Bull. Malays. Math. Sci. Soc.
https://doi.org/10.1007/s40840-021-01232-6

BULLETIN *of the* MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

A Condition on Hamilton-Connected Line Graphs

Lan Lei[1] · Jia Wei[2] · Yikang Xie[3] · Mingquan Zhan[4] · Hong-Jian Lai[3]

Received: 6 March 2021 / Revised: 19 November 2021 / Accepted: 20 December 2021
© Malaysian Mathematical Sciences Society and Penerbit Universiti Sains Malaysia 2022

Abstract

A graph G is Hamilton-connected if for any pair of distinct vertices $u, v \in V(G)$, G has a spanning (u, v)-path; G is 1-hamiltonian if for any vertex subset $S \subseteq V(G)$ with $|S| \leq 1$, $G - S$ has a spanning cycle. Let $\delta(G)$, $\alpha'(G)$ and $L(G)$ denote the minimum degree, the matching number and the line graph of a graph G, respectively. The following result is obtained. Let G be a simple graph with $|E(G)| \geq 3$. If $\delta(G) \geq \alpha'(G)$, then each of the following holds. (i) $L(G)$ is Hamilton-connected if and only if $\kappa(L(G)) \geq 3$. (ii) $L(G)$ is 1-hamiltonian if and only if $\kappa(L(G)) \geq 3$.

Keywords Chvátal-Erdős condition · Hamilton-connected · Essential edge connectivity · Matching · Minimum degree · 1-hamiltonian

Mathematics subject classification 05C45 · 05C75

Communicated by Sandi Klavžar.

✉ Jia Wei
jiawei_math@163.com

Lan Lei
leilan@ctbu.edu.cn

Yikang Xie
yx0010@mix.wvu.edu

Mingquan Zhan
mingquan.zhan@millersville.edu

Hong-Jian Lai
hjlai@math.wvu.edu

1 School of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing 400067, People's Republic of China

2 School of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an 710129, Shaanxi, People's Republic of China

3 Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA

4 Department of Mathematics, Millersville University of Pennsylvania, Millersville, PA 17551, USA

1 The Problem

Graphs considered here are finite and loopless. Terms and notation in this paper follow generally those in [3]. As in [3], for a graph G, let $\alpha(G)$, $\alpha'(G)$, $\kappa(G)$ and $\kappa'(G)$ denote the stability number (also called the independence number), matching number, connectivity and edge connectivity of G, respectively. This research is motivated by the following well-known theorem of Chvátal and Erdős on hamiltonian graphs.

Theorem 1.1 (Chvátal and Erdős [9]) Let G be a simple graph with at least three vertices.

(i) If $\kappa(G) \geq \alpha(G)$, then G has a Hamilton cycle.
(ii) If $\kappa(G) \geq \alpha(G) - 1$, then G has a Hamilton path.
(iii) If $\kappa(G) \geq \alpha(G) + 1$, then G is Hamilton-connected.

As shown in the survey of Saito in [29], there have been many extensions and variations of Theorem 1.1. A graph is supereulerian if it has a spanning Eulerian subgraph. There are quite a few investigations using similar conditions involving edge connectivity, stability number or matching number to study supereulerian graphs, as seen in [10,20,21,31,34], among others.

Another motivation of this research comes from Thomassen's conjecture [32] that every 4-connected line graph is hamiltonian. The line graph of a graph G, denoted by $L(G)$, is a simple graph with vertex set $E(G)$, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. A number most fascinating conjectures in this area are presented below. By an ingenious argument of Ryjáček [27], Conjecture 1.2(i) below is equivalent to a seeming stronger conjecture of Conjecture 1.2(ii). In [28], it is shown that all conjectures stated in Conjecture 1.2 are equivalent to each other.

Conjecture 1.2

(i) (Thomassen [32]) Every 4-connected line graph is hamiltonian.
(ii) (Matthews and Sumner [26]) Every 4-connected claw-free graph is hamiltonian.
(iii) (Kužel and Xiong [15]) Every 4-connected line graph is Hamilton-connected.
(iv) (Ryjáček and Vrána [28]) Every 4-connected claw-free graph is Hamilton-connected.

