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On the s-hamiltonianicity of an hourglass-free line graph

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ABSTRACT

Fault-tolerant networks are often modeled as s-hamiltonian graphs. Thus it is of interests to find graph families in which whether a graph is s-hamiltonian can be determined in polynomial time. An hourglass is a graph obtained from K_5 by deleting the edges in a cycle of length 4, and an hourglass-free graph is one that has no induced subgraph isomorphic to an hourglass. Kriesell in [J. Combin. Theory Ser. B, 82 (2001), 306-315] proved that every 4-connected hourglass-free line graph is Hamilton-connected, and Kaiser, Ryjáček and Vrána in [Discrete Mathematics, 321 (2014) 1-11] extended it by showing that every 4-connected hourglass-free line graph is 1-Hamilton-connected. We characterize all essentially 4-edge-connected graphs whose line graph is hourglass-free. Consequently we prove that for any integer s and for any hourglass-free line graph $L(G)$, each of the following holds.
(i) If $s \geq 2$, then $L(G)$ is s-hamiltonian if and only if $\kappa(L(G)) \geq s+2$;
(ii) If $s \geq 1$, then $L(G)$ is s-Hamilton-connected if and only if $\kappa(L(G)) \geq s+3$.

1. Introduction

We consider finite graphs without loops but permitting multiple edges, and follow [1] for undefined terms and notations. In particular, for a graph G, $\kappa(G)$, $\kappa'(G)$ and $\delta(G)$ denote the connectivity, edge-connectivity and the minimum degree of G, respectively. For any integer s with $0 \leq s \leq n-3$, a graph G of order $n \geq 3$ is **s-hamiltonian** (or **s-Hamilton-connected**, respectively) if for any $X \subseteq V(G)$ with $|X| \leq s$, $G-X$ is hamiltonian ($G-X$ is Hamilton-connected, respectively). The line graph of a graph G, denoted by $L(G)$, is a simple graph with $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent.

Certain fault-tolerant networks are modeled as s-hamiltonian graphs [8]. Thus it is of interests to find graph families in which whether a graph is s-hamiltonian can be determined in polynomial time. It has been shown in [11] that 1-Hamilton-connectedness is polynomial-time decidable in the class of hourglass-free line graphs. A few most fascinating problems in this area are presented below. In [21], Ryjáček uses an ingenious argument to show that Conjecture 1.1(i) below is equivalent to a seeming stronger statement in Conjecture 1.1(ii). Later, Ryjáček and Vrána in [22] indicated that all the statements in Conjecture 1.1 are mutually equivalent.

Conjecture 1.1. *(i) (Thomassen [24]) Every 4-connected line graph is hamiltonian.*
(ii) (Matthews and Sumner [19]) Every 4-connected claw-free graph is hamiltonian.

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(iii) (Kučzel and Xiong [13]) Every 4-connected line graph is Hamilton-connected.
(iv) (Ryjáček and Vrána [22]) Every 4-connected claw-free graph is Hamilton-connected.

Towards Conjecture 1.1, Zhan gave a first result in this direction. The best result so far is obtained by Kaiser and Vrána [10] and Kaiser, Ryjáček and Vrána [11]. The following are known.

Theorem 1.2. *Let G be a graph.*
(i) (Zhan, Theorem 3 in [26]) If $\kappa(L(G)) \geq 7$, then $L(G)$ is Hamilton-connected.
(ii) (Kaiser and Vrána [10]) If $\kappa(L(G)) \geq 5$ and $\delta(L(G)) \geq 6$, then $L(G)$ is hamiltonian.
(iii) (Kaiser, Ryjáček and Vrána [11]) If $\kappa(L(G)) \geq 5$ and $\delta(L(G)) \geq 6$, then $L(G)$ is 1-Hamilton-connected.

By definitions, if $s \geq 1$, then s-hamiltonian graphs are $(s-1)$-hamiltonian, and s-hamiltonian graphs are hamiltonian. It is well known that if a graph G is s-hamiltonian, then G is $(s+2)$-connected. Broersma and Veldman in [3] consider the problem of determining the range of s such that a line graph $L(G)$ is s-hamiltonian if and only if $L(G)$ is $(s+2)$-connected. They define, for an integer $k \geq 0$, a graph G is **k-triangular** if every edge of G lies in at least k triangles of G.

