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Hamiltonian line graphs with local degree conditions ☆

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ABSTRACT

Let $N_{1,1,1}$ be the graph formed by attaching a pendant edge to each vertex of a triangle, and $B_{1,2}$ be a graph obtained by attaching end vertices of two disjoint paths of lengths 1, 2 to two vertices of a triangle. Broersma (1993) [2] and Čada et al. (2016) [3] conjectured that for a 2-connected claw-free simple graph G and for a fixed graph $\Gamma \in \{N_{1,1,1}, B_{1,2}, P_6\}$, if $\delta_\Gamma(G) = \min\{d_G(v) : d_H(v) = 1$ for any induced subgraph $H \cong \Gamma$ in $G\} \geq \frac{|V(G)|-2}{3}$, then G is Hamiltonian. While Chen settles this conjecture recently, the following two results of the conjecture for 3-connected line graphs are proved.

(i) For real numbers a, b with $0 < a < 1$, there exists a family $\mathcal{F}(a, b)$ of finitely many nonsupereulerian graphs, such that for any 3-connected line graph $H = L(G)$ of a simple graph G, if $\delta_{N_{1,1,1}}(H) \geq a|V(H)| + b$, then either H is Hamiltonian or G is contractible to a member in $\mathcal{F}(a, b)$.

(ii) Let $H = L(G)$ be a 3-connected line graph of a simple graph G with $|V(H)| \geq 116$. If $\delta_{N_{1,1,1}}(H) \geq \frac{|V(H)|+5}{10}$, then either H is Hamiltonian or G is isomorphic to the graph $P(10)'$, which is formed from the Petersen graph $P(10)$ by attaching $\frac{|V(H)|-15}{10}$ pendant edges to every vertex of $P(10)$.

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1. Introduction

We consider finite loopless graphs and follow [1] for undefined notation and terms. Let $\kappa(G)$, $\kappa'(G)$, $\alpha'(G)$, $\delta(G)$ and $g(G)$ denote the *vertex connectivity*, the *edge connectivity*, the *matching number*, *the minimum degree* and the *girth* of a graph G, respectively. For a vertex $v \in V(G)$, let $E_G(v) = \{f \in E(G) : v \in V(f)\}$, $d_G(v) = |E_G(v)|$, $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$. For a vertex subset S of G, define $N_G(S) = (\cup_{v \in S} N_G(v)) \setminus S$ and $N_G[S] = N_G(S) \cup S$. Let $i \geq 0$ be an integer and define $D_i(G) = \{v \in V(G) : d_G(v) = i\}$, $D_{\leq i}(G) = \{v \in V(G) : d_G(v) \leq i\}$. Vertices in $D_1(G)$ are the *leaves* of G, and edges incident with vertices in $D_1(G)$ are the *pendant edges* of G. For an edge $e = uv \in E(G)$, define $E_G(e) = E_G(u) \cup E_G(v)$. Thus $|E_G(e)| = d_G(u) + d_G(v) - 1$. For a given graph H, a graph G is *H-free* if G does not contain an induced subgraph isomorphic to H. A $K_{1,3}$-free graph is often referred as to a *claw-free* graph. The *line graph* of a graph G, denoted by $L(G)$, is a simple graph with vertex set $E(G)$, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. In [21], Ryjáček defined the *closure* $cl(H)$ of a claw-free graph H to be one obtained by recursively adding edges to join two nonadjacent vertices in the neighborhood of any locally connected vertex of H, as long as this is possible. Consequently, $cl(H)$ is a line graph.

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Let $\ell \geq 1$ be an integer, and let $P(10)$ denote the Petersen graph. The graph $P(10,\ell)$ is obtained from $P(10)$ by attaching ℓ pendant edges at every vertex of $P(10)$. If we do not emphasize the value of ℓ, we use $P(10)'$ for $P(10,\ell)$. For nonnegative integers i, j, k, we use $N_{i,j,k}$ to denote the graph formed by attaching a path of order $i+1, j+1, k+1$ to each of the three vertices of K_3, respectively. It is common to use Z_i instead of $N_{i,0,0}$ if $i > 0$ and $B_{i,j}$ instead of $N_{i,j,0}$ if $i, j > 0$. Let P_i (or C_i, respectively) denote a path (or a cycle, respectively) on i vertices.

A graph is *Hamiltonian* if it contains a spanning cycle. Sufficient conditions for a 2-connected or 3-connected claw-free graph to be Hamiltonian have been the subjects of many papers. The following are classical results due to degree conditions.

Theorem 1.1. *Let H be a simple claw-free graph on $n \geq 3$ vertices. Each of the following holds.*
(i) (Matthews and Sumner, [20]) If $\kappa(H) \geq 2$ and $\delta(H) \geq \frac{n-2}{3}$, then H is Hamiltonian.
(ii) (Favaron and Fraisse, [13]) If $\kappa(H) \geq 3$ and $\delta(H) \geq \frac{n+38}{10}$, then H is Hamiltonian.
(iii) (Lai, Shao and Zhan, [16]) If $n \geq 196$, $\kappa(H) \geq 3$ and $\delta(H) \geq \frac{n+5}{10}$, then H is Hamiltonian, unless $cl(H) \cong L(P(10)')$.

As $H = L(P(10)')$ is a 3-connected non-Hamiltonian claw-free graph with $\delta(H) \geq \frac{|V(H)|+5}{10}$, Theorem 1.1 (iii) settles the conjecture, posed by Kuipers and Veldman (see [13]), that for sufficiently large n, every 3-connected claw-free graph on n vertices with $\delta(H) \geq \frac{n+6}{10}$ is Hamiltonian. Faudree, Flandrin and Ryjáček, in Section 2(d) of their frequently quoted survey [12], listed a number of forbidden induced subgraphs in the study of Hamiltonian claw-free graphs. Among them, members in the family $N_{i,j,k}$, $B_{i,j}$ with $i+j+k \leq 3$ are included. For a connected graph Γ, define

$$\delta_\Gamma(G) = \min\{d_G(v) : v \in D_1(H) \text{ for any induced subgraph } H \cong \Gamma \text{ in } G\}.$$

In 1993, Broersma considered to combine the forbidden induced subgraph conditions and degree conditions in the study of Hamiltonian claw-free graphs. He proposed the following conjecture by considering a local degree condition of induced $N_{1,1,1}$.

Conjecture 1.2. *(Broersma, [2]) A 2-connected claw-free simple graph H with $\delta_{N_{1,1,1}}(H) \geq \frac{|V(H)|-2}{3}$ is Hamiltonian.*

Fujisawa and Yamashita [14] obtained a result for $\delta_{Z_1}(G) \geq \frac{n-2}{3}$ and Čada et al. [3] obtained a result for $\delta_\Gamma(H) \geq \frac{n+3}{3}$ where $\Gamma \in \{P_6, B_{1,2}, N_{1,1,1}\}$. They then proposed the following conjecture.

Conjecture 1.3. *(Čada, Li, Ning and Zhang, [3]) For fixed $\Gamma \in \{P_6, B_{1,2}\}$, every 2-connected claw-free simple graph H with $\delta_\Gamma(H) \geq \frac{|V(H)|-2}{3}$ is Hamiltonian.*

Conjectures 1.2 and 1.3 have been proved affirmatively by Chen recently.

Theorem 1.4. *(Chen, [8,9]) Every 2-connected claw-free simple graph on n vertices with $\delta_\Gamma(H) \geq \frac{n-2}{3}$ for a fixed $\Gamma \in \{P_6, B_{1,2}, N_{1,1,1}\}$ is Hamiltonian.*

It is natural to extend Theorem 1.4 to 3-connected claw-free graphs. Utilizing Theorem 1.1 (ii) and (iii), we prove the following result for 3-connected line graphs.

Theorem 1.5. *Let $H = L(G)$ be a 3-connected line graph of a simple graph G on $n \geq 116$ vertices. If $\delta_{N_{1,1,1}}(H) \geq \frac{n+5}{10}$, then either H is Hamiltonian or $G \cong P(10)'$.*

A more general question extending Conjecture 1.2 can be posed as follows: given a graph Γ in the list in Section 2(d) of [12], determine best possible linear function $c(n,\Gamma)$ such that for any claw-free graph G on n vertices, if $\delta_\Gamma(G) \geq c(n,\Gamma)$, then when n is sufficiently large, G is Hamiltonian. We also obtain the following result in this direction.

Theorem 1.6. *Let a and b be real numbers with $0 < a < 1$. There exists a family $\mathcal{F}(a,b)$ of finitely many nonsupereulerian graphs, such that for any a 3-connected line graph $H = L(G)$ of a simple graph G on n vertices, if $\delta_{N_{1,1,1}}(H) \geq an+b$, then either H is Hamiltonian or G is contractible to a member in $\mathcal{F}(a,b)$.*

Theorem 1.6 reveals that, under any nontrivial linear function lower bound for the local degree condition involving $N_{1,1,1}$, there are only finitely many contractional obstacles for the line graph to be Hamiltonian. Theorem 1.5 strengthens Theorem 1.1 (iii) and indicates that a better bound can be obtained in Conjecture 1.2 within 3-edge-connected line graphs. We will present some definitions and results that will be used in the next section. The justification of the main result will be given in the last section.

2. Preliminaries

For notational convenience, in the paper, if G is a graph and $X \subseteq E(G)$ is an edge subset, then we also use X to denote both an edge subset of $E(G)$ and $G[X]$, the subgraph induced by X in G. Thus $V(X)$ is the set of vertices in G incident with an edge in X. If $X = \{e\}$, we write $V(e)$ for $V(X)$. For a subgraph H of G and $X \subseteq E(G)$, we often use $H \cup X$ to denote the subgraph $G[E(H) \cup X]$. A path with end vertices u and v is often referred as to a (u, v)-path (or $P[u, v]$). For two disjoint subsets X, Y of $V(G)$, an (X, Y)-path is a path linking a vertex in X and a vertex in Y, and whose internal vertices belong to neither X nor Y. When $X = \{v\}$, we write (v, Y)-path for (X, Y)-path.

Let $O(G) = \bigcup_{i\geq 0} D_{2i+1}(G)$ denote the set of odd degree vertices of a graph G. If $O(G) = \emptyset$ and G is connected, then G is *eulerian*; if G contains a spanning eulerian subgraph, then it is *supereulerian*. An eulerian subgraph H of G is *dominating* if $V(G) - V(H)$ is a stable set of G. Harary and Nash-Williams proved a useful relationship between dominating eulerian subgraphs and Hamiltonian line graphs.

Theorem 2.1. *(Harary and Nash-Williams, [15]) Let G be a connected graph with at least 3 edges. The line graph $L(G)$ is Hamiltonian if and only if G has a dominating eulerian subgraph.*

Let $X \subseteq E(G)$ be an edge subset of a graph G. The *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. By definition, even if G is a simple graph, G/X may have multiple edges. We define $G/\emptyset = G$. When K is a connected subgraph of G, we write G/K for $G/E(K)$ with v_K denoting the vertex in G/K onto which K is contracted. The preimage of v_K in G, denoted by $PI_G(v_K)$, is the induced subgraph $G[V(K)]$. The vertex v_K is *nontrivial* if $PI_G(v_K)$ has at least one edge. For a connected subgraph $\Gamma \subseteq G/K$, we denote $PI_G(\Gamma)$ to be the induced subgraph $G[\cup_{u\in V(\Gamma)} V(PI_G(u))]$. Thus if $P' \subseteq G/K$ is a path (or a cycle, respectively), then $PI_G(P')$ contains a path P (or a cycle, respectively). The following result is useful.