Many researches have been conducted towards these conjectures, as can be found in the surveys in [4,12,13], among others. The best result by far is obtained by Kaiser, Ryjáček and Vrána in [14]. Recently, Algefari et al. (Corollary 1.1 of [1]) proved that every connected simple graph G with $|E(G)| \geq 3$ and with $\delta(G) \geq \alpha'(G)$ has a hamiltonian line graph. For an integer $s \geq 0$, a graph G is s-hamiltonian if for any vertex subset $X \subseteq V(G)$ with $|X| \leq s$, $G - X$ has a Hamilton cycle. The current research is to investigate similar relationship between the minimum degree and the matching number of a graph that would warrant Hamilton-connected line graphs and 1-hamiltonian line graphs. As Hamilton-connected graphs and 1-hamiltonian graphs must be 3-connected, it is natural to conduct the investigation within 3-connected line graphs. The following is our main result.

Theorem 1.3 *Let G be a simple graph with $|E(G)| \geq 3$ and $\delta(G) \geq \alpha'(G)$. Then, each of the following holds.*

(i) $L(G)$ is Hamilton-connected if and only if $\kappa(L(G)) \geq 3$.
(ii) $L(G)$ is 1-hamiltonian if and only if $\kappa(L(G)) \geq 3$.

2 Preliminaries

A cycle on n vertices is often called an n-cycle. For a subset $X \subseteq V(G)$ or $X \subseteq E(G)$, $G[X]$ is the subgraph of G induced by X. A path from a vertex u to a vertex v is referred to as a (u, v)-path. An edge subset X of G is an essential cut if $G - X$ has at least two nontrivial components or if $|X| = |E(G)| - 1$. For an integer $k \geq 0$, a connected graph G is essentially k-edge-connected if G does not have an essential edge cut X with $|X| < k$. For a connected graph G, let $ess'(G)$ be the largest integer k such that G is essentially k-edge-connected. By the definition of a line graph, we have the following observation for a graph G and its line graph $L(G)$:

$$\kappa(L(G)) = ess'(G). \tag{1}$$

2.1 Maximum Matching of a Graph

Let M be a matching in G. We use $V(M)$ to denote the set $V(G[M])$. A path P in G is an M**-augmenting path** if the edges of P are alternately in M and in $E(G) - M$, and if both end vertices of P are not in $V(M)$. We start with a fundamental theorem of Berge.

Theorem 2.1 (Berge [2]) A matching M in G is a maximum matching if and only if G does not have M-augmenting paths.

Applying Theorem 2.1, the following results are proved in [1], which will be utilized in our arguments in the proof of Theorem 1.3.

Lemma 2.2 (Lemma 2.1 of [1]) Let $k > 0$ be an integer and G be a graph with a matching M such that $|M| = k$. Suppose that $V(G) - V(M)$ has a subset X with $|X| \geq 2$ such that for any $v \in X$, $d(v) \geq k$. If X has at least one vertex u such that $d(u) \geq k + 1$, then M is not a maximum matching of G.

Theorem 2.3 (Theorem 2.2 of [1]) Let G be a connected simple graph with $n = |V(G)| \geq 2$ and $k = \alpha'(G)$. If $\delta(G) \geq k$, then $\kappa'(G) \geq k$.

2.2 Collapsible Graphs and Strongly Spanning Trailable Graphs

We use a definition of collapsible graphs [18] that is equivalent to Catlin's original definition in [6]. For a graph G, we use $O(G)$ to denote the set of all vertices of odd degree in G. A graph G is collapsible if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph H such that $O(H) = R$. If G is collapsible, then by definition with $R = \emptyset$, G is supereulerian and so $\kappa'(G) \geq 2$. As examples, Catlin [6] observed that cycles of length at most 3 are collapsible. In [6], Catlin showed that for any graph G, every vertex of G lies in a unique maximal collapsible subgraph of G. The reduction of G, denoted by G', is obtained from G by contracting all nontrivial maximal collapsible subgraphs of G. A graph is **reduced** if it is the reduction of some graph. As shown in [6], a reduced graph is simple.

Theorem 2.4 *Let G be a graph.*

(i) (Catlin, Theorem 3 of [6]) Suppose that H is a collapsible subgraph of G. Then, G is collapsible if and only if G/H is collapsible.
(ii) (Catlin, Lemma 3 of [6]) If G is collapsible, then any contraction of G is also collapsible.
(iii) (Catlin, Theorem 5 of [6]) A graph G is reduced if and only if G does not contain a nontrivial collapsible subgraph.
(iv) If G has a spanning connected subgraph Q, such that for any edge $e \in E(Q)$, G has a collapsible subgraph J_e with $e \in E(J_e)$, then G is collapsible.