Theorem 1.3. *(Broersma and Veldman, [3]) Let $k \geq s \geq 0$ be integers and let G be a k-triangular simple graph. Then $L(G)$ is s-hamiltonian if and only if $L(G)$ is $(s+2)$-connected.*

Broersma and Veldman in [3] proposed an open problem of determining the range of integral values s such that within triangular graphs, $L(G)$ is s-hamiltonian if and only if $L(G)$ is $(s+2)$-connected. This problem was first settled by Chen et al. in [7]. Later, it is extended in [15].

Theorem 1.4. *Let G be a connected graph and let k, s be nonnegative integers. Each of the following holds.*
(i) (Chen et al., Theorem 1.2 in [7]) Suppose that $0 \leq s \leq \max\{2k, 6k-16\}$, and G is a k-triangular simple graph. Then $L(G)$ is s-hamiltonian if and only if $L(G)$ is $(s+2)$-connected.
(ii) (Theorem 3.1 of [14]) Suppose $s \geq 2$ and $\kappa'(G) \geq s+2$. Then $L(G)$ is s-Hamilton-connected if and only if $\kappa(L(G)) \geq s+3$.
(iii) (Theorem 1.3 in [15]) Suppose $s \geq 5$. Then $L(G)$ is s-hamiltonian if and only if $L(G)$ is $(s+2)$-connected.

Similar problem for s-Hamilton-connectedness is also considered by researchers. In addition to the result by Kaiser, Ryjáček and Vrána (Theorem 1.2(iii)), the following are also known.

Theorem 1.5. *Let G be a claw-free graph and $s \geq 2$ be an integer.*
(i) (Kriesell [12]) If $\kappa(L(G)) \geq 4$, then $L(G)$ is Hamilton-connected.
(ii) (Theorem 1.6 of [18]) The line graph $L(G)$ is s-hamiltonian if and only if $\kappa(L(G)) \geq s+2$; and $L(G)$ is 1-hamiltonian connected if and only if $\kappa(L(G)) \geq 4$.

Theorem 1.5(i) is further extended to Quasi claw-free graphs in [17] and to almost claw-free graphs in [16]. In view of the Conjecture 1.1 and, results presented in Theorems 1.2, 1.4 and 1.5, the following has been considered in [14,15,18].

Conjecture 1.6. *Let G be a connected graph and let s be an integer.*
(i) ([15]) If $s \geq 2$, then $L(G)$ is s-hamiltonian if and only if $\kappa(L(G)) \geq s+2$.
(ii) ([14]) If $s \geq 1$, then $L(G)$ is s-Hamilton-connected if and only if $\kappa(L(G)) \geq s+3$.

Let $P(10,\ell)$ be the graph obtained from $P(10)$, the Petersen graph by attaching $\ell > 0$ pendant edges at every vertex of $P(10)$. It is known that $L(P(10,\ell))$ is 3-connected but not hamiltonian. Hence the values of s in Conjecture 1.6 cannot be smaller.

An hourglass is the graph consisting of two triangles meeting in exactly one vertex. Thus if C denotes a cycle of length 4 in a K_5, then $K_5 - E(C)$ is the hourglass graph. A graph G is **hourglass-free** if G contains no induced subgraph isomorphic to an hourglass. The following have been proved.

Theorem 1.7. *Let G be an hourglass-free and claw-free graph.*
(i) (Broersma et al., Theorem 6 in [2]) If $\kappa(G) \geq 4$, then G is hamiltonian.
(ii) (Kriesell, Corollary 4 in [12]) If $\kappa(G) \geq 4$, then G is Hamilton-connected.
(iii) (Kaiser, Ryjáček and Vrána [11]) If $\kappa(G) \geq 4$, then G is 1-Hamilton-connected.

In this paper, we prove the following theorem, which provides support to Conjecture 1.6.

Theorem 1.8. *Let $L(G)$ be an hourglass-free line graph and s be an integer. Each of the following holds.*
(i) If $s \geq 2$, then $L(G)$ is s-hamiltonian if and only if $\kappa(L(G)) \geq s+2$.
(ii) If $s \geq 1$, then $L(G)$ is s-Hamilton-connected if and only if $\kappa(L(G)) \geq s+3$.