Theorem 2.2. *(Chen et al., Theorem 1.1 of [11]) Let G be a 3-edge-connected graph and let $A \subseteq V(G)$ with $|A| \leq 12$. Then either G has an eulerian subgraph H with $A \subseteq V(H)$, or G can be contracted to the Petersen graph $P(10)$ in such a way that the preimage of each vertex of the Petersen graph contains at least one vertex in A.*

2.1. Catlin reduction method

As in [1], $K_{m,n}$ denotes the complete bipartite graph with partite sets of size m and n. By $H \subseteq G$, we mean that H is a subgraph of G. If $H \subseteq G$, then the set of *vertices of attachments* of H in G is defined as

$$A_G(H) = \{v \in V(H) : N_G(v) \nsubseteq V(H)\}.$$

In [4], Catlin introduced collapsible graphs. By Proposition 1 of [17], a graph G is *collapsible* if for any $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph Γ_R with $O(\Gamma_R) = R$. Catlin showed in [4] that every vertex of G lies in a unique maximal collapsible subgraph of G. For any graph G, let $H_1, H_2, \cdots, H_c$ be the collection of all maximal collapsible subgraphs of G. The graph $G/(H_1 \cup H_2 \cup \cdots \cup H_c)$ is the *reduction* of G. A graph G is *reduced* if G equals its reduction.

Let $F(G)$ be the minimum number of additional edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. Catlin (Theorem 2 of [5]) shows that every graph G with $F(G) = 0$ is collapsible. We summarize some results on Catlin's reduction method and other related tools in Theorem 2.3, and use $2K_1$ to denote the edgeless graph with two vertices.

Theorem 2.3. *Let G be a graph, $H \subseteq G$ be a collapsible graph and let G' be the reduction of G. Then each of the following holds.*
(i) (Catlin, Theorem 8 of [4]) G is collapsible (or supereulerian) if and only if G/H is collapsible (or supereulerian). In particular, G is collapsible if and only if $G' = K_1$.
(ii) (Catlin, Theorem 5 of [4]) G is reduced if and only if G has no nontrivial collapsible subgraphs.
(iii) (Catlin, Theorem 8 of [4]) $g(G') \geq 4$.
(iv) (Catlin, Theorem 7 of [5], see also Theorem 3.4 of [18]) If G is reduced, or if $E(G)$ is the union of the edge sets of two spanning trees in G, then $F(G) = 2|V(G)| - 2 - |E(G)|$.
(v) (Catlin et al., Theorem 1.3 of [6]) If $F(G) \leq 1$, then $G' \in \{K_1, K_2\}$; if $F(G) \leq 2$, then $G' \in \{K_1, 2K_1, K_2, K_{2,t}\}$ for some $t \geq 1$; if $F(G) \leq 2$ and $\kappa'(G) \geq 3$, then G is collapsible.
(vi) (Chen, [7]) If $\kappa'(G') \geq 3$ and $|V(G')| \leq 11$, then $G \in \{K_1, P(10)\}$.

Theorem 2.4. *(Chen et al., Theorem 4.4 of [10]) Let G be a connected reduced graph with n vertices and $\delta(G) \geq 3$. Then $\alpha'(G) \geq \min\{\frac{n}{2}, \frac{n+5}{3}\}$.*

Lemma 2.5. *(Xiong et al., Lemma 2.4 of [24]) Let G be a 3-edge-connected graph, and let $H \subset G$ be an induced connected subgraph of G with v_H as its contraction image in G/H such that $d_{G/H}(v_H) = 3$. Then each of the following holds.*

(i) If $|V(H)| \le 5$, then H is collapsible unless $H \cong K_{2,3}$ with $A_G(H) = D_2(H)$.
(ii) If H is not collapsible, then for any vertex $u \in A_G(H)$, H has a path of length at least 4 with u as its end vertex.

As an application of Lemma 2.5, the following is obtained.

Lemma 2.6. *Let G be a 3-edge-connected reduced graph, and let $H \subset G$ be an induced connected subgraph of G with v_H being its contraction image in G/H and $d_{G/H}(v_H) = 3$. Each of the following holds.*
(i) If $|V(H)| \le 5$, then $H \cong K_{2,3}$ with $A_G(H) = D_2(H)$.
(ii) If $|V(H)| \ge 6$, then $\alpha'(H) \ge 3$.

Proof. As $\kappa'(G) \ge 3$ and $d_{G/H}(v_H) = 3$, it follows that $\kappa'(H) \ge 2$. If $|V(H)| \le 5$, then by Lemma 2.5 (i), Lemma 2.6(i) holds. Hence we assume that $|V(H)| \ge 6$. If $\alpha'(H) \ge 3$, then Lemma 2.6(ii) holds. By contradiction, we assume that $\alpha'(H) \le 2$.

Choose a vertex $u \in A_G(H)$. By Theorem 2.3(ii) and (iii), H is reduced with $g(H) \ge 4$. By Lemma 2.5 (ii), H contains a $Q_1 = ux_1 \cdots x_\ell$ with $\ell \ge 4$ and ℓ maximized among all paths and cycles. If $\ell \ge 6$, or if $\ell \ge 5$ and $x_\ell \ne u$, then $\alpha'(H) \ge 3$, and so Lemma 2.6(ii) holds. Hence we assume that either $\ell = 4$ and $u \ne x_4$, or $\ell = 5$ with $u = x_5$.

Suppose first that $\ell = 5$ with $u = x_5$. Then Q_1 is a 5-cycle. If $V(H)\backslash V(Q_1) \ne \emptyset$, then since H is connected, there exists a vertex $v \in V(H)\backslash V(Q_1)$ and a vertex $x \in V(Q_1)$ such that $vx \in E(H)$. It follows that H contains a matching consisting of vx and two edges of $E(Q_1)$, and so $\alpha'(H) \ge 3$, contrary to the assumption that $\alpha'(H) \le 2$. Hence we must have $V(H) = V(Q_1)$, contrary to the fact that $|V(H)| \ge 6$.

Therefore, we must have $\ell = 4$ and $u \ne x_4$. As $\kappa'(H) \ge 2$, x_4 is adjacent to a vertex $w \in V(H)\backslash\{x_3\}$. Since $\alpha'(H) \le 2$ and H is reduced, we must have $w = x_1$ and $ux_3 \in E(H)$. Thus $H[V(Q_1)] \cong K_{2,3}$ with $D_2(H[V(Q_1)]) = \{u, x_2, x_4\}$. Choose a largest integer $t \ge 3$ such that $K = K_{2,t}$ is a subgraph in H. Since H is reduced, K is induced. If there exists a vertex $z \in V(H)\backslash V(K)$ which is adjacent to a vertex $z' \in D_2(K)$, then zz' together with a 2-matching in the 4-cycle of $K - z'$ forces that $\alpha'(H) \ge 3$, a contradiction. Hence $D_2(K) \subseteq D_2(H) \subseteq A_G(H)$ since $\kappa'(G) \ge 3$, implying that $t = 3$. Since $|V(H)| \ge 6$, there exists a vertex $z \in V(H)\backslash V(K)$ which is adjacent to a vertex $z' \in D_3(K)$. As $\alpha'(H) \le 2$, $N_H(z) \subseteq D_3(K)$. Thus $H[V(K) \cup \{z\}] \cong K_{2,4}$, contrary to the choice of K. This final contradiction justifies the lemma. □

2.2. The core of a graph

An edge-cut of a graph G is an *essential edge-cut* if $G - X$ has at least two nontrivial components. A connected graph G is *essentially k-edge-connected* if G does not have an essential edge-cut of size less than k. Let $ess'(G)$ be the smallest k such that G has an essentially k-edge-cut, if G has an essential edge-cut, or $ess'(G) = |E(G)| - 1$ if G does not have an essential edge-cut. With this definition, it is routine to verify that for a connected graph G with $|E(G)| \ge 2$, $\kappa(L(G)) = ess'(G)$.

Let G be a graph with $|E(G)| \ge 4$ and $ess'(G) \ge 3$. As $ess'(G) \ge 3$, $D_{\le 2}(G)$ is a stable set of G. For each $v \in D_2(G)$, let $E_G(v) = \{e_1^v, e_2^v\}$ and $X_2(G) = \{e_2^v : v \in D_2(G)\}$. Thus for each vertex $v \in D_2(G)$, $|X_2(G) \cap E_G(v)| = 1$. Define the *core* of G to be the graph G_0 in (1):

$$\begin{aligned} &G_1 = G - D_1(G),\\ &G_0 = G/((\cup_{v\in D_1(G)} E_G(v)) \cup X_2(G)) = G_1/X_2(G), \qquad (1)\\ &NE(G) = \cup_{v\in D_2(G)} E_G(v) - X_2(G). \end{aligned}$$

The *nontrivial edges* in G_0 are the edges in $NE(G)$. For notational convenience, the vertices in G adjacent to a vertex in $D_{\le 2}(G)$ can be viewed as vertices in G_0. Then $V(G_0) \subseteq V(G_1) \subseteq V(G)$. Let G_0' be the reduction of G_0. Then G_0' is a contraction of G_0 as well as G, and so we can view $E(G_0') \subseteq E(G_0) \subseteq E(G)$. Denote the sets of nontrivial vertices in G_0 and G_0' as follows:

$$\begin{aligned} &\Lambda(G_0) = \{v \in V(G_0) : PI_G(v) \ne K_1 \text{ or } PI_G(v) \cap V(NE(G)) \ne \emptyset\},\\ &\Lambda'(G_0) = \{v \in V(G_0') : PI_G(v) \ne K_1 \text{ or } PI_G(v) \cap V(NE(G)) \ne \emptyset\}. \end{aligned}$$

Applying Theorem 2.1, Shao proved the following.

Theorem 2.7. *(Shao, Section 1.4 of [23], see also Theorem 4.2 of [19]) Let G be a graph with $|E(G)| \ge 3$ and $ess'(G) \ge 3$, and let G_0 be the core of graph G. Then each of the following holds.*
(i) G_0 is well defined and nontrivial with $\delta(G_0) \ge \kappa'(G_0) \ge 3$.
(ii) $L(G)$ is Hamiltonian if and only if G_0 has a dominating eulerian subgraph H such that $\Lambda(G_0) \subseteq V(H)$.
(iii) $L(G)$ is Hamiltonian if and only if G_0' has a dominating eulerian subgraph H' such that $\Lambda'(G_0) \subseteq V(H')$.

3. Proofs of the main results

Let H_1, H_2 be two graphs. Define $H_1 \cup H_2$ to be the graph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$, and $H_1 - H_2 = H_1[E(H_1) \backslash E(H_2)]$. If X, Y are two vertex subsets of a graph G, define $E_G(X, Y) = \{xy \in E(G) : x \in X, y \in Y\}$. When $X = \{x\}$ or $Y = \{y\}$, we use $E_G(x, Y)$ or $E_G(X, y)$ for $E_G(X, Y)$, respectively. When G is understood from the context, we often omit the subscript G. For positive integers i, j, k, let $T_{i,j,k}$ denote the tree obtained from the disjoint union of three paths P_{i+1}, P_{j+1} and P_{k+1} by identifying an end vertex of each of these three paths into the only degree 3 vertex of $T_{i,j,k}$.