Proof We argue by induction on $n = |V(G)|$ to prove (iv). As (iv) holds for $n = 1$, we assume that $n \geq 2$. For any $e \in E(Q)$, let J_e denote a collapsible subgraph of G with $e \in E(J_e)$. We fix an edge $e_0 \in E(Q)$ and let $J = J_{e_0}$ be a collapsible subgraph of G that contains e_0. Define $G_1 = G/J$. As Q is a spanning subgraph in G, $Q_1 = Q/(Q \cap J)$ is a spanning subgraph of G_1. For any edge $e \in E(Q_1) \subseteq E(Q)$, there exists a collapsible subgraph J_e of G with $e \in E(J_e)$. By Theorem 2.4(ii), $J'_e = J_e/(J \cap J_e)$ is a collapsible subgraph of G_1 with $e \in E(J'_e)$. It follows by induction that G_1 is collapsible. By Theorem 2.4(i), G is collapsible. □

For $u, v \in V(G)$, a (u, v)-trail is a trail of G from u to v. For $e, e' \in E(G)$, an (e, e')-trail is a trail of G having end-edges e and e'. An (e, e')-trail T is dominating if each edge of G is incident with at least one internal vertex of T, and T is spanning if T is a dominating trail with $V(T) = V(G)$. A graph G is spanning trailable if for each pair of edges e_1 and e_2, G has a spanning (e_1, e_2)-trail. Suppose that $e = u_1v_1$ and $e' = u_2v_2$ are two edges of G. If $e \neq e'$, then the graph $G(e, e')$ is obtained from G by replacing $e = u_1v_1$ with a path $u_1v_ev_1$ and by replacing $e' = u_2v_2$ with a path $u_2v_{e'}v_2$, where $v_e, v_{e'}$ are two new vertices not in $V(G)$. If $e = e'$, then $G(e, e')$, also denoted by $G(e)$, is obtained from G by replacing $e = u_1v_1$ with a path $u_1v_ev_1$. For the recovering operation, we let $c_e(G(e, e'))$ be the graph obtained from $G(e, e')$ by replacing the path $u_1v_ev_1$ with the edge $e = u_1v_1$. Thus, $c_{e'}(c_e(G(e, e'))) = G$.

By the definition of $G(e', e'')$, we have the following observation.

$$\text{If } G(e', e'') \text{ is collapsible, then } G(e', e'') \text{ has a spanning } (v_{e'}, v_{e''})\text{-trail.} \tag{2}$$

In fact, if $G(e', e'')$ is collapsible, then $G(e', e'')$ has a spanning connected subgraph J with $O(J) = \{v_{e'}, v_{e''}\}$. Hence J is a spanning $(v_{e'}, v_{e''})$-trail.

As defined in [23], a graph G is **strongly spanning trailable** if for any $e, e' \in E(G)$, $G(e, e')$ has a $(v_e, v_{e'})$-trail T with $V(G) = V(T) - \{v_e, v_{e'}\}$. Since $e = e'$ is possible, strongly spanning trailable graphs are both spanning trailable and supereulerian.

Theorem 2.5 (Luo et al. [24], see also Theorem 4 of [7]) *If $\kappa'(G) \geq 4$, then G is strongly spanning trailable.*

Harary and Nash-Williams showed that there is a close relationship between a graph and its line graph concerning Hamilton cycles.

Theorem 2.6 (Harary and Nash-Williams [11]) *Let G be a graph with $|E(G)| \geq 3$. Then, $L(G)$ is hamiltonian if and only if G has an Eulerian subgraph H with $E(G - V(H)) = \emptyset$.*

Let G be a graph with $|V(G)| \geq 3$. For each integer $i \geq 0$, define $D_i(G) = \{v \in V(G) : d_G(v) = i\}$. Suppose that $ess'(G) \geq 3$. The **core** of this graph G, denoted by G_0, is obtained from $G - D_1(G)$ by contracting exactly one edge xy or yz for each path xyz in G with $d_G(y) = 2$. By the definition of $D_i(G)$, $G - D_1(G)$ is connected if G is connected. As contraction does not decrease the edge connectivity, G_0 is connected if G is connected. Lemma 2.7 (iii) below is proved by using a similar argument in the proof of Theorem 2.6.

Lemma 2.7 (Shao [30]) Let G be a connected nontrivial graph such that $\kappa(L(G)) \geq 3$, and let G_0 denote the core of G.