Consequently, for sufficiently large value of s, whether an hourglass-free graph is s-hamiltonian and s-Hamilton-connected is polynomial-time decidable. Preliminaries will be provided in the next section and the structural characterization is stated and proved in Section 3. The proof of Theorem 3.2 will be given in the last section.

2. Preliminaries

A graph is trivial if it has no edges. Unless otherwise stated, we always assume that the graph G under discussion is nontrivial. For integers $i, j \geq 0$, let $D_i(G) = \{v \in V(G) : d_G(v) = i\}$, and $D_{\geq j}(G) = \cup_{i \geq j} D_i(G)$. Define $E_G(v) = \{e \in E(G) : e$ is incident with v in $G\}$, $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$. We write $H \subseteq G$ to mean that H is a subgraph of G.

2.1. *s-hamiltonian line graphs*

For a graph G, let $O(G)$ denote the set of odd degree vertices in G. A graph G is **eulerian** if G is connected with $O(G) = \emptyset$, and is **supereulerian** if G has a spanning eulerian subgraph. A subgraph H of a graph G is **dominating** if $G - V(H)$ is edgeless. Harary and Nash-Williams proved a very useful relationship between hamiltonian cycles in the line graph $L(G)$ and dominating eulerian subgraphs in G.

Theorem 2.1. *(Harary and Nash-Williams [9]) For a connected graph G with $|E(G)| \geq 3$, $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

As deleting vertices in $L(G)$ amount to deleting the corresponding edges in G and then removing the resulting isolated vertices, for simplicity, we use $G - S$ in the discussions instead of $G - S - D_0(G - S)$. Throughout this article, isolated vertices arising from edge deletion will be deleted automatically unless otherwise specified. With the same arguments used in [9], the following is observed.

Proposition 2.2. *(Theorem 2.7 of [15]) Let $s \geq 0$ be an integer and G be a connected graph with $|E(G)| \geq s+3$. The line graph $L(G)$ is s-hamiltonian if and only if for any $S \subseteq E(G)$ with $|S| \leq s$, $G - S$ has a dominating eulerian subgraph.*

Let G be a graph G and $k > 0$ be an integer. An edge-cut X of G is an **essential k-edge-cut** of G if $|X| = k$ and each side of $G - X$ has at least one edge. A connected graph G is **essentially k-edge-connected** if G does not have an essential k'-edge-cut for any $k' < k$. Let $\mathcal{K}_0$ denote the family of graphs such that a graph $G \in \mathcal{K}_0$ if and only if both $\kappa'(G) \geq 2$ and G is spanned by a K_3, or contains a vertex incident with all edges. We have the following observation.

Observation 2.3. *Each of the following holds.*
(i) A graph G does not have an essential edge cut if and only if $G \in \mathcal{K}_0$.
(ii) If $G \in \mathcal{K}_0$, then $L(G)$ is a complete graph.
(iii) (Proposition 1.1.3 of [23]) Let G be a graph. Then $\kappa(L(G)) \geq k$ if and only if $ess'(G) \geq k$.

Observation 2.3 leads to the following definition: if $G \in \mathcal{K}_0$, then define $ess'(G) = |E(G)| - 1$; otherwise let $ess'(G)$ be the largest integer k such that G is essentially k-edge-connected. Observation 2.3(ii) suggests that when discussing hamiltonicity of $L(G)$ for a graph G, we may assume that $G \notin \mathcal{K}_0$.

Definition 2.4. Let $X_1(G) = \{e \in E(G)$: e is incident with a vertex in $D_1(G)\}$. For each vertex $v \in D_2(G)$, let $E_G(v) = \{e_v, e'_v\}$ be the set of edges incident with v and define

$$X_2(G) = \{e_v : v \in D_2(G)\}, X'_2(G) = \{e'_v : v \in D_2(G)\}, \tag{2.1}$$

and

$$Y(G) = E(G) - (X_1(G) \cup X_2(G) \cup X'_2(G)). \tag{2.2}$$

Then $(X_1(G), X_2(G), X'_2(G), Y(G))$ is a partition of $E(G)$. The **core** of G is defined as

$$G_0 = G/(X_1(G) \cup X'_2(G)).$$

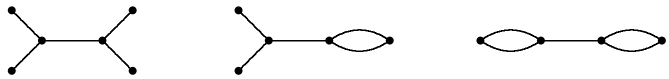

Fig. 1. Graphs in $\mathcal{F}_0$.