3.1. Lemmas

A vertex $v \in D_1(G)$ is a *good-leaf* of a graph G if G has a subgraph $L_v \in \{T_{2,2,1}, T_{2,1,1}, T_{1,1,1}\}$ and a vertex $u \in D_3(L_v)$ such that $uv \in E(L_v)$ and $N_{L_v}(u) \cap D_1(L_v) \subseteq D_1(G)$. For each vertex $x_i \in D_1(G)$ that is not a good-leaf, an ordered pair (H_i, e_i) is an x_i-*net* if G has a subgraph $H_i \in \{T_{2,2,2}, T_{2,2,1}, T_{2,1,1}\}$ and an edge $e_i = v_i u_i$ such that $v_i u_i w_i \subseteq H_i$ for $w_i \in D_3(H_i)$, $u_i \in D_{\le 3}(G)$ and $N_{H_i}(w_i) \cap D_1(H_i) \subseteq D_1(G) \backslash \{x_i\}$. For an integer $t \ge 1$, define $K_{2,t} + e$ to be the graph obtained from $K_{2,t}$ by adding an edge e joining any two nonadjacent vertices of degree t. As an example, $K_{2,1} + e \cong K_3$.

Lemma 3.1. *Let G be a graph such that $g(G) \ge 3$, $\kappa'(G - D_1(G)) \ge 2$ and $D_1(G) = \{x_1, x_2, \cdots, x_k\}$ with $k \ge 3$. Then for any integers $\{i, j\} \subseteq \{1, \cdots, k\}$, each of the following holds.*

(i) If x_i is not a good-leaf of G, then G has an x_i-net (H_i, e_i) and a block Γ_i, which depends on x_i (see Fig. 1), satisfying both of the following properties.
  (α) There exist disjoint subsets $R_0, R_1, \cdots, R_p$ in $D_{\le 3}(G)$ and a set $\{y_1, w_1, \cdots, w_{p+1}\}$ of cut-vertices of G such that for any $\ell \in \{0, 1, \cdots, p\}$, if $|R_\ell| \ge 3$, then $E(G[R_\ell]) = \emptyset$, and $\Gamma_i = G[V(R_0 \cup \cdots \cup R_p) \cup \{x_i, y_1, w_1, \cdots, w_{p+1}\}]$,
  (β) $e_i \in E(G[R_p \cup \{w_p\}])$ with $E_G(e_i) \subseteq E(\Gamma_i)$ and, if $|E_G(e_i)| \ge 6$, then $V(e_i) \cap D_2(G) \ne \emptyset$.

(ii) If $x_i, x_j \in D_1(G)$ are not good-leaves, then G has a x_i-net (H_i, e_i) and a x_j-net (H_j, e_j) such that $V(e_i) \cap V(e_j) = \emptyset$ and $E_G(e_i) \cap E_G(e_j) = \emptyset$.

Proof. (i) For each s with $1 \le s \le k$, as $x_s \in D_1(G)$, there exists an unique vertex y_s with $x_s y_s \in E(G)$. Without loss of generality, we assume that x_1 is not a good-leaf of G. Choose a shortest $(x_1, \{x_2, \cdots, x_k\})$-path P_1 in G, say (x_1, x_2)-path, and then choose a shortest $(\{x_3, \cdots, x_k\}, V(P_1))$-path P_2 in G, say (x_3, y_0)-path for some vertex $y_0 \in V(P_1) \cap V(P_2)$. As $x_1, x_2, x_3 \in D_1(G)$, we have $x_1 y_1, x_2 y_2 \in E(P_1)$ and $x_3 y_3 \in E(P_2)$. We first claim that for any path $P \subseteq G$ whose end vertices belong to $\{x_2, \cdots, x_k\}$, $E_0 = E(x_1, V(P)) = \emptyset$. Since otherwise, $G[E_0 \cup E(P)] \cong T_{\ell_1,\ell_2,\ell_3}$ for some integer $\ell_1, \ell_2, \ell_3 \ge 1$ with x_1 and two of $\{x_2, \cdots, x_k\}$ as its leaves, which implies that x_1 is a good-leaf, a contradiction. Hence, $y_1 \notin V(P_2) \cup V(y_0 P_1 x_2)$ and $E(y_1, V(P_2 - y_0)) = \emptyset$. Then there is a vertex $z_1 \ne x_1$ with $y_1 z_1 \in E(P_1)$. By the choice of P_1, it follows that $E(y_1, V(P_1 - \{z_1, x_1\})) = \emptyset$.

Let $V_0 = N_G(y_1) \backslash \{z_1, x_1\}$ and $t_0 = |V_0|$. Then $V_0 \cap V(P_1 \cup P_2) = \emptyset$. Since $\kappa'(G - D_1(G)) \ge 2$, we have $d_G(y_1) \ge 3$, and so $t_0 \ge 1$. Let $z_2 \in N_{P_1}(z_1) \backslash \{y_1\}$. Then for any vertex $v \in V_0$, as x_1 is not a good-leaf, we have $N_G(v) \subseteq \{y_1, z_1, z_2\}$, implying that V_0 is a stable set of G. If there are vertices $v_1, v_2 \in V_0$ such that $v_1 z_1, v_2 z_2 \in E(G)$, then $G[\{x_1 y_1, y_1 v_1, y_1 v_2, v_1 z_1, v_2 z_2\}] \cong T_{2,2,1}$,implying that x_1 is a good-leaf, a contradiction. Hence either $|N_G(V_0)| = 2$ or $N_G(V_0) = \{y_1, z_1, z_2\}$ with $|V_0| = 1$. Let $L_0 = G[N_G[V_0] \cup \{z_1\}]$. We then have the following claim.

Claim 1. *One of the following holds:*

(i) $L_0 \in \{K_{2,t_0} + e, K_{2,t_0+1}\}$ with $V_0 \subseteq D_2(G)$ if $|N_G(V_0)| = 2$, or
(ii) $L_0 \cong K_{2,2} + e$ with $V_0 \cup \{z_1\} \subseteq D_3(G)$ if $|N_G(V_0)| = 3$ and $|V_0| = 1$.

If there exists a vertex $v_0 \in V_0$ with $y_1 v_0 z_2 \subseteq G$ and a vertex $v \notin \{y_1, z_2\} \cup V_0$ with $z_1 v \in E(G)$, then $G[\{x_1 y_1, y_1 v_0, v_0 z_2, y_1 z_1, z_1 v\}] \cong T_{2,2,1}$, implying that x_1 is a good leaf of G, a contradiction. This implies that $N_G(z_1) \subseteq \{y_1, z_2\} \cup V_0$ if $z_2 \in N_G(V_0)$. Let $w_1 = z_2$ if $z_2 \in N_G(V_0)$, and $w_1 = z_1$ if $z_2 \notin N_G(V_0)$. Hence y_1, w_1 are cut-vertices of G. Let $R_0 = V_0 \cup \{z_1\}$ if $z_1 \ne w_1$, and $R_0 = V_0$ if $z_1 = w_1$. Choose a vertex $v_1 \in R_0$ and let $\Gamma_0 = G - N_G[y_1] \backslash \{v_1, w_1\}$. Then $D_1(\Gamma_0) = (D_1(G) - \{x_1\}) \cup \{v_1\}$ and $\kappa'(\Gamma_0 - D_1(\Gamma_0)) \ge 2$. If v_1 is not a good-leaf of Γ_0, then replace graph G by Γ_0 and repeat the discussion above. Set $w_0 = y_1$. We have obtained a sequence of induced graphs $L_0, \cdots, L_p$ (see Fig. 1 for an illustration) such that

- $L_\ell \in \{K_{2,t_\ell} + e, K_{2,t_\ell+1}\}$, $w_0, w_1, \cdots, w_{p+1}$ are cut-vertices of G, $R_\ell = V(L_\ell) \backslash \{w_\ell, w_{\ell+1}\}$, $N_G(R_\ell) = \{w_\ell, w_{\ell+1}\}$, $R_\ell \subseteq D_{\le 3}(G)$, and $E(G[R_\ell]) = \emptyset$ if $|R_\ell| \ge 3$ for each $\ell \in \{0, \cdots, p\}$ and some integer $t_\ell \ge 1$.
- $N_G(w_0) \subseteq R_0 \cup \{x_1\}$, $N_G(w_\ell) \subseteq R_{\ell-1} \cup R_\ell$ and $(V(L_0) \cup \cdots \cup V(L_p)) \cap V(P_2) \subseteq \{y_0\}$, $\ell \in \{1, \cdots, p\}$.
- For any vertex $v_0 \in R_p$, v_0 is a good-leaf of Γ_0 for the block $\Gamma_1 = G[V(L_0 \cup \cdots \cup L_p) \cup \{x_1\}]$ and the subgraph $\Gamma_0 = (G - V(\Gamma_1)) \cup \{v_0 w_{p+1}\}$ of G.

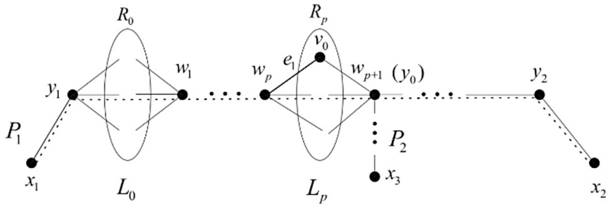

Fig. 1. An induced subgraph $\Gamma_1 = G[V(L_0 \cup \cdots \cup L_p) \cup \{x_1\}]$ and a subgraph $\Gamma_0 = (G - V(\Gamma_1)) \cup \{v_0 w_{p+1}\}$ of G.

Then Γ_1 is an induced subgraph of G satisfying the assumption of Lemma 3.1 $(i)(\alpha)$. Moreover, Γ_0 has a subgraph $T_0 \in \{T_{2,2,1}, T_{2,1,1}, T_{1,1,1}\}$ such that v_0 is a good-leaf of T_0. Choose $e_1 \in E(G[\{w_p\} \cup R_p]) \subseteq E(\Gamma_1)$. We then set $H_1 = T_0 \cup \{e_1\}$, and so $H_1 \in \{T_{2,2,2}, T_{2,2,1}, T_{2,1,1}\}$. Hence (H_1, e_1) is an x_1-net with $E_G(e_1) \subseteq E(L_{p-1} \cup L_p) \subseteq E(\Gamma_1)$. If $|E_G(e_1)| \geq 6$, then $L_p \ncong K_{2,2} + e$, and so $V(e_1) \cap D_2(G) \neq \emptyset$ by Claim 1. Hence Lemma 3.1$(i)(\beta)$ holds.

