4$,

$$\chi_3(C_m\square C_n)\begin{cases}=4, & \text{if } m\equiv 0 \pmod 4 \text{ and } n\equiv t \pmod 4 \text{ for } t\in\{0,1,2\};\\ \leq 5, & \text{if } m\equiv s \pmod 4 \text{ and } n\equiv 3 \pmod 4 \text{ for } s\in\{0,1\};\\ \leq 6, & \text{otherwise.}\end{cases}$$

(ii) (O [112]) $5\leq\chi_4(C_m\square C_n)\leq 9$.
(iii) (O [112]) $\chi_4(C_m\square C_n)=5$ when $m\equiv 0 \pmod 5$ and $n\equiv 0 \pmod 5$.
(iv) (O [112]) $\chi_4(C_3\square C_3)=9$.

The r-hued chromatic numbers of Cartesian product of a path and its square are studied by Shao and Zuo [121].

Theorem 6.4 (Shao and Zuo [121]). *Let $m,n\geq 3$ be integers. Then*

$$\chi_3(P_m^2\square P_n)=\begin{cases}6, & \text{if } m\in\{3,5\};\\ 5, & \text{if } m=7;\\ 4, & \text{otherwise.}\end{cases}$$

$$\chi_4(P_m^2\square P_n)=\begin{cases}7, & \text{if } m\in\{5,6,7\}, \text{ or both } m=4 \text{ and } n \text{ is odd;}\\ 6, & \text{otherwise.}\end{cases}$$

and for $r\geq 5$,

$$\chi_r(P_m^2\square P_n)=\begin{cases}6, & \text{if } m=3;\\ 7, & \text{otherwise.}\end{cases}$$

Kaliraj et al. [68] studied 2-hued chromatic numbers of Cartesian product of complete graphs and wheels.

Theorem 6.5 (Kaliraj, Kumar and Vivin [68]).
(i) For positive integers t and s,

$$\chi_2(K_t\square K_s)=\begin{cases}4, & \text{if } t=s=2;\\ \max(t,s), & \text{otherwise.}\end{cases}$$

(ii) For positive integers $s\geq 2$ and n,

$$\chi_2(K_n\square K_{1,s})=\begin{cases}3, & \text{if } n=1;\\ 4, & \text{if } n=2;\\ n, & \text{otherwise.}\end{cases}$$

(iii) For positive integers $\ell \geq 4$ and n, $\chi_2(W_\ell \square K_n) = \max\{\chi_2(W_\ell), \chi_2(K_n)\}$.

6.2. Tensor product

For two graphs G and H, the **tensor product** of G and H (sometimes is called **direct product**, **Kronecker product**, **categorical product**, **cardinal product**, **relational product**, **weak direct product** or **conjunction**), denoted by $G \times H$, has the vertex set as the Cartesian product $V(G) \times V(H)$, where (u_1, v_1) and (u_2, v_2) are adjacent if and only if $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$. Deepa et al. [40] recently obtained several results on the r-hued chromatic numbers of the tensor product of a path with either a path or a cycle as follows.

Theorem 6.6 (*Deepa, Venkatachalam and Falcón [40]*)**.** *Let m, n and r be positive integers with $m, n \geq 3$. Each of the following holds.*
(i)

$$\chi_r(P_m \times P_n) = \begin{cases} 2, & \text{if } r = 1; \\ 4, & \text{if } r \in \{2, 3\}; \\ 5, & \text{otherwise}. \end{cases}$$