(i) G_0 is uniquely determined by G with $\kappa'(G_0) \geq 3$.
(ii) (see also Lemma 2.9 of [16]) If G_0 is strongly spanning trailable, then $L(G)$ is Hamilton-connected.
(iii) (see also Proposition 2.2 of [16]) $L(G)$ is Hamilton-connected if and only if for any pair of edges $e', e'' \in E(G)$, G has a dominating (e', e'')-trail.

3 Proof of the Main Results

Theorem 1.3 will be proved in this section. As every Hamilton-connected graph must be 3-connected, and every 1-hamiltonian graph must be 3-connected, it suffices to prove that if G is a graph satisfying $\delta(G) \geq \alpha'(G)$ and $\kappa(L(G)) \geq 3$, then $L(G)$ is Hamilton-connected for Theorem 1.3(i) and $L(G)$ is 1-hamiltonian for Theorem 1.3(ii).

3.1 Proof of Theorem 1.3(i).

As $\kappa(L(G)) \geq 3$, we have $ess'(G) \geq 3$, and so by Lemma 2.7(i), the core G_0 of G is well-defined with $\kappa'(G_0) \geq 3$. We shall prove a slightly stronger Theorem 3.1, which implies the sufficiency of Theorem 1.3(i).

Theorem 3.1 *Let G be a connected simple graph with $|E(G)| \geq 3$ and $ess'(G) \geq 3$, and let G_0 denote the core of G.*

(i) If $\delta(G_0) \geq \alpha'(G_0)$, then G_0 is strongly spanning trailable.
(ii) Suppose that $\delta(G) \geq \alpha'(G)$. Then, $L(G)$ is Hamilton-connected if and only if $\kappa(L(G)) \geq 3$.

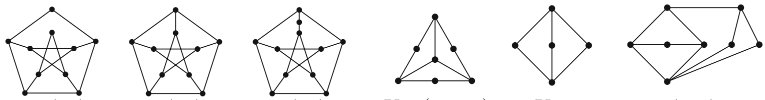

Fig. 1 nontrivial reduced graphs in Theorem 3.3(ii)

To prove Theorem 3.1, we begin with some tools that would be used in the arguments. For a graph G, let $circ(G)$ denote the length of a longest cycle of G.

Proposition 3.2 *Let G be a connected simple graph with $|E(G)| \geq 3$ and $\delta(G) \geq \alpha'(G) = k$.*

(i) If $k \geq 4$, then $\kappa'(G) \geq k \geq 4$ and G is strongly spanning trailable.
(ii) (Lemma 3.1 of [1]) If $k = 1$, then $G \in \{K_3, K_{1,n-1}\}$.
(iii) If $k \geq 4$ or $k = 1$, then $L(G)$ is Hamilton-connected.

Proof To prove Proposition 3.2(i), we apply Theorem 2.3 to conclude that $\kappa'(G) \geq k \geq 4$. Hence by Theorem 2.5, G is strongly spanning trailable. It remains to justify Proposition 3.2(iii). If $k \geq 4$, then as G is strongly spanning trailable, by Lemma 2.7(iii), $L(G)$ is Hamilton-connected. If $k = 1$, then $L(G)$ is a complete graph and so it is also Hamilton-connected. □

We define $P^{-}(10)$, $P(10)$, $P(11)$, $K_{1,3}(1,1,1)$, $K_{2,3}$, $T(1,2)$ to be the graphs as respectively depicted in Fig. 1.

Theorem 3.3 *Let G be a connected graph with $n = |V(G)|$, and let G' denote the reduction of G.*

(i) (Ma et al., Theorem 3.2 of [25], See also Theorem 4.5.4 of [33]) If $G = G'$, and G satisfies $\kappa'(G) \geq 2$, $circ(G) \leq 8$, $|D_2(G)| \leq 2$ and $ess'(G) \geq 3$, then G is collapsible.
(ii) (Theorem 1.7 of [19]) If $ess'(G) \geq 3$, $n \leq 11$, $|D_1(G)| = 0$ and $|D_2(G)| \leq 2$, then $G' \in \{K_1, K_{2,3}, K_{1,3}(1,1,1), T(1,2), P^{-}(10), P(10), P(11)\}$.

Corollary 3.4 *Each of the following holds.*

(i) Every graph G with $\kappa'(G) \geq 2$, $circ(G) \leq 8$, $|D_2(G)| \leq 2$ and $ess'(G) \geq 3$ is collapsible.
(ii) Every graph G with $\kappa'(G) \geq 3$ and $circ(G) \leq 6$ is strongly spanning trailable.
(iii) Let G be a graph with $ess'(G) \geq 3$ and $circ(G) \leq 6$, and let G_0 be the core of G. Then, G_0 is strongly spanning trailable.