(ii) If x_i and x_j are not good-leaves, then by Lemma 3.1(i), G has x_i-net (H_i, e_i), x_j-net (H_j, e_j) and two blocks Γ_i, Γ_j such that $E_G(e_i) \subseteq E(\Gamma_i)$ and $E_G(e_j) \subseteq E(\Gamma_j)$. Hence $V(e_i) \cap V(e_j) = \emptyset$, $E_G(e_i) \cap E_G(e_j) \subseteq E(\Gamma_i) \cap E(\Gamma_j) = \emptyset$, and so Lemma 3.1$(ii)$ follows. □

Lemma 3.2. *Let G be a 3-edge-connected reduced graph. Then for $\{i, j\} = \{1, 2\}$, any edge $v_1 v_2 \in E(G)$ and any vertex $u_i \in N_G(v_i)\backslash\{v_j\}$, G has subgraphs $T_1 \cong T_2 \cong T_{2,2,1}$ and $T_3 \cong T_4 \cong T_{2,2,2}$ such that $v_1 v_2 \subseteq T_i$, $v_i \in D_3(T_i)$, $v_j \in D_1(T_i)$, $u_i v_i v_j \subseteq T_{i+2}$ and $u_i \in D_3(T_{i+2})$.*

Proof. Without loss of generality, we consider the case when $i = 1$. By Theorem 2.3(iii), $g(G) \geq 4$. As $\delta(G) \geq 3$, there are vertices $w_1, z_1 \in N_G(v_1) \setminus \{v_2\}$ with $w_1 v_2, z_1 v_2 \notin E(G)$ and vertices $w_2 \in N_G(w_1) \setminus \{v_1\}$, $z_2 \in N_G(z_1) \setminus \{v_1\}$. We then set $T_1 = G[\{v_1 v_2, v_1 w_1, v_1 z_1, w_1 w_2, z_1 z_2\}]$ with $T_1 \cong T_{2,2,1}$.

Since $g(G) \geq 4$, there are two vertices $w_1, z_1 \in N_G(u_1)\backslash\{v_1, v_2\}$ such that $\{v_1, w_1, z_1\}$ is a stable set of G. As $\delta(G) \geq 3$, there is a vertex $w_2 \in N_G(w_1)\backslash\{u_1, v_2\}$. If there is a vertex $z_2 \in N_G(z_1)\backslash\{u_1, v_2, w_2\}$, then we set $T_3 = G[\{u_1 v_1, u_1 w_1, u_1 z_1, v_1 v_2, w_1 w_2, z_1 z_2\}]$ with $T_3 \cong T_{2,2,2}$. Otherwise, we must have $\{z_1 v_2, z_1 w_2\} \subseteq E(G)$. If $w_1 v_2 \in E(G)$, then $G[\{u_1, v_1, v_2, w_1, w_2, z_1\}] \cong K^-_{3,3}$, where $K^-_{3,3}$ is a graph obtained from the complete bipartite graph $K_{3,3}$ via deleting one edge. As $F(K^-_{3,3}) = 2$, it follows by Theorem 2.3(v) that $G[\{u_1, v_1, v_2, w_1, w_2, z_1\}]$ is collapsible, contrary to the assumption that G is reduced by Theorem 2.3(ii).

Then $w_1 v_2 \notin E(G)$, w_1 has a neighbor w'_1 outside $\{u_1, v_2, w_2\}$ and we have $T_3 = G[\{u_1 v_1, u_1 w_1, u_1 z_1, v_1 v_2, w_1 w'_1, z_1 w_2\}] \cong T_{2,2,2}$. □

Lemma 3.3. *Let a, b be any two real numbers with $a > 0$, and let $M(a, b) = \max\{\frac{1}{a}, \frac{3-b}{a} - 4\}$. If $ax + b \geq 3$, then $f(x) = \frac{x-4}{ax+b-2} \leq M(a, b)$.*

Proof. As $f'(x) = \frac{b+4a-2}{(ax+b-2)^2}$ and $\lim_{x\to\infty} f(x) = \frac{1}{a}$, it follows that if $b + 4a = 2$, then $f(x) \equiv \frac{1}{a}$. If $b + 4a > 2$, then $f'(x) > 0$, implying that $f(x) \leq \frac{1}{a}$. If $b + 4a < 2$, then $f'(x) < 0$ and since $x \geq \frac{3-b}{a}$, we have $f(x) \leq f(\frac{3-b}{a}) = \frac{3-b}{a} - 4$. □

3.2. Proofs

Let a, b be two given real numbers with $0 < a < 1$. Throughout this section, we assume that $H = L(G)$ is 3-connected graph with $n = |V(H)|$ and $\delta_{N_{1,1,1}}(H) \geq an + b \geq 3$ for a simple graph G. Then $ess'(G) \geq 3$, $|E(G)| = |V(H)| = n$. Define

$E_L(G) = \{uv \in E(G) : \{u, v\} \cap D_1(G) \neq \emptyset\}$ to be the pendant edge set of G.

Then for any subgraph $T \cong T_{2,2,2}$ of G and any edge $xy \in E(G) \cap E_L(T)$, as $L(xy)$ is a leaf of an induced subgraph $L(T) \cong N_{1,1,1}$ of H, we must have

$$d_G(x) + d_G(y) = d_H(L(xy)) + 2 \geq an + b + 2. \tag{2}$$

As in (1), $G_1 = G - D_1(G)$ and G_0 is the core of G. Let G'_0 be the reduction of G_0. By Theorem 2.7, we assume that $|V(G'_0)| > 1$. Then $\kappa'(G'_0) \geq \kappa'(G_0) \geq 3$. For any vertex $v \in V(G'_0)$, define $\ell(v) = |E(PI_G(v))| + d_{G'_0}(v)$. Then

$$|E(G)| = |E(G'_0)| + \sum_{v \in V(G'_0)} |E(PI_G(v))| = \sum_{v \in V(G'_0)} \ell(v) - |E(G'_0)|. \tag{3}$$

A vertex v is *k-heavy* if $\ell(v) \geq k(an + b + 1)$. Define

$$X^k = \{v \in V(G_0') \text{ is } k\text{-heavy and not } (k+1)\text{-heavy}\},$$
$$X_0 = \cup_{k\geq 1} X^k,\ X_1 = \Lambda'(G_0)\backslash X_0. \tag{4}$$

Let $Y = \{uv \in E(G_0') : \ell(u) + \ell(v) \geq an + b + 2\}$. Choose a maximal matching M of $G - (X_0 \cup X_1)$ such that

$$|Y \cap M| \text{ maximized.} \tag{5}$$

Let $Y_0 = Y \cap M$, $Y_1 = M\backslash Y_0$. Then

$$X_0, X_1, V(Y_0), V(Y_1) \text{ are four mutually disjoint subsets of } V(G_0'). \tag{6}$$

We have the following discussions on heavy vertices.

Lemma 3.4. *For any vertex $v \in V(G_0')$, if $H_v = PI_{G_1}(v) \cup E_{G_0'}(v)$ has k leaves which are not good-leaves of H_v, then $v \in X^\ell$ for some integer $\ell \geq k$.*

Proof. Assume that $v_1, \cdots, v_k \in D_1(H_v)$ that are not good-leaves of H_v. If $v \in V(G_0)$, then $v \in V(G_1)$ and $H_v \cong K_{1,t}$ for some integer $t = d_{G_0}(v) \geq 3$ with $v_1, \cdots, v_k$ as its good-leaves, a contradiction. Hence $PI_{G_0}(v) \ncong K_1$ is a non-trivial collapsible subgraph, implying that $g(PI_{G_1}(v)) \geq g(G) \geq 3$ and $\kappa'(PI_{G_1}(v)) \geq 2$. By Lemma 3.1(*ii*), H_v has v_1-net (H_1, e_1), $\cdots$, v_k-net (H_k, e_k) such that for any $\{i, j\} \subseteq \{1, \cdots, k\}$, $V(e_i) \cap V(e_j) = \emptyset$ and $E_G(e_i) \cap E_G(e_j) = \emptyset$.

Then $H_i \in \{T_{2,2,2}, T_{2,2,1}, T_{2,1,1}\}$ with $D_3(H_i) = \{u_i\}$. Furthermore, for any vertex $w_i \in N_{H_i}(u_i) \cap D_1(H_i)$, we have $w_i \in N_{G_0}(v)\backslash\{v_i\}$. Then there is an edge $w_i z_i \in E(G_0' - v)$ such that $z_i w_i v \subseteq G_0'$. As $E(G_0') \subseteq E(G)$, the subgraph $PI_G(w_i z_i v)$ of G is connected, and so it contains an edge $w_i' z_i'$ with $u_i w_i' z_i' \subseteq G$. We set $H_i := (H_i - u_i w_i) \cup u_i w_i' z_i'$. Hence G always has a subgraph $H_i \cong T_{2,2,2}$ with $e_i \in E_L(H_i)$. Assume that $e_i = x_i y_i$. By (2), $|E_G(e_i)| \geq d_G(x_i) + d_G(y_i) - 1 \geq an + b + 1$. Thus $\ell(v) \geq (|E_G(e_1)| + \cdots + |E_G(e_k)| + k - d_{G_0'}(v)) + d_{G_0'}(v) \geq k(an + b + 1)$. By (4), $v \in X^\ell$ for some integer $\ell \geq k$. □

Lemma 3.5. *Each of the following holds.*

(i) For any vertex $v \in \Lambda'(G_0)$, either $v \in X_0$ or $N_{G_0'}(v) \subseteq X_0$.
(ii) For any edge $uv \in E(G_0')$, either $uv \in Y$ or $N_{G_0'}(\{u, v\}) \subseteq X_0$.
(iii) For any vertex $v \in X_0$, if $|N_{G_0'}(v) \cap (X_1 \cup V(Y_1))| = k$, then $v \in X^\ell$ for some integer $\ell \geq k$.
(iv) For any path $u_1 v_1 v_2 u_2$ with $\{u_1, u_2\} \subseteq X_1 \cup V(Y_1)$, $\max\{\ell(v_1), \ell(v_2)\} \geq 2(an + b)$.

Proof. (*i*) Assume that $v \in \Lambda'(G_0)\backslash X_0$. Choose a vertex $w \in N_{G_0'}(v)$. It suffices to prove that $w \in X_0$. By Lemma 3.2, G_0' has a subgraph $T_1 = G_0'[\{wv, wx_1, x_1 y_1, wx_2, x_2 y_2\}] \cong T_{2,2,1}$ for some vertices x_1, x_2, y_1, y_2. Let $L_1 = PI_{G_1}(w) \cup \{wv, wx_1, wx_2\} \subseteq G$. Suppose first that v is a good-leaf of L_1. As $E(G_0') \subseteq E(G)$, $PI_G(L_1) \cup PI_G(y_1) \cup PI_G(y_2) \cup \{x_1 y_1, x_2 y_2\}$ is connected and it has a subgraph $T_2 \cong T_{2,2,1}$ such that $v \in D_1(T_2) \cap N_{T_2}(D_3(T_2))$. Hence for any edge $vv_1 \subseteq PI_G(v)$, $T_2 \cup \{vv_1\} \cong T_{2,2,2}$. It follows by (2) that $d_G(v) + d_G(v_1) \geq an + b + 2$, and so $\ell(v) = |E(PI_G(v))| + d_{G_0'}(v) \geq (d_G(v) + d_G(v_1) - 1 - d_{G_0'}(v)) + d_{G_0'}(v) \geq an + b + 1$. This implies that $v \in X_0$, a contradiction. Thus v is not a good-leaf of L_1. By Lemma 3.4, $w \in X_0$. This proves Lemma 3.5(*i*).