(ii)

$$\chi_r(P_m \times C_n) = \begin{cases} 2, & \text{if } r = 1; \\ 3, & \text{if } r = 2 \text{ and } n = 3t, \text{ for some } t \geq 1; \\ 4, & \text{if } \begin{cases} r = 2 \text{ and } n \neq 3t, \text{ for all } t \geq 1; \\ r = 3 \text{ and } n \in \{3t, 6t+1, 6t+2, 6t+5\}, \text{ for some } t \geq 1; \\ r = 3 \text{ and } n \in \{6t+4\}, \text{ for some } t \geq 2; \end{cases} \\ 5, & \text{if } \begin{cases} r = 3 \text{ and } n \in \{4, 5, 10\}, \text{ for all } t \geq 1; \\ r \geq 4 \text{ and } n = 5t, \text{ for some } t \geq 1; \\ r \geq 4,\ m \in \{3, 4\} \text{ and } n \notin \{3, 4, 6, 7, 8, 14\}; \end{cases} \\ 6, & \text{if } \begin{cases} r \geq 4,\ m \in \{3, 4\} \text{ and } n \in \{3, 4, 6, 7, 8, 14\}; \\ r \geq 4,\ m \geq 5 \text{ and } n \neq 5t, \text{ for all } t \geq 1. \end{cases} \end{cases}$$

6.3. Corona product

The **corona** [38] of two graphs G and H is denoted by $G \odot H$, which is obtained by taking one copy of G and $|V(G)|$ copies of H where ith vertex of G is adjacent to every vertex in the ith copy of H. For any integer $\ell \geq 2$, a graph $G \odot^\ell H$ is called ℓ-**corona product** [38] of G and H such that $G \odot^\ell H = (G \odot^{\ell-1} H) \odot H$. Agustin, Alfarisi, Dafik, Harsya, Kristiana and Utoyo [2,81–84] obtained a list of results on r-hued chromatic numbers of coronas of two graphs, including complete graphs K_n, paths P_n, cycles C_n, star graphs S_n, wheel graphs W_n, and fan graphs F_n with $n \geq 3$. Dafik et al. [38] extended some of those results to 2-corona products.

For positive integers n, m and r, define f_1, f_2 and f_3 as follows:

$$f_1 = \begin{cases} n, & \text{if } 1 \leq r \leq n-1; \\ r+1, & \text{if } n \leq r \leq m+n; \\ m+n+1, & \text{otherwise}. \end{cases}$$
$$f_2 = \begin{cases} r+1, & \text{if } n \leq r \leq m+n; \\ m+n+1, & \text{otherwise}. \end{cases}$$
$$f_3 = \begin{cases} m+1, & \text{if } 1 \leq r \leq m; \\ r+1, & \text{if } m+1 \leq r \leq m+n; \\ m+n+1, & \text{otherwise}. \end{cases}$$

Theorem 6.7. *Let m, n, r be integers. For a graph G, let $\Delta = \Delta(G)$. Each of the following holds.*
(i) (Agustin, Dafik and Harsya [2]) If $n \geq 3$ and $m \geq 2$, then for $1 \leq r \leq 2$,

$$\chi_r(W_n \odot P_m) = \begin{cases} 3, & \text{if } n \text{ is even}; \\ 4, & \text{if } n \text{ is odd}. \end{cases}$$

(ii) (Agustin, Dafik and Harsya [2]) If $n \geq 3$ and $m \geq 3$, then for $1 \leq r \leq 2$,

$$\chi_r(C_n \odot S_m) = \begin{cases} 3, & \text{if } n \text{ is even}; \\ 4, & \text{if } n \text{ is odd}. \end{cases}$$

(iii) (Kristiana, Dafik, Utoyo, Agustin [81]) If $n, m \geq 2$, then

$$\chi_r(P_n \odot P_m) = \begin{cases} 3, & \text{if } 1 \leq r \leq 2; \\ r+1, & \text{if } 3 \leq r \leq \Delta - 1; \\ m+3, & \text{otherwise}. \end{cases}$$

(iv) (Kristiana, Dafik, Utoyo, Agustin [81]) If $n, m \geq 3$, then[2]

$$\chi(P_n \odot C_m) = \chi_2(P_n \odot C_m) = \begin{cases} 3, & \text{if } m \text{ is even;} \\ 4, & \text{if } m \text{ is odd.} \end{cases}$$

$$\chi_3(P_n \odot C_m) = \begin{cases} 4, & \text{if } m \equiv 0 \pmod 3; \\ 6, & \text{if } m = 5; \\ 5, & \text{otherwise.} \end{cases}$$

$$\chi_r(P_n \odot C_m) = \begin{cases} r+1, & \text{if } 4 \leq r \leq \Delta - 1; \\ m+3, & \text{otherwise.} \end{cases}$$