Shao in her dissertation [23] indicated that if $G \notin \mathcal{K}_0$ and $ess'(G) \geq 3$, then

$$G_0\text{, the core of } G\text{, is uniquely determined and } \delta(G_0) \geq \kappa'(G_0) \geq 3. \tag{2.3}$$

Lemma 2.5. *Suppose $G \notin \mathcal{K}_0$ with $ess'(G) \geq 3$ and let G_0 be the core of G. Let $s \geq 0$ be an integer, $S \subset E(G_0)$ with $|S| \leq s$, and $G_S = (G_0 - S) - D_1(G_0 - S)$. If for any $S \subset E(G_0)$ with $|S| \leq s$, G_S is supereulerian, then for any edge subset $Z \subset E(G)$ with $|Z| \leq s$, $G - Z$ has a dominating eulerian trail. Consequently, $L(G)$ is s-hamiltonian.*

Proof. Fix a subset $Z \subseteq E(G)$ with $|Z| \leq s$. Define $X_1(G)$ and $X_2(G)$ as in Definition 2.4. Let $S = Z - (X_1(G) \cup X_2'(G))$. We adopt the convention that

$$\text{if } v \in D_2(G) \text{ and } (Z - X_1(G)) \cap E_G(v) = \{e\}\text{, then we assume that } e \in S \cap X_2(G). \tag{2.4}$$

Thus $S \subset E(G_0)$ with $|S| \leq |Z| \leq s$.

Suppose that G_S is supereulerian, which implies that $G - Z$ has a dominating eulerian trail. By Proposition 2.2, $L(G)$ is s-hamiltonian. □

2.2. Catlin's reduction method

In [4], Catlin defined collapsible graphs. A graph is **collapsible** if for every subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph Γ such that $O(\Gamma) = R$. As one can take $R = \emptyset$, it follows by definition that every collapsible graph is supereuerlian. For a graph G and an edge subset $X \subseteq E(G)$, G/X denotes the graph obtained from G by contracting each edge in X and then deleting resulting loops. If H is a subgraph of G, then we use G/H for $G/E(H)$. If H is connected, and if v_H is the vertex in G/H onto which H is contracted, then H is the **pre-image** of v_H in G/H. If $H_1, H_2, ..., H_k$ are the list of all maximal collapsible subgraphs of G, then $G' = G/(\cup_{i=1}^k H_i)$ is the **reduction** of G; a graph is **reduced** if it is the reduction of some graph. The next theorem briefs some of the useful properties related to collapsible graphs.

Theorem 2.6. *Let G be a connected graph. Each of the following holds.*
(i) (Catlin, Corollary of Theorem 3 in [4]) Let H be a collapsible subgraph of G. Then G is supereulerian (collapsible, respectively) if and only if G/H is supereulerian (collapsible, respectively). In particular, if G' is the reduction of G, then G is supereulerian (collapsible, respectively) if and only if G' is supereulerian (a K_1, respectively).
(ii) (Catlin [4]) Every cycle of length at most 3 is collapsible.

For a graph G, let $F(G)$ be the minimum number of additional edges that must be added to G to result in a graph G_1 with two edge-disjoint spanning trees. Thus $F(G) = 0$ if and only if G contains two edge-disjoint spanning trees. Theorem 2.7(iii) below is an application of the well-known spanning tree packing theorem of Nash-Williams ([20]) and Tutte ([25]).