(*ii*) Assume that $uv \notin Y$. If $\{u, v\} \nsubseteq V(G)$, then by Lemma 3.5(*i*), $\{u, v\} \cap X_0 \neq \emptyset$. Thus $\ell(u) + \ell(v) \geq an + b + 4$, implying that $uv \in Y$, a contradiction. Hence $\{u, v\} \subseteq V(G)$. Choose a vertex $w \in N_{G_0'}(u) \cup N_{G_0'}(v)$ (say $w \in N_{G_0'}(u)$). By Lemma 3.2, G_0' has a subgraph $T_3 = G_0'[\{wu, uv, wx_1, x_1 y_1, wx_2, x_2 y_2\}] \cong T_{2,2,2}$ for some vertices x_1, x_2, y_1, y_2. Let $L_2 = PI_{G_1}(w) \cup \{wu, wx_1, wx_2\} \subseteq G$. If u is a good-leaf of L_2, then as $PI_G(L_2) \cup PI_G(y_1) \cup PI_G(y_2) \cup \{x_1 y_1, x_2 y_2\}$ is connected, it has a subgraph $T_4 \cong T_{2,2,1}$ such that $u \in D_1(T_4) \cap N_{T_4}(D_3(T_4))$. Hence $T_4 \cup \{uv\} \cong T_{2,2,2}$. Then $d_G(u) + d_G(v) \geq an + b + 2$ and $\ell(u) + \ell(v) = d_{G_0'}(u) + d_{G_0'}(v) = d_G(u) + d_G(v) \geq an + b + 2$, implying that $uv \in Y$, a contradiction. Thus u is not a good-leaf of L_2, and so $w \in X_0$ by Lemma 3.4. This proves Lemma 3.5(*ii*).

(*iii*) If there is a vertex $v_0 \in N_{G_0'}(v) \cap (X_1 \cup V(Y_1))$ which is a good-leaf of $H_v = PI_{G_1}(v) \cup G_0'[N_{G_0'}[v]]$, then by the same analysis above, there is a subgraph $T_5 \cong T_{2,2,1}$ and an edge $v_0 u_0 \in Y_1 \cup E(PI_G(v_0))$ such that $T_5 \cup \{v_0 u_0\} \cong T_{2,2,2}$, forcing v_0 is heavy if $u_0 \in V(PI_G(v))$ or $v_0 u_0 \in Y_0$ if $v_0 u_0 \in Y_1$, which is impossible. Thus $N_{G_0'}(v) \cap (X_1 \cup V(Y_1))$ are not good-leaves of H_v with $|N_{G_0'}(v) \cap (X_1 \cup V(Y_1))| = k$. By Lemma 3.4, $v \in X^\ell$ for some integer $\ell \geq k$.

(*iv*) By Lemma 3.5(*i*) and (*ii*), $\{v_1, v_2\} \subseteq X_0$. For $i \in \{1, 2\}$, let $L_i = G[V(PI_G(v_i)) \cup N_{G_0'}(v_i)]$. As $u_i \in X_1 \cap V(Y_1)$, using arguments similar to those in the proof for Lemma 3.5(*i*) and (*ii*), we conclude that u_i is not a good-leaf of L_i. If v_2 is not a good leaf of L_1, then by Lemma 3.4, $\ell(v_1) \geq 2(an + b + 1)$, and so Lemma 3.5(*iv*) follows. We then assume that v_2 is a good leaf of L_1 and v_1 is a good leaf of L_2. Then there exists an edge $vv_1 \subseteq E_L(T_{2,2,2}) \cap E(L_2)$. As u_2 is not a good-leaf of L_2 and by Lemma 3.1(*i*), L_2 has an edge $v_2 v_3 \in E_L(T_{2,2,2}) \cap E(L_2)$ with $v_2 \in D_{\leq 3}(G)$ and $v_3 \notin N_G(v)$. Then $\ell(v) + \ell(v_1) \geq an + b + 2$, $\ell(v_2) + \ell(v_3) \geq an + b + 2$ and $\{v, v_1\} \cap \{v_2, v_3\} \subseteq \{v_2\}$, and so $\ell(v) \geq |E(PI(v))| - d_{G_0'}(v) \geq d_G(v) + d_G(v_1) + d_G(v_2) + d_G(v_3) - 2 - (d_G(v_2) - 1) \geq 2(an + b + 2) - d_G(v_2) - 1 \geq 2(an + b)$. Thus Lemma 3.5(*iv*) always holds. □

By Lemma 3.5(i), $E(G_0'[X_1]) = \emptyset$ and for any vertex $x \in X_1$, $N_{G_0'}(x) \subseteq X_0$. Throughout the rest of Subsection 3.2, we assume that X_0, X_1 are defined as in (4), and a maximum matching satisfying (5) is fixed and so Y_0, Y_1 are defined as before. Thus (6) holds. By Lemma 3.5(ii), P_3 is not a subgraph of $G_0'[X_1 \cup V(Y_0 \cup Y_1)]$ and $N_{G_0'}(Y_1) \subseteq X_0$. Define

$$H_0 = G_0'[X_0 \cup V(Y_0)], H_1 = G_0'[X_0 \cup X_1 \cup V(Y_0) \cup V(Y_1)]. \tag{7}$$

Let $\bar{X} = V(G_0') \backslash V(H_1)$. By the choice of M, $E(G_0'[\bar{X}]) = \emptyset$. By Lemma 3.5($i$) and ($ii$), there is always an edge in $E(H_0)$. As G_0' is reduced, we conclude that

$$\text{both } H_0 \text{ and } H_1 \text{ are reduced graphs with } |E(H_0)| > 0. \tag{8}$$

Let $|X_0| = x_0$, $|X^k| = x^k$, $|X_1| = x_1$, $|Y_0| = y_0$ and $|Y_1| = y_1$. As $k \geq 1$ is an integer, we have $x_0 = \sum_k x^k \leq \sum_k kx^k$.

Lemma 3.6. $x_0 + y_0 + y_1 \geq \alpha'(G_0')$.

Proof. Let M' be any maximum matching of G_0', $M_1 = \{e \in M' : V(e) \cap X_0 \neq \emptyset\}$ and $M_2 = M' \backslash M_1$. Then $|M_1| \leq |X_0|$. If there is an edge $uv \in M_2$ with $\{u, v\} \cap X_1 \neq \emptyset$, then by Lemma 3.5($i$), $\{u, v\} \cap X_0 \neq \emptyset$, and so $uv \in M_1$, a contradiction. Hence $M_2 \subseteq E(G_0'[V(Y_0 \cup Y_1) \cup \bar{X}])$. By the choice of M, it must have $|M_2| \leq |M| = |Y_0| + |Y_1|$, implying that $\alpha'(G_0') = |M_1| + |M_2| \leq x_0 + y_0 + y_1$. □

3.2.1. Proof of Theorem 1.6

We assume that $L(G)$ is not Hamiltonian. By Theorem 2.7(iii), G_0' is not supereulerian. It suffices to prove the existence of $\mathcal{F}(a, b)$ such that $G_0' \in \mathcal{F}(a, b)$. Define $M(a, b)$ as in Lemma 3.3, and let $B(a, b) = \max\{\lceil \frac{5M(a,b)}{2} \rceil, \lceil 4M(a, b) - 5 \rceil, \lceil \frac{2-b}{a} \rceil, 10\}$.

Claim 2. $|V(G_0')| \leq B(a, b)$.

Proof. We argue by contradiction and assume that $|V(G_0')| > B(a, b)$. Then $|V(G_0')| \geq 11$ implies that $\alpha'(G_0') \geq 5$ by Theorem 2.4. If $Y_1 = \emptyset$, then by Lemma 3.6, $x_0 + y_0 \geq \alpha'(G_0') \geq 5$; if $Y_1 \neq \emptyset$, then $x_0 \geq 4$. Hence we always have $|V(H_0)| = x_0 + 2y_0 \geq 4$ and $H_0 \notin \{2K_1, K_2\}$. By (8) and Theorem 2.3(iv) and (v), $2 \leq F(H_0) \leq 2(x_0 + 2y_0) - |E(H_0)| - 2$, and so

$$|E(H_0)| \leq 2x_0 + 4y_0 - 4 \leq 2\sum_k kx^k + 4y_0 - 4. \tag{9}$$

As $N_{G_0'}(V(Y_1)) \subseteq X_0$ and by Lemma 3.5(iii), $\sum_k kx^k \geq |E(V(Y_1), X_0)| \geq 4|Y_1|$, implying that $|Y_1| \leq \frac{1}{4}\sum_k kx^k \leq \frac{1}{4}(\sum_k kx^k + y_0)$. As $n \geq |V(G_0')| > \lceil \frac{2-b}{a} \rceil$, $an + b > 2$. By Lemma 3.3, we have $M(a, b) \geq \frac{n-4}{an+b-2}$. As $|V(G_0')| > \max\{\lceil \frac{5M(a,b)}{2} \rceil, \lceil 4M(a, b) - 5 \rceil\}$ and by Theorem 2.4, we have

$$\sum_k kx^k + y_0 \geq \alpha'(G_0') - |Y_1| > M(a, b) \geq \frac{n-4}{an+b-2}. \tag{10}$$

However, by (3), (4), (9) and (10), we obtain the following contradiction:

$$\begin{aligned}
|E(G)| &= \sum_{v \in V(G_0')} \ell(v) - |E(G_0')| \geq \sum_{v \in X_0 \cup V(Y_0)} \ell(v) - |E(H_0)| \\
&\geq (\sum_k kx^k)(an + b + 1) + y_0(an + b + 2) - (2\sum_k kx^k + 4y_0 - 4) \\
&\geq (\sum_k kx^k + y_0)(an + b + 1) - 2\sum_k kx^k - 3y_0 + 4 \\
&\geq (\sum_k kx^k + y_0)(an + b - 2) + 4 > n = |E(G)|.
\end{aligned}$$

Hence $|V(G_0')| \leq B(a, b)$. □

Let $\mathcal{F}(a, b) = \{F\colon F$ is a reduced nonsupereulerian graph such that $\kappa'(F) \geq 3$ and $|V(F)| \leq B(a, b)\}$. By Claim 2, $\mathcal{F}(a, b)$ is a finite family with $G_0' \in \mathcal{F}(a, b)$. This completes the proof of Theorem 1.6. □

3.2.2. Proof of Theorem 1.5

Let $m > 1$ be an integer. For circular indexing purpose, we shall use $\mathbb{Z}_m$ to denote the (additive) cyclic group or order m. Assume that $H = L(G)$ satisfies the hypotheses of Theorem 1.5 with $a = \frac{1}{10}$ and $b = \frac{1}{2}$. If $G_0' = K_1$, then by Theorem 2.3(i), G_0 is collapsible and also supereulerian. By Theorem 2.7(ii) and (iii), $L(G)$ is Hamiltonian. We then assume that $|V(G_0')| > 1$ and G_0' is not supereulerian with $\kappa'(G_0') \geq \kappa'(G_0) \geq 3$. If $|V(G_0')| < 10$, then by Theorem 2.3(vi), G_0 must be supereulerian. By Theorem 2.3(v) and $\kappa'(G_0') \geq 3$, and by Theorem 2.4, we may assume that

$$F(G_0') \geq 3, |V(G_0')| \geq 10 \text{ and } \alpha'(G_0') \geq 5. \tag{11}$$