Proof Let G be a graph with $\kappa'(G) \geq 2$, $circ(G) \leq 8$, $|D_2(G)| \leq 2$ and $ess'(G) \geq 3$, and let G' be the reduction of G. By the definition of contraction, we have $\kappa'(G') \geq \kappa'(G) \geq 2$, $circ(G') \leq circ(G) \leq 8$ and $ess'(G') \geq ess'(G) \geq 3$. Let $v \in D_2(G')$ be a vertex. Since $ess'(G) \geq 3$, v must be a trivial vertex and so $v \in D_2(G)$. This implies that $|D_2(G')| \leq |D_2(G)| \leq 2$. It follows by Theorem 3.3, G' is collapsible which implies that $G' = K_1$ and so G is collapsible. This proves (i).

To prove (ii), we assume that G with $\kappa'(G) \geq 3$ and $circ(G) \leq 6$. Let $e', e'' \in E(G)$ be two edges and let $H = G(e', e'')$. Then, as $\kappa'(G) \geq 3$ and $circ(G) \leq 6$, we conclude that $\kappa'(H) \geq 2$, $circ(H) \leq 8$, $|D_2(H)| \leq 2$ and $ess'(H) \geq 3$. It follows by (i) that H is collapsible. Let $v_{e'}$ and $v_{e''}$ denote the only vertices in $D_2(H)$. As H is collapsible, H has a spanning connected subgraph T with $O(T) = \{v_{e'}, v_{e''}\}$. Thus, T is a spanning $(v_{e'}, v_{e''})$-trail of H, and so by the randomness of e', e'', G is strongly spanning trailable. This proves (ii).

Now we assume that G is a graph with $ess'(G) \geq 3$ and $circ(G) \leq 6$. Let G_0 denote the core of G. By Lemma 2.7(i), $\kappa'(G_0) \geq 3$. As G_0 is a contraction of G, we have $circ(G_0) \leq circ(G) \leq 6$. By (ii), G_0 is strongly spanning trailable. $\square$

3.1.1 Proof of Theorem 3.1(i)

We assume that $\delta(G_0) \geq \alpha'(G_0)$. Let $n = |V(G_0)|$ and $k = \alpha'(G_0)$. As G is connected, by the definition of G_0, G_0 is also connected. Thus, if $k = 0$, then $n = 1$, and so by definition, G_0 is strongly spanning trailable. Hence, we assume that $k > 0$. Then, $|V(G)| \geq n = |V(G_0)| \geq 2\alpha'(G_0) = 2k \geq 2$. Thus, G is a connected nontrivial graph. As $ess'(G) \geq 3$, by (1) and Lemma 2.7(i), $\kappa'(G_0) \geq 3$. Thus, $|E(G_0)| \geq 3$. If $k = 1$, then applying Proposition 3.2(ii) to G_0, G_0 is spanned either by a K_3 or by a $K_{1,n-1}$ with $\kappa'(G_0) \geq 3$. If G_0 is spanned by a K_3, then this K_3 must have at least two edges each of which lies in a 2-cycle. For any $e', e'' \in G_0$, if there exists a 2-cycle C in $G_0(e', e'')$, then after contracting this 2-cycle C in $G_0(e', e'')$, every edge of $G_0(e', e'')/C$ lies in a cycle of length at most 3. As C is collapsible, by Theorem 2.4(i) that $G_0(e', e'')$ is collapsible. If there does not exist a 2-cycle in $G_0(e', e'')$, every edge of $G_0(e', e'')$ lies in a cycle of length at most 3 in $G_0(e', e'')$. It follows by Theorem 2.4(iv) that $G_0(e', e'')$ is collapsible. When G_0 is spanned by a $K_{1,n-1}$, since $\kappa'(G_0) \geq 3$, every edge must be in a parallel class of at least three edges. In this case, every edge of $G_0(e', e'')$ lies in a cycle of length at most 3. It follows by Theorem 2.4(iv) that $G_0(e', e'')$ is collapsible. By (2), G_0 is strongly spanning trailable. If $k \geq 4$, then by Proposition 3.2(i), G_0 is strongly spanning trailable. Therefore, we assume that $k \in \{2, 3\}$. Suppose that $k = 2$. Then, G_0 does not have a cycle of length longer than 5, and so by Corollary 3.4(iii), G_0 is strongly spanning trailable.