Theorem 2.7. *Let G be a connected graph. Each of the following holds.*
(i) (Catlin, Theorem 7 in [4]) If $F(G) \leq 1$, then G is collapsible if and only if $\kappa'(G) \geq 2$.
(ii) (Catlin et al., Theorem 1.3 in [5]) If $F(G) \leq 2$, then either G is collapsible, or the reduction of G is a K_2 or a $K_{2,t}$ for some integer $t \geq 1$.
(iii) (Theorem 1.1 in [6]) For any integer $k > 0$, $\kappa'(G) \geq 2k$ if and only if for any edge subset X with $|X| \leq k$, $G - X$ contains k edge-disjoint spanning trees.

3. Proof of Theorem 1.8

For a graph G, if e is an edge not in $E(G)$ but $V(e) \subseteq V(G)$, then we define $G + e$ to be the graph with vertex set $V(G)$ and edge set $E(G) \cup \{e\}$. Like wise, if X is an edge subset consisting of edges whose end vertices are in $V(G)$, then $G + X$ is the graph spanned by G with edge set $E(G) \cup X$.

Define $\mathcal{F}_0$ to be the family consisting of the three graphs depicted in Fig. 1. A graph G is $\mathcal{F}_0$**-clear** if G does not have a (not necessarily induced) subgraph isomorphic to a member in $\mathcal{F}_0$. Thus

$$L(G) \text{ is hourglass-free if and only if } G \text{ does not contain a member in } \mathcal{F}_0 \text{ as a subgraph.} \tag{3.5}$$

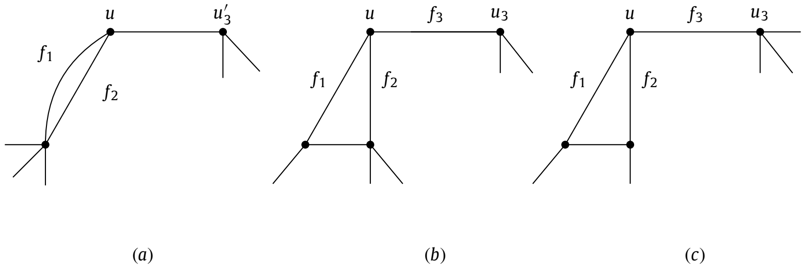

Fig. 2. Illustration of the proof of Lemma 3.1 and Lemma 3.2.

Throughout the rest of this section, we let G be a connected graph not in $\mathcal{K}_0$ and with $ess'(G) \geq 4$, and let G_0 be the core of G as define in Definition 2.4. An edge $e \in E(G)$ is an X_2-edge if $e \in X_2(G)$. By the definition of G_0, we can view $E(G_0)$ as a subset of $E(G)$. Thus we define an edge $e \in E(G_0)$ to be an X_2-edge of G_0 if e is an X_2-edge of G. Let $u \in D_3(G_0)$ be an arbitrary vertex with $E_{G_0}(u) = \{f_1, f_2, f_3\}$, where for each $i \in \{1, 2, 3\}$, $f_i = uu_i$. Then we have the following lemmas.

Lemma 3.1. *Suppose G does not contain a subgraph isomorphic to a member in $\mathcal{F}_0$. Each of the following holds.*
(i) Every edge in $E_{G_0}(u)$ is not an X_2-edge of G_0.
(ii) There is a cycle C_u of G_0 with $|E(C_u)| \leq 3$ such that $|E(C_u) \cap E_{G_0}(u)| = 2$.
*(iii) For any $w \in D_3(G_0)$, let C_w be a fixed cycle (called the **short cycle of** w) of length at most 3 whose existence is warranted by (ii). Then for any $Y \subseteq E_{G_0}(u)$ with $|Y| \leq 2$, there are at most three vertices in $N_G[u]$ whose short cycles containing an edge in Y.*

Proof. Let H_u denote the preimage of u in G. If $|V(H_u)| \geq 2$, then $E_{G_0}(u)$ is an essential edge-cut of G, contrary to $ess'(G) \geq 4$. Hence we must have $V(H_u) = \{u\}$. Next we assume that f_1 is a X_2-edge of G_0 by symmetry. Then by definition, there exists a vertex $v' \in D_2(G)$ such that $E_G(v') = \{f_1, f_1'\}$ for some edge $f_1' \in X_2'(G)$. As $G \notin \mathcal{K}_0$, it follows that either $\{f_1, f_2, f_3\}$ or $\{f_1', f_2, f_3\}$ is an essential edge-cut of G, contrary to the assumption that $ess'(G) \geq 4$. This implies Lemma 3.1(i).