(v) (Kristiana, Dafik, Utoyo, Agustin [81]) If $n, m \geq 3$, then

$$\chi(P_n \odot W_m) = \chi_2(P_n \odot W_m) = \chi_3(P_n \odot W_m) = \begin{cases} 4, & \text{if } m \text{ is even;} \\ 5, & \text{if } m \text{ is odd.} \end{cases}$$

$$\chi_4(P_n \odot W_m) = \begin{cases} 5, & \text{if } m \equiv 0 \pmod 3; \\ 7, & \text{if } m = 5; \\ 6, & \text{otherwise.} \end{cases}$$

$$\chi_r(P_n \odot W_m) = \begin{cases} r+1, & \text{if } 5 \leq r \leq \Delta - 1; \\ m+4, & \text{otherwise.} \end{cases}$$

(vi) (Kristiana, Utoyo, Dafik [84]) If $n \geq 4$ and $m \geq 3$, then $\chi_r(K_n \odot S_m) = \chi_r(K_n \odot F_m) = f_1$.
(vii) (Kristiana, Utoyo, Dafik [84]) If $n \geq 3$ and $m \geq 4$, then $\chi_r(S_n \odot K_m) = f_2$ and $\chi_r(F_n \odot K_m) = f_3$.
(viii) (Kristiana, Utoyo, Dafik [84]) If $n, m \geq 4$, then for $n \leq m$, $\chi_r(K_n \odot K_m) = f_3$; for $n \geq m+1$, $\chi_r(K_n \odot K_m) = f_1$.
(ix) (Kristiana, Utoyo, Dafik [83]) Let $G \in \{S_n \odot W_m, W_n \odot W_m\}$ with $n, m \geq 3$. Then

$$\chi(G) = \chi_2(G) = \chi_3(G) = \begin{cases} 4, & \text{if } m \text{ is even,} \\ 5, & \text{if } m \text{ is odd.} \end{cases}$$

$$\chi_4(G) = \chi_5(G) = \begin{cases} r+1, & \text{if } m \equiv 0 \pmod 3; \\ 7, & \text{if } m = 5; \\ 6, & \text{otherwise.} \end{cases}$$

$$\chi_r(G) = \begin{cases} r+1, & \text{if } 6 \leq r \leq n+m+1; \\ n+m+2, & \text{otherwise.} \end{cases}$$

Theorem 6.8 (*Kristiana, Utoyo, Alfarisi and Dafik [82]*). *For integers $m, n \geq 3$, $r \geq 1$,*
(i)

$$\chi_r(W_n \odot S_m) = \begin{cases} 3, & \text{if } 1 \leq r \leq 2, n \text{ is even;} \\ 4, & \text{if } 1 \leq r \leq 2, n \text{ is odd;} \\ r+1, & \text{if } 3 \leq r \leq n+m+1; \\ m+n+2, & \text{otherwise.} \end{cases}$$

(ii)

$$\chi_r(S_n \odot F_m) = \begin{cases} 4, & \text{if } 1 \leq r \leq 3; \\ r+1, & \text{if } 4 \leq r \leq n+m+1; \\ m+n+2, & \text{otherwise.} \end{cases}$$

(iii)

$$\chi_r(W_n \odot F_m) = \begin{cases} 3, & \text{if } 1 \leq r \leq 3, n \text{ is even;} \\ 4, & \text{if } 1 \leq r \leq 3, n \text{ is odd;} \\ r+1, & \text{if } 4 \leq r \leq n+m+1; \\ m+n+2, & \text{otherwise.} \end{cases}$$

(iv) Let $G \in \{F_n \odot S_m, S_n \odot S_m\}$.