Hence, we assume that $k = 3$, and so $circ(G_0) \leq 7$. If $circ(G_0) \leq 6$, then by Corollary 3.4(iii), G_0 is strongly spanning trailable, and we are done. Therefore, we assume that $circ(G_0) = 7$. Let C be a cycle of G_0 with $|V(C)| = 7$. If $V(G_0) - V(C) \neq \emptyset$, then as G_0 is connected, there must be a vertex $v \in V(G_0) - V(C)$ such that v is adjacent to a vertex on C, implying that $3 = \alpha'(G_0) \geq 4$, a contradiction. Thus, $V(G_0) = V(C)$ and so $|V(G_0)| = 7$ and C is a Hamilton cycle of G_0.

For any $e', e'' \in E(G_0)$, let $H = G_0(e', e'')$ and let $v_{e'}, v_{e''}$ denote the new vertices newly added in the process of subdividing e' and e'', respectively. Then, $|V(H)| = 9$. As $\kappa'(G_0) \geq 3$, we have $|D_1(H)| = 0$ and $|D_2(H)| = 2$. Let H' be the reduction of H. We claim that $H' = K_1$ and so H is collapsible. By contradiction, we assume that $1 < |V(H')| \leq |V(H)| = 9$. By Theorem 3.3 (ii), $H' \in \{K_{2,3}, K_{1,3}(1, 1, 1), T(1, 2)\}$. Since $\kappa'(G_0) \geq 3$, $|D_2(H')| \leq |D_2(H)| \leq 2$. It follows that $H' \notin \{K_{2,3}, K_{1,3}(1, 1, 1), T(1, 2)\}$, as any of these graphs have at least 3 vertices of degree 2. This contradiction implies that $H' = K_1$ and so H is collapsible.

By (2), G_0 is strongly spanning trailable. This completes the proof of Theorem 3.1(i). □

3.1.2 Proof of Theorem 3.1(ii)

In this subsection, we assume Theorem 3.1(i) to prove Theorem 3.1(ii). It suffices to show that if $ess'(G) \geq 3$ and $\delta(G) \geq \alpha'(G)$, then $L(G)$ is Hamilton-connected. Let G_0 denote the core of G, $k = \alpha'(G)$ and $n = |V(G_0)|$.

By Proposition 3.2(i), if $\alpha'(G) \geq 4$, then G is strongly spanning trailable. By definition, any spanning $(v_{e'}, v_{e''})$-trail induces a spanning (e', e'')-trail in G. It follows by Lemma 2.7 that $L(G)$ is Hamilton-connected.

Hence, we assume that $k \leq 3$. As G_0 is a contraction of G, we have $\alpha'(G_0) \leq \alpha'(G) \leq 3 \leq \kappa'(G_0) \leq \delta(G_0)$. By Theorem 3.1(i), G_0 is strongly spanning trailable. By Lemma 2.7(ii), $L(G)$ is Hamilton-connected. This completes the proof. □

3.2 Proof of Theorem 1.3(ii)

For a vertex $u \in V(G)$, define $N_G(u) = \{v \in V(G) : uv \in E(G)\}$ to be the set of neighbors of u in G. The main purpose of this subsection is to prove Theorem 1.3(ii). As remarked at the beginning of this section, it suffices to assume that G is a graph satisfying $\delta(G) \geq \alpha'(G)$ and $\kappa(L(G)) \geq 3$ to show that $L(G)$ is 1-hamiltonian. In the proof, we will need the following former results.

Lemma 3.5 *Let G be a connected graph, and let $K_{3,3}^-$ denote the graph obtained from $K_{3,3}$ by deleting an edge. Each of the following holds.*

- *(i)* (Catlin et al., Theorem 1.1 of [8]) If $\kappa'(G) \geq 4$, then for any edge subset $X \subseteq E(G)$ with $|X| \leq 2$, $G-X$ has two edge-disjoint spanning trees and is collapsible.
- (ii) (Catlin [5]) $K_{3,3}^-$ is collapsible, and so $K_{3,3}$ is collapsible.
- (iii) (Li et al., Lemma 2.1 of [22]) If $|V(G)| \leq 8$ with $|D_1(G)| = 0$ and $|D_2(G)| \leq 2$. Then, the reduction of G is in $\{K_1, K_2, K_{2,3}\}$.

By the definition of the core, and imitating the arguments in [11,30] and in Theorem 2.7 of [17], we have the following observation.