We argue by contradiction to show (ii) and assume that no two edges in $\{f_1, f_2, f_3\}$ lie in a cycle of length at most 3 in G_0. Thus there exist distinct vertices $u_1, u_2, u_3 \in V(G_0)$ such that $f_i = uu_i$. For each vertex $w \in V(G_0)$, recall that H_w denote the contraction preimage of w in G. Assume that $v, v_1 \in V(G)$ such that $f_1 = vv_1 \in E(G)$. (Thus $v \in V(H_u)$ and $v_1 \in V(H_{u_1})$). By Lemma 3.1(i), each of v and v_1 has degree at least 3 in G, and so G contains a subgraph isomorphic to a member in $\mathcal{F}_0$, contrary to the assumption of the lemma. Hence Lemma 3.1(ii) must hold.

To prove (iii), suppose first that $E(C_u) = \{f_1, f_2\}$ with $u_1 = u_2$. (See Fig. 2(a) for an illustration.) As $u \in D_3(G_0)$ and $ess'(G_0) \geq 4$, $d_{G_0}(u_1) \geq 5$. By (i), f_3 is not an X_2-edge, and so there exists a vertex $u_3' \in D_{\geq 3}(G)$ such that $f_3 = uu_3'$. If $E_G(u_3') - \{f_3\}$ contains two edges f_3', f_3'' that are not in $E_{G_0}(u_1)$, then $G[\{f_1, f_2, f_3, f_3', f_3''\}] \in \mathcal{F}_0$, a contradiction to the assumption of the lemma. Hence we may assume that $f_3' \in E_{G_0}(u_1)$. Since $u \in D_3(G_0)$, for any $w \in D_3(G_0) - \{u, u_1, u_3\}$, C_w does not contain u and so Lemma 3.1(iii) follows. Hence we assume that $E_{G_0}(u)$ contains no cycles of length 2. By symmetry, assume that $f_1 \in E(C_u)$. (See Fig. 2(b) and (c) for an illustration.) By $ess'(G_0) \geq 4$, at most two vertices in $\{u_1, u_2, u_3\}$ are in $D_3(G_0)$, and so for any $w \in D_3(G_0) - N_{G_0}[u]$, C_w does not contain an edge in Y. $\square$

Lemma 3.2. *Let $G \notin \mathcal{K}_0$ be a connected graph with $ess'(G) \geq 4$ such that G does not have a subgraph isomorphic to a member in $\mathcal{F}_0$, and let G_0 be the core of G. Then for any edge subset $Y \subseteq E(G_0)$ with $|Y| \leq 2$, $(G_0 - Y) - D_1(G_0 - Y)$ is supereulerian.*

Proof. Suppose that $|D_3(G_0)| = 0$. Then as $\kappa'(G_0) \geq 3$ and $ess'(G_0) \geq 4$, we conclude that $\kappa'(G_0) \geq 4$, and so by $|Y| \leq 2$ and by Theorem 2.7(i) and (iii), $G_0 - Y$ is collapsible, and supereulerian. Hence we assume that $|D_3(G_0)| > 0$. We firstly justify the following claim.

Claim 1. *If for some $u \in D_3(G_0)$, $Y \subseteq E_{G_0}(u)$, then $(G_0 - Y) - D_1(G_0 - Y)$ is supereulerian.*