$$\chi_r(G) = \begin{cases} 3, & \text{if } 1 \leq r \leq 2; \\ r+1, & \text{if } 3 \leq r \leq n+m+1; \\ m+n+2, & \text{otherwise.} \end{cases}$$

Theorem 6.9 (*Dafik, Agustin, Wardani, Septory, Kristiana and Kurniawati [38]*). *Let $G \in \{K_n \odot^2 P_m, K_n \odot^2 S_m, K_n \odot^2 F_m, K_n \odot^2 K_m\}$ and n, m be integers with $n, m \geq 3$, then*

$$\chi_r(G) = \begin{cases} n, & \text{if } 1 \leq r \leq n-1; \\ r+1, & \text{if } n \leq r \leq 2m+n-2; \\ 2m+n, & \text{otherwise.} \end{cases}$$

[2] This result is revised by authors.

6.4. Join operation

Given two graphs G and H. The **join** $G+H$ [2] is defined as follows: $V(G+H) = V(G) \cup V(H)$, $E(G+H) = E(G) \cup E(H) \cup \{(u,v) : u \in V(G) \text{ and } v \in V(H)\}$. Agustin et al. [2], Dafik et al. [39], Reddy et al. [117] and Kaliraj et al. [68] studied r-hued chromatic numbers of joins of two graphs.

Theorem 6.10. *For positive integers n, m, r, each of the following holds.*
(i) (Agustin, Dafik and Harsya [2]) If $n \geq 2$ and $m \geq 3$, then for $1 \leq r \leq 3$,

$$\chi_r(P_n + C_m) = \begin{cases} 4, & \text{if } m \text{ is even;} \\ 5, & \text{if } n \text{ is odd.} \end{cases}$$

and

$$\chi_4(P_n + C_m) = \begin{cases} 5, & \text{if } m \equiv 0 \pmod 3; \\ 6, & \text{otherwise.} \end{cases}$$

(ii) (Dafik, Meganingtyas, Purnomo, Tarmidzi and Agustin [39]) If $n, m \geq 3$, then

$$\chi_5(P_n + C_m) = \begin{cases} 6, & \text{if } m = 3; \\ 8, & \text{if } m = 5; \\ 7, & \text{otherwise.} \end{cases}$$

(iii) (Agustin, Dafik and Harsya [2]) If $n \geq 3$ and $m \geq 2$, then for $1 \leq r \leq 4$,

$$\chi_r(W_n + P_m) = \begin{cases} 5, & \text{if } n \text{ is even;} \\ 6, & \text{if } n \text{ is odd.} \end{cases}$$

(iv) (Reddy and Iyer [117]) Let T_1, T_2 be two non trivial trees with $|V(T_1)| \leq |V(T_2)|$. Then $\chi_r(T_1 + T_2) = 2(r-1)$, where $4 \leq r \leq |V(T_1)| + 1$.
(v) (Kaliraj, Kumar and Vivin [68]) For any two connected graphs G and H, $\chi_2(G+H) = \chi(G) + \chi(H)$.

Agustin et al. also studied r-hued chromatic numbers of some other operations of two graphs in [2], such as the lexicographic product of a cycle with a star and shackles of the Cartesian product of a path with either a cycle or a star.

7. The sensitivity problem

Let $k > 0$ be an integer. A graph G is k-**critical** if $\chi(G) = k$ but any proper subgraph H of G satisfies $\chi(H) < k$. It is very common to study critical graphs in graph coloring researches. However, there exist graphs G with a proper subgraph H satisfying $\chi_2(H) > \chi_2(G)$. For example, let H be the 5-cycle and let G be the 5-cycle plus a chord. Then we have $\chi_2(G) = 4$ whereas $\chi_2(H) = 5$. Thus, in r-hued colorings, it is of interest to investigate tight bounds for the changes of χ_r when an edge or a vertex is being removed or added. Montgomery [110] studied the case when $r = 2$, the effect of deleting a vertex.