Observation 3.6 *Let $s \geq 0$ be an integer, G be a connected graph with $|E(G)| \geq s+3$ and $ess'(G) \geq 3$, and G_0 be the core of G.*

- *(i) (Theorem 2.7 of [17]) The line graph $L(G)$ is s-hamiltonian if and only if for any $S \subseteq E(G)$ with $|S| \leq s$, $G - S$ has a dominating Eulerian subgraph.*
- *(ii) If for any $S \subseteq E(G_0)$ with $|S| \leq s$, $G_0 - S$ is supereulerian, then $L(G)$ is s-hamiltonian.*

Proof It suffices to justify Observation 3.6(ii). By Observation 3.6(i), we need to prove that for any $X \subseteq E(G)$ with $|X| \leq s$, $G - X$ has a dominating Eulerian subgraph. Let G_0 denote the core of G. Define

$$S_1 = \{e \in E(G) : e \text{ is incident with a vertex in } D_1(G)\},$$
$$S_2' = \{e \in X : e \text{ is incident with a vertex in } D_2(G)\}.$$

We shall adopt the following convention in our arguments. If $e', e'' \in S_2'$ are incident with a vertex $v \in D_2(G)$, then we may always assume that e'' is being contracted in the construction of G_0 and e' remains in $E(G_0)$. With this convention, for any $v \in D_2(G)$, we may denote $E_G(v) = \{e_v', e_v''\}$, and define $X_2 = \{e_v'' : v \in D_2(G)\}$. Hence by the definition of G_0, we may assume that $G_0 = G/(S_1 \cup X_2)$.

Let $S_2 = S_2' - X_2$, $S_3 = X - (S_1 \cup S_2')$. Then, $S = S_2 \cup S_3 \subseteq E(G_0)$. As $S \subseteq X$, we have $|S| \leq |X| \leq s$. By the assumption of Observation 3.6(ii), $G_0 - S$ has a spanning Eulerian subgraph H'. Let $S_2'' = \cup_{v \in D_2(G), e_v' \in E(H')}\{e_v', e_v''\}$. Define $H = G[E(H') \cup S_2'']$. Since H' is an Eulerian subgraph of G_0, by the definition of G_0, every vertex in H not incident with an edge in $X_2 \cap S_2''$ has the same (even) degree as in H'. As H is obtained from $G[E(H')]$ by adding the edges in $X_2 \cap S_2''$, which amounts to subdividing the edges in $(\cup_{v \in D_2(G)} E_G(v)) \cap E(H')$ to form H, it follows that H is an Eulerian subgraph of G. For any edge $e \in E(G)$, if $e \in E(G) - (S_1 \cup X_2) = E(G_0)$, then since H' is a spanning Eulerian subgraph of G_0, e is incident with a vertex in $V(H)$. If $e \in S_1$, then by $ess'(G) \geq 3$, e is also incident with a vertex of degree at least 4 in G. Hence, e is incident with a vertex in $V(H)$ as well. Finally, we assume that $e \in X_2$. As $X_2 = \{e_v'' : v \in D_2(G)\}$, there exists a vertex $v \in D_2(G)$ with $e = e_v''$. Let u, w be the neighbors of v in G, and so uvw is a path of length 2 in G. As $ess'(G) \geq 3$, it follows that $d_G(u) \geq 3$ and $d_G(w) \geq 3$. By the definition that $G_0 = G/(S_1 \cup X_2)$ and since H' spans G_0, we have $u, w \in V(G_0) = V(H')$. As $H = G[E(H') \cup S_2'']$, this implies that $u, w \in V(H)$, and so e must be incident with a vertex in $V(H)$. It follows by definition that H is a dominating Eulerian subgraph of $G - X$, and so by Observation 3.6(i), $L(G) - X$ is hamiltonian. This proves Observation 3.6(ii). $\square$

To prove Theorem 1.3(ii), we let $k = \alpha'(G)$ and G_0 denote the core of G. Then, we will justify the following claim.

Claim 1 *If $k = 1$ or $k \geq 4$, then $L(G)$ is 1-hamiltonian.*

Suppose first that $k = 1$. By Proposition 3.2(ii), $G \in \{K_3, K_{1,n-1}\}$. As $\kappa(L(G)) \geq 3$, $G \in \{K_{1,n-1}\}$ where $n \geq 5$. By the definition of a line graph, $L(G) = K_{n-1}$ is 1-hamiltonian. Next we assume that $k = \alpha'(G) \geq 4$. By Theorem 2.3, $\kappa'(G) \geq 4$. By Lemma 3.5(i), for any $e \in E(G)$, $G - e$ is collapsible, and so is supereulerian. Thus by Observation 3.6, $L(G)$ is 1-hamiltonian. This proves Claim 1.