Denote $N_G(u) = \{u_1, u_2, u_3\}$. By Lemma 3.1(iii) and by symmetry, we may assume that every C_w with $w \in D_3(G_0) - \{u, u_1, u_2\}$ is still a cycle in $G_0 - Y$, and that $u_3 \in D_{\geq 4}(G_0)$. Suppose first that $Y = \{f_1\}$. Then let f_1', f_2' be distinct edges not in G_0 but with $V(f_i') = V(f_i)$, for $i \in \{1, 2\}$, and let $X' = \cup_{w \in D_3(G_0) - \{u, u_1, u_2\}} E(C_w)$. As $E_{G_0/X'}(u)$, $E_{G_0/X'}(u_1)$ and $E_{G_0/X'}(u_2)$ are the only possible edge cuts of size 3, it follows that $\kappa'((G_0 + \{f_1', f_2'\})/X') \geq 4$, and so by Theorem 2.7(iii), $(G_0 + \{f_1', f_2'\})/X' - \{f_1', f_2'\}$ has two edge-disjoint spanning trees. It follows by $\kappa'(G_0) \geq 3$ that $F((G_0 + \{f_1', f_2'\})/X' - \{f_1, f_1', f_2'\}) \leq 1$ and $\kappa'((G_0 + \{f_1', f_2'\})/X' - \{f_1, f_1', f_2'\}) \geq 2$. By Theorem 2.7(i), $(G_0 - f_1)/X' = (G_0 + \{f_1', f_2'\})/X' - \{f_1, f_1', f_2'\}$ is collapsible, and so supereulerian. By Theorem 2.6, $G_0 - f_1$ is supereulerian.

Hence we assume that $|Y| = 2$ and by symmetry, $f_1 \in Y$. By Lemma 3.1(iii), we may assume that $u_1, u_2 \in N_{G_0}(u)$ such that $D_3(G_0) \cap N_{G_0}(u) \subseteq \{u_1, u_2\}$ (see Fig. 2(c)). Now let f'' be an edge not in $E(G_0)$ and with $V(f'') = \{u_1, u_2\}$. Then as $E_{G_0/(X' \cup \{f_3\})}(u_1)$ and $E_{G_0/(X' \cup \{f_3\})}(u_2)$ are the only possible edge cuts of size 3(see Fig. 2(c)), it follows that $\kappa'((G_0 + \{f''\})/(X' \cup \{f_3\})) \geq 4$, and so by Theorem 2.7(iii), $(G_0 + \{f''\})/(X' \cup \{f_3\}) - Y$ has two edge-disjoint spanning trees. Thus again we have $F((G_0+\{f''\})/(X'\cup\{f_3\})-(Y\cup\{f''\}))\le 1$ and $\kappa'((G_0+\{f''\})/(X'\cup\{f_3\})-Y)\ge 2$. By Theorem 2.7(i), $(G_0-u)/X'=(G_0+\{f''\})/(X'\cup\{f_3\})-(Y\cup\{f''\})$ is collapsible, and supereulerian. By Theorem 2.6, G_0-Y is supereulerian. This proves Claim 1.

By Claim 1, we may assume that $|Y|=2$ and Y is a matching of G_0. Let $Y=\{e_1,e_2\}$ with $e_i=u_iv_i$ for $i\in\{1,2\}$. Define $X''=\cup_{w\in D_3(G_0)-\{u_1,v_1,u_2,v_2\}}E(C_w)$, and let e_1',e_2' denote two edges not in $E(G_0)$ such that $V(e_i')=\{u_i,v_i\}$. Then as $ess'(G_0)\ge 4$, we have $\kappa'((G_0+\{e_1',e_2'\})/X'')\ge 4$. It follows by Theorem 2.7(iii) that $F(((G_0+\{e_1',e_2'\})/X'')-\{e_1',e_2'\})=0$, and $F(G_0/X''-Y)=F(((G_0+\{e_1',e_2'\})/X'')-(Y\cup\{e_1',e_2'\}))\le 2$. By Theorem 2.7(ii), either $G_0/X''-Y$ is collapsible, or $G_0/X''-Y$ is contracted to a $K_{2,t}$ for some integer $t\ge 2$. Since the only edge cuts of size 2 in $G_0/X''-Y$ are of the form $E_{G_0/X''-Y}(u_i)$ and $E_{G_0/X''-Y}(v_i)$ with $i\in\{1,2\}$, we conclude that $t\le 4$. As $ess'(G_0)\ge 4$, we must have $t=4$. It follows that the reduction of $(G_0-Y)/X''=G_0/X''-Y$ is either K_1 or $K_{2,4}$, and so supereulerian. By Theorem 2.6, G_0-Y is also supereulerian. □

Proof of Theorem 1.8. It suffices to prove the sufficiency in either statement of the theorem. If $L(G)$ is a complete graph, then $L(G)$ is s-hamiltonian (in (i)) and $L(G)$ is s-Hamilton-connected (in (ii)). Thus we assume that $G\notin\mathcal{K}_0$.