Claim 3. $\sum_k kx^k + y_0 \leq 10.$

Proof. We argue by contradiction and assume that $\sum_k kx^k + y_0 \geq 11$. If $Y_1 = \emptyset$, then by Lemma 3.6, $\sum_k x^k + y_0 \geq \alpha'(G_0') \geq 5$. If $Y_1 \neq \emptyset$, then by Lemma 3.5(ii), $\sum_k x^k \geq 4$. Hence $|V(H_0)| = \sum_k x^k + 2y_0 \geq 4$ and $H_0 \notin \{2K_1, K_2\}$. By Theorem 2.3(v), we have $F(H_0) \geq 2$. By (8) and by Theorem 2.3(v), $2 \leq F(H_0) = 2(\sum_k x^k + 2y_0) - |E(H_0)| - 2$, where $F(H_0) = 2$ only if $H_0 \cong K_{2,t}$ for some integer $t > 0$. We have show first that $|E(H_0)| \leq 3\sum_k kx^k + 4y_0 - 5$. If $y_0 \leq 2$, then as $\sum_k kx^k \geq 11 - y_0 > 1$, $|E(H_0)| \leq 2\sum_k x^k + 4y_0 - 4 \leq 2\sum_k kx^k + 4y_0 - (5 - \sum_k kx^k) = 3\sum_k kx^k + 4y_0 - 5$. Assume that $y_0 \geq 3$. Then $H_0 \ncong K_{2,t}$ for any integer $t \geq 2$ and so by Theorem 2.3(v), $F(H_0) \geq 3$, leading also to $|E(H_0)| \leq 2\sum_k kx^k + 4y_0 - 5 \leq 3\sum_k kx^k + 4y_0 - 5$. However, by (3), (4) and as $|E(G)| \geq 116$, we obtain the following contradiction:

$$\begin{aligned}
|E(G)| &= \sum_{v \in V(G_0')} \ell(v) - |E(G_0')| \geq \sum_{v \in V(H_0)} \ell(v) - |E(H_0)| \\
&\geq (\sum_k kx^k) \times \frac{|E(G)| + 15}{10} + y_0 \times \frac{|E(G)| + 25}{10} - (3\sum_k kx^k + 4y_0 - 5) \\
&\geq |E(G)| + (\sum_k kx^k + y_0 - 10) \times \frac{|E(G)| - 15}{10} - 10 > |E(G)|.
\end{aligned}$$

Hence $\sum_k kx^k + y_0 \leq 10$. □

Claim 4. $|V(G_0')| \geq 10 + 9y_0.$

Proof. For any edge $uv \in Y_0$, as $|E(G)| \geq 116$, we have $d_{G_0'}(u) + d_{G_0'}(v) = \ell(u) + \ell(v) \geq \frac{|E(G)|+25}{10} > 14$. Then $d_{G_0'}(u) + d_{G_0'}(v) \geq 15$, and so $2|E(G_0')| = \sum_{v \in V(Y_0)} d_{G_0'}(v) + \sum_{v \in V(G_0') \setminus V(Y_0)} d_{G_0'}(v) \geq 15|Y_0| + 3(|V(G_0')| - 2|Y_0|) = 3|V(G_0')| + 9|Y_0|$. By Theorem 2.3($v$) and (11), $3 \leq F(G_0') \leq 2|V(G_0')| - \frac{1}{2}(3|V(G_0')| + 9|Y_0|) - 2$, implying that $|V(G_0')| \geq 10 + 9y_0$. □

We shall distinguish the following cases to finish our proof.

Case 1. $X_1 \cup Y_1 = \emptyset$.

By Lemma 3.6, $\alpha'(G_0') \leq |X_0| + |Y_0| \leq 10$. By Theorem 2.4, $|V(G_0')| \leq 25$ and by Claim 4, $|Y_0| \leq 1$. Let $S_0 = \emptyset$ if $Y_0 = \emptyset$, or $S_0 = \{v_e \in V(e)\}$ if $Y_0 = \{e\}$. Let $A = X_0 \cup S_0$. Then $\Lambda'(G_0) \subseteq A$ with $|A| \leq 10$. If G_0' has a dominating eulerian subgraph containing A, then $L(G)$ is Hamiltonian by Theorem 2.7(iii). If not, then by Theorem 2.2, G_0' can be contracted to a graph $L_0 \cong P(10)$ such that the preimage of each vertex of L_0 contains at least one vertex in A. Then $|X_0| + |Y_0| = 10$ and for any vertex $u \in V(L_0)$, we have $\ell(u) = d_{L_0}(u) + |E(PI_G(u))| \geq \frac{|E(G)|+15}{10}$. As $|E(G)| \geq \sum_{u \in V(L_0)} \ell(u) - |E(L_0)| \geq 10 \times \frac{|E(G)|+15}{10} - 15$, we have $\ell(u) = \frac{|E(G)|+15}{10}$ and $|E(PI_G(u))| = \frac{|E(G)|-15}{10}$. Hence $|A_G(PI_G(u))| = 1$. Assume that $A_G(PI_G(u)) = \{u\}$. If $PI_G(u) - u$ has an edge xy, then as $xy \in E_L(T_{2,2,2})$, $|E(PI_G(u))| \geq d_G(x) + d_G(y) - 1 \geq \frac{|E(G)|+15}{10}$, a contradiction. Thus $PI_G(u) \cong K_{1,\frac{|E(G)|-15}{10}}$ and $G \cong P(10)'$.

Case 2. $X_1 \cup Y_1 \neq \emptyset$.

Claim 5. $x_0 + y_0 \leq \sum_k kx^k + y_0 \leq 9.$

Proof. By Claim 3, $\sum_k kx^k + y_0 \leq 10$. We then assume by contradiction that $\sum_k kx^k + y_0 = 10$, and so $x_0 + y_0 \leq 10$. Recall that $H_1 = G_0'[X_0 \cup X_1 \cup V(Y_0 \cup Y_1)]$, $|X_1| = x_1$ and $|Y_1| = y_1$. For any vertex $x \in X_1$ and edge $uv \in Y_1$, by Lemma 3.5 (i), (ii), $N_{G_0'}(x) = N_{H_1}(x) \subseteq X_0$ and $N_{G_0'}(\{u, v\}) = N_{H_1}(\{u, v\}) \subseteq X_0$. We obtain the following conclusions.

(a) $|E(H_0)| \geq 15 + x_1 + y_0 + y_1$.

Since $|E(G)| \geq |X_0| \times \frac{|E(G)|+15}{10} + |Y_0| \times \frac{|E(G)|+25}{10} + |X_1| + |Y_1| - |E(H_0)| = |E(G)| + 15 + x_1 + y_0 + y_1 - |E(H_0)|$, it follows that $|E(H_0)| \geq 15 + x_1 + y_0 + y_1$.

(b) $|E(H_1)| \geq 15 + 4x_1 + y_0 + 6y_1 \geq 19$.

It follows by (a) that $|E(H_1)| \geq |E(H_0)| + |E(X_1 \cup Y_1, X_0)| + |Y_1| \geq |E(H_0)| + 3x_1 + 5y_1 \geq 15 + 4x_1 + y_0 + 6y_1 \geq 19$.

(c) $H_1 \ncong K_{2,t}$ **for any** $t \geq 1$.

If $H_1 \cong K_{2,t}$ for some integer $t \geq 1$, then as $\kappa'(G'_0) \geq 3$, we have $\bar{X} \neq \emptyset$ with $N_G(\bar{X}) \subseteq X_0$, and so G'_0 is collapsible, a contradiction.

(d) $y_0 \geq 2$.

By (b), $H_1 \notin \{2K_1, K_2\}$ and so by Theorem 2.3(v), we have $F(H_1) \geq 2$, with equality only if $H_1 \cong K_{2,t}$ for some integer $t \geq 1$. By (c), we must have $F(H_1) \geq 3$. Thus by (8) and Theorem 2.3(v), as well as the assumption $x_0 + y_0 \leq 10$, we have $3 \leq F(H_1) = 2|V(H_1)| - |E(H_1)| - 2 = 2(x_0 + x_1 + 2(y_0 + y_1)) - |E(H_1)| - 2 \leq 2x_0 + 3y_0 - 2(x_1 + y_1) - 17 \leq 20 + y_0 - 17 - 2(x_1 + y_1) = y_0 + 3 - 2(x_1 + y_1)$. This implies that $y_0 \geq 2(x_1 + y_1) \geq 2$.

(e) $|V(H_1)| \leq 18$.

By (d) and as $\sum_k kx^k + y_0 = 10$, we have $x_0 \leq sum_k kx^k \leq 8$. Then $3x_1 + 4y_1 \leq |E(X_1 \cup V(Y_1), X_0)| \leq \sum_k kx^k \leq 8$ by Lemma 3.5 (iii), implying that $x_1 + y_1 \leq 2$. If $y_1 = 2$, then $x_0 = 8$, $y_0 = 2$ and $x_1 = 0$, whence $|V(H_1)| = 4 + 8 + 4 = 16$. If $y_1 = 1$, then either $x_1 = 1$, $x_0 \geq 7$ and $y_0 \leq 3$, whence $|V(H_1)| = 16$; or $x_1 = 0$, $x_0 \geq 4$ and $y_0 \leq 3$, whence $|V(H_1)| \leq 18$. If $y_1 = 0$, then either $x_1 = 2$, $x_0 \geq 6$ and $y_0 \leq 4$, whence $|V(H_1)| \leq 16$; or $x_1 = 1$, $x_0 \geq 3$ and $y_0 \leq 7$, whence $|V(H_1)| \leq 18$.

(f) $y_0 = 2$.

If $y_0 \geq 3$, then by Claim 4, $|V(G'_0)| \geq 37$. By (b) and (e), $|E(H_1)| \geq 19, |V(H_1)| \leq 18$, and so $|E(G'_0)| = |E(H_1)| + |E_{G'_0}(V(H_1), \bar{X})| \geq 19 + 3(|V(G'_0)| - 18) = 3|V(G'_0)| - 35$ as $E(G'_0[\bar{X}]) = \emptyset$. Hence $F(G'_0) = 2|V(G'_0)| - |E(G'_0)| - 2 \leq 33 - |V(G'_0)| < 0$, contrary to the fact that G'_0 is reduced. Hence $y_0 \leq 2$. By (d), $y_0 = 2$.

(g) $|V(H_1)| \leq 16$.

As $3x_1 + 4y_1 \leq |E(X_1 \cup V(Y_1), X_0)| \leq \sum_k kx^k \leq 8$, $x_1 + y_1 \leq 2$. By (f) and as $\sum_k kx^k + y_0 = 10$, we have $y_0 = 2$, $x_0 \leq 8$, and so $|V(H_1)| = x_0 + x_1 + 2(y_0 + y_1) \leq x_0 + 2y_0 + 2(x_1 + y_1) \leq 16$.