Theorem 7.1 (*Montgomery [110]*)**.** *For any graph G, $\chi_2(G - v) \geq \chi_2(G) - 2$ for any vertex $v \in V(G)$. The only graphs for which $\chi_2(G-v) = \chi_2(G) - 2$ for at least one vertex are C_5, the 5-cycle, and $K_{1,n-1}$, the star on n vertices with $n \geq 3$.*

Miao et al. [108] have found an interesting phenomenon. They showed that for any integer $M \geq 1$, there exists a graph G such that G possess a vertex v satisfying $\chi_2(G - v) \geq \chi_2(G) + M$. Thus, to bound the difference $\chi_2(G-v) - \chi_2(G)$ for generic graphs G would be impossible. They also studied the impact of removing an edge when $r = 2$.

Theorem 7.2 (*Miao, Lai, Guo and Miao [108]*)**.** *Each of the following holds.*
(i) Let G be a connected graph with $|V(G)| \geq 3$. Then for every edge $e \in E(G)$, $\chi_2(G) - 2 \leq \chi_2(G-e) \leq \chi_2(G) + 2$.
(ii) There exists a graph G such that $\chi_2(G-e) = \chi_2(G) + 2$ for at least one edge $e \in E(G)$.
(iii) If a connected graph G is such that $\chi_2(G-e) = \chi_2(G) - 2$ for at least one edge $e \in E(G)$, then $G = C_5$.

Theorem 7.3 (*Miao, Lai, Guo and Miao [108]*)**.** *Let G be a connected graph with $|V(G)| \geq 2$. If G does not contain a subdivision of $K_{3,3}$, then $\chi_2(G-e) \leq \chi_2(G) + 1$ for every $e \in E(G)$.*

Song et al. investigated the general case when $r \geq 2$ in [126], extending some of the findings in [108] from $r = 2$ to generic values of r. The effect of a topological $K_{3,3}$ minor seems to deserve the attention of research. Further investigations might be needed to fully understand the impact of a topological $K_{3,3}$ minor in this direction of studies.

Theorem 7.4 (*Song, Miao, Miao and Lai [126]*)**.** *Let $r \geq 2$ be an integer, and G be a connected graph with $|V(G)| \geq 2$.*
(i) For any edge $e \in E(G)$, $\chi_r(G) - 2 \leq \chi_r(G-e) \leq \chi_r(G) + 2$.
(ii) For every graph G, there exists an edge $e \in E(G)$ such that $\chi_r(G-e) \neq \chi_r(G) + 2$.
(iii) If $\chi_r(G-e) = \chi_r(G) + 2$ for some $e \in E(G)$, then G must contain a subdivision of $K_{3,3}$.

Theorem 7.5 (*Song, Miao, Miao and Lai [126]*)**.** *Let $r \geq 2$ be an integer, and G be a connected graph with $|V(G)| \geq 2$, and u, v be any pair of nonadjacent vertices in $V(G)$.*
(i) If u, v are in the same component of G, then $\chi_r(G) - 2 \leq \chi_r(G + uv) \leq \chi_r(G) + 2$.
(ii) If u, v are in different components of G, then $\chi_r(G) - 1 \leq \chi_r(G + uv) \leq \chi_r(G) + 1$.

8. Other remarks

We have reviewed related literature regarding r-hued colorings and list r-hued colorings of graphs, which are natural generalizations of graph colorings and list graph colorings. As this survey is intended to review the literature and development for r-hued colorings with $r \geq 2$, many of the results in the area of classical colorings and list colorings are not included. A resourceful monograph by Jensen and Toft [67] would be a much better source for graph colorings in general. As mentioned in the introduction section, distance colorings is an important component of r-hued colorings. Kramer and Kramer [80] reviewed the literature and development of distance colorings, which is also a very resourceful reference for this topic. For more literary works and references concerning coloring problems on products of graphs, see the surveys of Klavžar [75] and N. Sauer [119]. Finally, we would like to mention the excellent survey of Zhu [142] on circular colorings of graphs. While vertex circular colorings refine vertex colorings, it is of interest to seek a similar refinement of r-hued colorings of graphs.

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