We argue by induction on s to prove (i) and (ii) of Theorem 1.8. Suppose that $\kappa(L(G))\ge 4$. If $s=1$, then by Theorem 1.7(iii), $L(G)$ is 1-Hamilton-connected. If $s=2$, then let G be a graph whose line graph $L(G)$ is 4-connected. By Lemmas 2.5 and 3.2, $L(G)$ is 2-hamiltonian.

Assume that $s\ge 3$ for (i) and $s\ge 2$ for (ii), and that Theorem 1.8 holds for smaller values of s. Since $L(G)$ is hourglass-free, G is $\mathcal{F}_0$-clear. Let $X\subseteq E(G)$ be an edge set with $1\le|X|\le s$. Choose a subset $X'\subseteq X$ such that $|X'|\le s-1$. Let $X''=X-X'$, and $G'=G-X''$. Since G is $\mathcal{F}_0$-clear, G' is also $\mathcal{F}_0$-clear. By the definition of a line graph, $L(G')=L(G)-X''$ satisfies $\kappa(L(G'))\ge\kappa(L(G))-1$.

For (i), assume that $\kappa(L(G))\ge s+2$. Then $\kappa(L(G'))\ge(s-1)+2$, and so by induction on s and as $|X'|\le s-1$, we deduce that $L(G)-X=L(G')-X'$ is hamiltonian. For (ii), assume that $\kappa(L(G))\ge s+3$. Then $\kappa(L(G'))\ge(s-1)+3$, and so by induction on s and as $|X'|\le s-1$, we deduce that $L(G)-X=L(G')-X'$ is Hamilton-connected. This proves the theorem. □

4. Remarks

Theorem 1.8 is motivated by Conjecture 1.6 and provides evidences for this conjecture. Given the relationship between claw-free graphs and line graphs, the following conjectures seem to be natural.

Conjecture 4.1. *Let G be a connected claw-free graph and let s be an integer.*
(i) If $s\ge 2$, then G is s-hamiltonian if and only if $\kappa(G)\ge s+2$.
(ii) If $s\ge 1$, then G is s-Hamilton-connected if and only if $\kappa(G)\ge s+3$.

Ryjáček [21] and Ryjáček and P. Vrána [22] ingeniously brought in the related closure concepts to show that Conjecture 1.6 and Conjecture 4.1 are equivalent for a few smaller values of s. In [22], Ryjáček and Vrána proved that all the conjectures stated in Conjecture 1.1 are equivalent. There results in fact can also be applied to show the equivalence between Thomassen's conjecture (stated as Conjecture 1.1(i)) and Conjecture 4.1(ii). It is obvious that the special case Conjecture 4.1(ii) when $s=1$ implies Thomassen's conjecture. On the other hand, for any integer $s\ge 1$, let G be a claw-free graph with $\kappa(G)\ge s+3$. For any nonempty vertex subset $S\subset V(G)$ with $|S|\le s$, choose a vertex $v\in S$. The graph $G-(S-\{v\})$ is 4-connected and claw-free. Assume the validity of Thomassen's conjecture, which implies, by a result of Ryjáček and Vrána in [22], that $G-(S-\{v\})$ is 1-Hamilton-connected. Thus if Thomassen's conjecture holds, then $G-S$ is also Hamilton-connected. This shows that Conjecture 4.1(ii) is also equivalent to any one stated in Conjecture 1.1. By the same reasoning, if one could prove that Thomassen's conjecture also implies that every 4-connected claw-free graph is 2-hamiltonian, then each of Conjecture 4.1(i) and (ii) would also be equivalent to any one stated in Conjecture 1.1.

As a final remark, a referee kindly indicated that the arguments to prove Theorem 1.8(ii) can also be applied, in conjunction with Theorem 1.7(iii), to prove the following seemingly stronger result.

Theorem 4.2. *Let $s\ge 1$ be an integer and let G be a hourglass-free and claw-free graph. If $\kappa(G)\ge s+3$, then G is s-Hamilton-connected.*

Declaration of competing interest

The authors declared that they have no conflicts of interest to this work. We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

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