By Claim 4 and (f), we have $|V(G'_0)| \geq 28$. It follows by (g) that $|\bar{X}| = |V(G'_0)| - |V(H_1)| \geq |V(G'_0)| - 16 \geq 12$, and so $F(G'_0) \leq 2|V(G'_0)| - (|E(H_1)| + 3|\bar{X}|) - 2 \leq 0$, contrary to the fact that G'_0 is reduced. Hence $\sum_k kx^k + y_0 \leq 9$. □

By Lemma 3.5, for any $S \subseteq X_1 \cup V(Y_1)$, $N_{G'_0}(S) \subseteq X_0$. Then $\sum_k kx^k \geq |E(V(Y_1), X_0)| + |E(X_1, X_0)| \geq 3x_1 + 4y_1$. By Claim 5, we have $3x_1 + 4y_1 \leq 9$, implying that

$$y_1 \leq 2 \text{ and if } Y_1 \neq \emptyset, \text{ then } x_1 + y_1 \leq 2; \text{ if } Y_1 = \emptyset, \text{ then } x_1 \leq 3. \tag{12}$$

By Lemma 3.6 and Claim 5, $\alpha'(G'_0) \leq x_0 + y_0 + y_1 \leq 9 + y_1 \leq 11$. By Theorem 2.4, $|V(G'_0)| \leq 28$. By Claim 4, $y_0 \leq 2$. Assume first that there exists an edge $e_1 = u_1v_1 \in Y_0$ and vertices $u_2, v_2 \in \bar{X}$ such that $u_1u_2, v_1v_2 \in E(G'_0)$. Then $u_1u_2 \in Y_0$, since otherwise, $v_1 \in X_0$ by Lemma 3.5, contrary to the fact that $V(Y_0) \cap X_0 = \emptyset$. By symmetry, $v_1v_2 \in Y_0$. We then obtain a subset $Y'_0 = (Y_0 \backslash \{u_1v_1\}) \cup \{u_1u_2, v_1v_2\}$ and a matching $M' = (M \backslash \{u_1v_1\}) \cup \{u_1u_2, v_1v_2\}$ such that $Y'_0 = Y \cap M'$ and $|Y'_0| = y_0 + 1$, contrary to (5). Hence for any edge $e \in Y_0$, there is a vertex $u_e \in V(e)$ such that $N_{G'_0}(u_e) \subseteq X_0 \cup V(e)$, and so

$$|E(X_0, V(Y_0))| \geq 2y_0. \tag{13}$$

In the rest of the arguments, choose a maximum stable set S_1 of $G'_0[Y_1]$ and a vertex $u_e \in V(e)$ with $N_{G'_0}(u_e) \subseteq X_0 \cup V(e)$ for some edge $e \in Y_0$. Set

$$S_0 = V(Y_0) \backslash \{u_e\} \text{ and } A = X_0 \cup X_1 \cup S_0 \cup S_1. \tag{14}$$

Then $|S_1| = |Y_1|$, $\Lambda'(G_0) \subseteq A$ and $E(V(G'_0) \backslash A) = \emptyset$. If $y_0 = 2$, then by Claim 4, $|V(G'_0)| = 28$. By Claim 5 and (12), (13), we have

$$\begin{aligned}
|E(G'_0)| &\geq |Y_0| + |E(X_0, V(Y_0))| + |Y_1| + |E_{G'_0}(X_0 \cup V(Y_0))| \\
&\geq 3|Y_0| + |Y_1| + 3(|X_1| + |\bar{X}|) + 4|Y_1| \\
&= 3y_0 + 5y_1 + 3(|V(G'_0)| - x_0 - 2y_0 - 2y_1) \\
&= 3|V(G'_0)| - 3(x_0 + y_0) - y_1 \geq 55.
\end{aligned}$$

However $F(G'_0) = 2|V(G'_0)| - |E(G'_0)| - 2 \leq 0$, contrary to the fact that G'_0 is reduced. Hence $y_0 \leq 1$, and so by Claim 5, we must have $|X_0| + |S_0| = x_0 + y_0 \leq 9$. It follows by (12) and (14) that $|A| = |X_0| + |S_0| + |X_1| + |S_1| \leq 12$. By Theorem 2.7(iii), $L(G)$ is Hamiltonian by the following claim.

Claim 6. *G_0' has a dominating eulerian subgraph containing A.*

Proof. We argue by contradiction. By Theorem 2.2, G_0' can be contracted to the graph $L_0 \cong P(10)$ such that $V(L_0) = \cup_{i\in\mathbb{Z}_5}\{u_i, v_i\}$, $E(L_0) = \cup_{i\in\mathbb{Z}_5}\{u_iv_i, v_iv_{i+1}, u_iu_{i+2}\}$ and the preimage of each vertex of L_0 contains at least one vertex in A, where $\mathbb{Z}_5$ is cyclic group of order 5. Let $V_1 \subseteq V(L_0)$ be the set such that for each vertex of V_1, its preimage in G_0' contains at least one vertex of $X_1 \cup S_0 \cup S_1$, and let $V_0 = V(L_0)\backslash V_1$. Then the preimage in G_0' of each vertex in V_0 contains at least one vertex of X_0, and by (14),

$$|V_1| \le |X_1| + |S_0| + |S_1| \text{ and } |V_0| \ge 10 - (|X_1| + |S_0| + |S_1|). \tag{15}$$

If $G_0' \ne L_0$, then for any vertex $v \in V(L_0)$, redefine $\ell(v) = |E(PI_G(v))| + d_{L_0}(v)$. For any vertex $v_1 \in V(L_0)$ with $H_1 = PI_{G_0'}(v_1)$ being nontrivial, we have the following conclusions.

(a) $Y_1 \cap E(H_1) = \emptyset$.
We argue by contradiction, and assume that $x_1y_1 \in Y_1 \cap E(H_1)$. Then $v_1 \in V_1$ and $|X_1| + |S_1| = |X_1| + |Y_1| \le 2$ by (12). As $|S_0| = |Y_0| \le 1$ and by (15), $|V_0\backslash\{v_1\}| = |V_0| \ge 10 - (|X_1| + |S_1| + |S_0|) \ge 7$. As $|X_0| \ge |V_0\backslash\{v_1\}| + |V(H_1) \cap X_0|$ and by Claim 5, we have $|V(H_1) \cap X_0| \le 2$. Then $|N_{G_0'}(\{x_1, y_1\}) \cap V(H_1)| \le 2$ since $N_{G_0'}(\{x_1, y_1\}) \subseteq X_0$. If $|N_{G_0'}(\{x_1, y_1\}) \cap V(H_1)| = 1$, then by symmetry, we may assume that $\{x_1u_1, x_1v_2, y_1v_5\} \subseteq E(L_0)$ and $y_1z_1 \in E(H_1)$ for some vertex z_1, and so y_1z_1 is an cut-edge of G_0', contrary to the fact that $\kappa'(G_0') \ge 3$. Hence $|N_{G_0'}(\{x_1, y_1\}) \cap V(H_1)| = |V(H_1) \cap X_0| = 2$. By Claim 5, $|V_0\backslash\{v_1\}| = 7$, $|X_0| + |Y_0| = |X_0| = 9$, and so $|Y_1| = 2$. This implies that there is an edge $uv \in Y_1 \cap E(L_0)$ with $\{u, v\} \subseteq V(L_0) \cap A$, contrary to the choice of A.

(b) $|V(H_1) \cap X_0| \ge 2$ **and** $|V(H_1) \cap X_0| \ge 3$ **if** $|V(H_1)| \ge 6$.
If $Y_0 \cap E(H_1) = \emptyset$, then for any edge $e \in E(H_1)$, it follows by *(a)* that $V(e) \cap X_0 \ne \emptyset$. Thus *(b)* holds by Lemma 2.6. We then assume that there is an edge $x_1y_1 \in Y_0 \cap E(H_1)$. By Claim 4, $|V(G_0')| \ge 19$. As $d_{G_0'}(x_1) + d_{G_0'}(y_1) \ge 15$ and $g(G_0') \ge 4$, we have $|N_{G_0'}(\{x_1, y_1\})| \ge 10$, and so $|V(H_1)| \ge 12$. By Lemma 2.6*(ii)*, $\alpha'(H_1) \ge 3$. If $\alpha'(H_1) \ge 4$, then $|V(H_1) \cap X_0| \ge 3$, and so *(b)* holds. We then assume that $\alpha'(H_1) = 3$ and $\{x_1y_1, x_2y_2, x_3y_3\} \subseteq E(H_1)$ with $\{x_2, x_3\} \subseteq X_0$. Then $H_1 - \{x_1, x_2, x_3, y_1, y_2, y_3\}$ has a stable set $\{z_1, z_2, z_3, z_4\} \subseteq \bar{X}$. If $E(\{z_1, z_2, z_3, z_4\}, \{y_2, y_3\}) = \emptyset$, then as $|N_{G_0'}(\bar{X}) \cap V(Y_0)| \le 1$, there is a collapsible subgraph $K_{3,4} \subseteq G_0'[\{z_1, z_2, z_3, z_4, x_2, x_3, x_1, y_1\}]$, contrary to the fact that G_0' is reduced. Hence $\{z_1, z_2, z_3, z_4, y_2, y_3\} \cap X_0 \ne \emptyset$, and so $|V(H_1) \cap X_0| \ge 3$.

(c) $|X_1| + |Y_1| \le 2$.
Assume by contradiction that $|X_1| + |Y_1| = 3$. Then $|X_1| = 3$ by (refeqa111). By *(b)*, $|E(PI_G(v_1))| \ge 2 \times \frac{|E(G)|+15}{10} - 1 - d_{L_0}(v_1) = \frac{|E(G)|-5}{5}$. If $X_1 = \{v_2, v_5, u_1\}$, then by Lemma 3.5*(i)* and *(ii)*, we have $\{v_3, v_4, u_2, u_3, u_4, u_5\} \subseteq V_0$. By Lemma 3.5*(iv)*, $\min\{\max\{\ell(v_3), \ell(u_3)\}, \max\{\ell(v_4), \ell(u_4)\}, \max\{\ell(u_2), \ell(u_5)\}\} \ge \frac{|E(G)|+5}{5}$. Without loss of generality, assume that $\min\{\ell(v_3), \ell(v_4), \ell(u_2)\} \ge \frac{|E(G)|+5}{5}$. However $|E(G)| \ge 3 \times \frac{|E(G)|+5}{5} + 3 \times \frac{|E(G)|+15}{10} + |E(PI_G(v_1))| - 15 > |E(G)|$, a contradiction. Hence $|X_1 \cap \{v_2, v_5, u_1\}| \le 2$. If $\{v_2, u_1\} \subseteq X_1$, then $\{v_3, u_2, u_3, u_4\} \subseteq V_0$ and either $v_4 \in X_1$ or $u_5 \in X_1$. If $v_4 \in X_1$, then by Lemma 3.5*(iii)*, $\{v_5, u_5\} \subseteq V_0$ and $v_3, u_4 \in X^t$ for some integer $t \ge 2$, implying that $\sum_k kx^k \ge 2 \times 2 + |\{v_5, u_2, u_3, u_5\}| + |V(H_1) \cap X_0| \ge 10$, contrary to Claim 5. If $u_5 \in X_1$, then by Lemma 3.5*(iii)*, $\{v_4, v_5\} \subseteq V_0$ and $u_2, u_3 \in X^t$ for some integer $t \ge 2$, implying that $\sum_k kx^k \ge 2 \times 2 + |\{v_3, v_4, v_5, u_4\}| + |V(H_1) \cap X_0| \ge 10$, contrary to Claim 5. So $|X_1 \cap \{v_2, v_5, u_1\}| \le 1$. Without loss of generality, we have $X_1 \in \{\{v_2, v_4, u_3\}, \{v_2, v_4, u_5\}, \{v_3, u_4, u_5\}\}$. If $X_1 = \{v_2, v_4, u_3\}$, then $\{v_5, u_1, u_2, u_4, u_5\} \subseteq V_0$ and $v_3 \in X^t$ for some integer $t \ge 3$, and so $\sum_k kx^k \ge 10$, a contradiction. If $X_1 = \{v_2, v_4, u_5\}$, then $\{u_1, u_2, u_3, u_4\} \subseteq V_0$ and $v_3, v_5 \in X^t$ for some integer $t \ge 2$, and so $\sum_k kx^k \ge 10$, a contradiction. If $X_1 = \{v_3, u_4, u_5\}$, then $\{v_2, v_5, u_1\} \subseteq V_0$ and $v_4, u_2, u_3 \in X^t$ for some integer $t \ge 2$, and so $\sum_k kx^k \ge 11$, a contradiction.

(d) $Y_0 \ne \emptyset$.
Assume by contradiction that $Y_0 = \emptyset$. By (15) and *(c)*, $|V_0| \ge 8$. It follows by Claim 5 that $|V(H_1) \cap X_0| = 2$, $|X_1| + |Y_1| = 2$, $|X_0| = 9$ and $X^t = \emptyset$ for any integer $t \ge 2$. By *(b)* and Lemma 2.6*(i)*, $|V(H_1)| \le 5$, $H_1 \cong K_{2,3}$ with $x_1, x_2 \in D_3(H_1) \cap X_0$ and $y_1, y_2, y_3 \in D_2(H_1)\backslash X_0$. By Lemma 3.5*(i)*, $\{v_2, v_5, u_1\} \cap X_1 = \emptyset$, and so $\{v_3, v_4, u_2, u_3, u_4, u_5\} \cap X_1 \ne \emptyset$. By symmetry, assume that $v_3 \in X_1$. Then $\{v_2, v_4, u_3\} \subseteq V_0$, and so $\{v_5, u_1, u_2, u_4, u_5\} \subseteq V_0$ since $X^t = \emptyset$ for any integer $t \ge 2$. However, $|X_0| \ge |V_0\backslash\{v_1\}| + 2 = 10$, a contradiction.

(e) $|V(H_1) \cap X_0| = 2$.
Assume by contradiction that $|V(H_1) \cap X_0| \ge 3$. As $|S_0| = |Y_0| = 1$ and by (15), *(c)* and Claim 5, $|V(H_1) \cap X_0| = 3$, $|X_1| + |Y_1| = 2$, $|V_0| = 7$, $|X_0| = 9$ and $X^t = \emptyset$ for any integer $t \ge 2$. If $\{v_2, v_5, u_1\} \cap (X_1 \cup V(Y_1)) \ne \emptyset$, then by symmetry, assume that $v_2 \in X_1 \cup V(Y_1)$. By Lemma 3.5 and as $X^t = \emptyset$ for any integer $t \ge 2$, we have $\{v_3, v_4, u_2, u_3, u_4, u_5\} \subseteq V_0$, and so $\{v_5, u_1\} \cap (X_1 \cup V(Y_1)) \ne \emptyset$. By symmetry, assume that $v_5 \in X_1 \cup V(Y_1)$. It follows by Lemma 3.5*(iv)* that $\min\{\max\{\ell(v_3), \ell(v_4)\}, \max\{\ell(u_2), \ell(u_5)\}\} \ge \frac{|E(G)|+5}{5}$. As $|V(H_1) \cap X_0| = 3$ and $g(G) \ge 3$, $|E(PI_G(v_1))| \ge 3 \times \frac{|E(G)|+15}{10} - 5 = \frac{3|E(G)|-5}{10}$. However $|E(G)| = |E(PI_G(v_1))| + 2 \times \frac{|E(G)|+5}{5} + 4 \times \frac{|E(G)|+15}{10} + 12 - 15 > |E(G)|$, a contradiction. Hence $\{v_2, v_5, u_1\} \cap (X_1 \cup V(Y_1)) = \emptyset$. Without loss of generality, we assume that $v_3 \in X_1 \cup V(Y_1)$. By Lemma 3.5 and as $X^t = \emptyset$ for any integer $t \ge 2$, we have $\{v_2, v_4, v_5, u_1, u_2, u_3, u_4, u_5\} \subseteq V_0$, and so $|V_0| \ge 8$, a contradiction.

To sum up, by (d) and Claim 4, $|Y_0| = 1$ and $|V(G_0')| \geq 19$; by (15), (c) and Claim 5, $|V_0| \geq 7$; by (b), (e) and Lemma 2.6(i), $|V(H_1) \cap X_0| = 2$ and $H_1 \cong K_{2,3}$. Those imply that L_0 has at least three vertices such that each of whose preimage in G_0' contains exactly 2 vertices in X_0, and so $|X_0| \geq (|V_0| - 3) + 2 \times 3 \geq 10$, a contradiction.

Hence $G_0' = L_0 \cong P(10)$. Then $X_0 = V_0$, $Y_0 = \emptyset$ by Claim 4. By the choice of A, $Y_1 = \emptyset$, and so $X_1 \neq \emptyset$. Without loss of generality, assume that $v_1 \in X_1$. If $|X_1| \geq 2$, by symmetry, assume that $v_3 \in X_1$. Thus $\{v_2, v_4, v_5, u_1, u_3\} \subseteq X_0$ and $v_2 \in X^t$ for some integer $t \geq 2$. Then $\{u_2, u_4, u_5\} \nsubseteq X_0$, since otherwise, $\sum_k k|X^k| \geq 10$, a contradiction. Without loss of generality, we must have $\{u_2, u_5\} \cap X_1 \neq \emptyset$. If $u_5 \in X_1$, then $\{v_4, u_1, u_2, u_4\} \subseteq X_0$ and $v_2, v_5, u_3 \in X^t$ for some integer $t \geq 2$, and so $\sum_k k|X^k| \geq 10$, a contradiction. Hence $u_2 \in X_1$. Then $\{v_4, v_5, u_1, u_3, u_4, u_5\} \subseteq X_0$ and $v_2 \in X^t$ for some integer $t \geq 3$. By Lemma 3.5(iv), $\min\{\max\{\ell(v_4), \ell(u_4)\}, \max\{\ell(v_5), \ell(u_5)\}, \max\{\ell(u_1), \ell(u_3)\}\} \geq \frac{|E(G)|+5}{5}$, and so we obtain a contradiction that $|E(G)| > |E(G)|$.

Thus $X_1 = \{v_1\}$ and $|X_0| = 9$. As $v_1 \in \Lambda'(G_0)$ and by the proof of Lemma 3.5(i), v_1 is not a good-leaf of $PI_G(v_2) \cup \{v_2v_1, v_2v_3, v_2u_2\}$ and $PI_G(v_5) \cup \{v_4v_5, v_5v_1, v_5u_5\}$. By Lemma 3.1$(i)$, $PI_G(v_2)$ has a v_1-net (x_1y_1, T_1) satisfying the assumption of Lemma 3.1(i). As $T_1 \in \{T_{2,2,2}, T_{2,2,1}, T_{2,1,2}\}$, $T_1 \cup PI_G(u_2v_2v_3)$ contains a subgraph $T_2 \cong T_{2,2,2}$ with $x_1y_1 \in E_L(T_{2,2,2})$. By (2), $d_G(x_1) + d_G(y_1) \geq \frac{|E(G)|+25}{10} \geq 15$, and so $\{x_1, y_1\} \cap D_2(G) \neq \emptyset$ by Lemma 3.1(β). Without loss of generality, assume that $x_1 \in D_2(G)$. Then $d_G(y_1) \geq \frac{|E(G)|+5}{10}$. If v_2 is a good-leaf of $PI_G(v_3) \cup \{v_2v_3, v_3v_4, v_3u_3\}$, then by the construction of Γ_1, $V(e_1) \cap V(e_2) \subseteq \{x_1\}$ and $PI_G(v_2)$ has an edge $e_2 = x_2y_2 \nsubseteq \Gamma_1 - w_{p+1}$ such that $e_2 \in E(G) \cap E_L(T_{2,2,2})$, and so $d_G(x_2) + d_G(y_2) \geq \frac{|E(G)|+25}{10}$. It follows that $\ell(v_2) \geq |E(PI_G(v_2))| + d_{G_0'}(v_2) \geq (|E_G(y_1)| + |E_G(e_2)| - 1 - d_{G_0'}(v_2)) + d_{G_0'}(v_2) \geq d_G(y_1) + d_G(x_2) + d_G(y_2) - 2 \geq \frac{2|E(G)|+10}{10}$. Since $v_1 \in X_1$, we have $|E(PI_G(v_1))| \geq 1$, and so $\ell(v_1) \geq 4$. Hence $|E(G)| \geq \sum_{v \in V(G_0')} \ell(v) - |E(G_0')| \geq \frac{2|E(G)|+10}{10} + 8 \times \frac{|E(G)|+15}{10} + 4 - 15 > |E(G)|$, a contradiction. This implies that v_2 is not a good-leaf of $PI_G(v_3) \cup \{v_2v_3, v_3v_4, v_3u_3\}$. By symmetry, v_3 is not a good-leaf of $PI_G(v_4) \cup \{v_3v_4, v_4v_5, v_4u_4\}$ and v_4 is not a good-leaf of $PI_G(v_5) \cup \{v_4v_5, v_5v_1, v_5u_5\}$. As v_1, v_4 are not good-leaves of $PI_G(v_5) \cup \{v_4v_5, v_5v_1, v_5u_5\}$ and by Lemma 3.4, $v_5 \in X^t$ for some integer $t \geq 2$, and so $\sum_k k|X^k| \geq 10$, a contradiction. Hence Claim 6 holds. □

By Claim 6 and by Theorem 2.7(iii), we conclude that Theorem 1.5 must be valid.

4. Remarks

For a claw-free graph H, a vertex $x \in V(H)$ is *eligible* if $H[N_H(x)]$ is a connected noncomplete subgraph of H. The *local completion* of H at x is the subgraph H_x^* obtained from H by adding all missing edges with both vertices in $N_H(x)$. The closure $cl(H)$ of H was defined in [21] as the graph obtained from H by recursively performing the local completion operation at eligible vertices as long as possible. In [22], the concept of an SM-closure H^M is obtained from H by performing local completions at some (but not all) eligible vertices, where these vertices are chosen in a special way such that the resulting graph is a line graph of a multigraph while still preserving the (non-)Hamilton-connectedness of H. The following result summarizes basic properties of $cl(H)$ and H^M.

Theorem 4.1. *Let H be a claw-free graph and $cl(H)$, H^M be its closures. Each of the following holds.*
(i) (Ryjáček, [21]) $cl(H)$ is well-defined, there is a triangle-free simple graph G_1 such that $cl(H) = L(G_1)$, and H is Hamiltonian if and only if $cl(H)$ is Hamiltonian.
(ii) (Ryjáček and Vrána, [22]) H^M is uniquely determined, there is a multigraph G_2 such that $H^M = L(G_2)$, and H^M is Hamilton-connected if and only if H is Hamilton-connected.

For a 3-connected claw-free graph H, by Theorem 4.1, both of its closures $cl(H)$ and H^M are line graphs. Our next step is to generalize Theorem 1.5 to the claw-free graph version, and leave it as Conjecture 4.2(i). Define H_8' to be the graph obtained from C_8 by adding four chords between four pairs of vertices of maximum distance in C_8, and by attaching $\frac{|E(H_8')|-12}{8}$ pendant edges at each vertex of degree 3. Then $H = L(H_8')$ is a 3-connected non-Hamilton-connected graph with $\delta_{N_{1,1,1}}(H) = \frac{|V(H)|+4}{8}$. We hence leave the claw-free Hamilton-connected graph version as Conjecture 4.2(ii).

Conjecture 4.2. *Let H be a 3-connected claw-free simple graph on n vertices.*
(i) If $\delta_{N_{1,1,1}}(H) \geq \frac{n+5}{10}$, then either H is Hamiltonian or $cl(H) \cong L(P(10)')$.
(ii) If $\delta_{N_{1,1,1}}(H) \geq \frac{n+4}{8}$, then either H is Hamilton-connected or $H^M \cong L(H_8')$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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