By Claim 1, it remains to discuss the cases when $k \in \{2, 3\}$. Suppose that $k = 2$. Let M be a maximum matching of G_0 and X be the set of vertices in G_0 not incident with any edges in M. As $\delta(G_0) \geq \kappa'(G_0) \geq 3$, it follows by Lemma 2.2 that $|X| \leq 1$ and so $|V(G_0)| \leq 5$. Thus for any edge $e \in E(G_0)$, we have $|V(G_0 - e)| \leq |V(G_0)| \leq 5$. As $\kappa'(G_0) \geq 3$, we have $|D_1(G_0 - e)| = 0$ and $|D_2(G_0 - e)| \leq 2$. By Lemma 3.5(iii), the reduction of $G_0 - e$ is in $\{K_1, K_2, K_{2,3}\}$. Again by $\kappa'(G_0) \geq 3$, $\kappa'(G_0 - e) \geq 2$ and $G_0 - e$ has at most two edge cuts of size 2. Thus, the reduction of $G_0 - e$ is 2-edge-connected and has at most two edge cuts of size 2. Then, the reduction of $G_0 - e$ is K_1 and so $G_0 - e$ is collapsible. By Observation 3.6, $L(G)$ is 1-hamiltonian.

Hence, we assume that $k = 3$. We shall show that

$$\text{for any } e \in E(G_0),\ G_0 - e \text{ is collapsible.} \tag{3}$$

We prove (3) by contradiction, and assume that for some $e_0 = z_1z_2 \in E(G_0)$, $G_0 - e_0$ is not collapsible. Let G_0' denote the reduction of $G_0 - e_0$. Since $G_0 - e_0$ is not collapsible, $|V(G_0')| \geq 2$.

Let w_1, w_2 be the vertices in $V(G_0')$, each of whose preimages in $G_0 - e_0$ contains an end vertex of e_0. We claim that $w_1 \neq w_2$. By contradiction, we assume that $w_1 = w_2$. As G_0' is the reduction of $G_0 - e_0$, there exists a collapsible subgraph H in $G_0 - e_0$ with $V(e_0) \subseteq V(H)$, and so $(G_0 - e_0)/H = G_0/H$. Since G_0/H is a contraction of G_0, we have $\kappa'(G_0/H) \geq \kappa'(G_0) \geq 3$ and $\alpha'(G_0/H) \leq \alpha'(G) = k \leq 3$. It follows that $\kappa'(G_0/H) \geq 3$ and $circ(G_0/H) \leq 7$. By Corollary 3.4(i), G_0/H is collapsible which implies that $(G_0 - e_0)/H$ is collapsible. Thus by Theorem 2.4(i), $G_0 - e_0$ is collapsible, which is contrary to the assumption that $G_0 - e_0$ is not collapsible. This proves $w_1 \neq w_2$.

Define G_0^+ to be the graph obtained from G_0' by adding a new edge linking w_1 and w_2. Thus, G_0^+ is a contraction of G_0, and $G_0' = G_0^+ - e_0$. As G_0 is a contraction of G and G_0' is a contraction of $G_0 - e_0$, it follows that $\alpha'(G_0') \leq \alpha'(G_0) \leq \alpha'(G) = k \leq 3$. Since $\kappa'(G_0) \geq 3$ and G_0^+ is a contraction of G_0, we have $\kappa'(G_0^+) \geq 3$. As $G_0' = G_0^+ - e_0$, we conclude that $\kappa'(G_0') \geq 2$, $|D_2(G_0')| \leq 2$ and $ess'(G_0') \geq 3$. Thus by Corollary 3.4(i), G_0' is collapsible. As G_0' is the reduction of $G_0 - e_0$, we have $G_0' = K_1$ and so $G_0 - e_0$ is collapsible. This leads to a contradiction to the assumption that $G_0 - e_0$ is not collapsible, and completes the proof of Theorem 1.3(ii). $\square$

Acknowledgements This research of Lan Lei is supported by General Project of Natural Science Foundation of Chongqing, China (No. cstc2019jcyj-msxmX0579). This research of Jia Wei is supported by China Scholarship Council (File No. 201906150105).

References

1. Algefari, M.J., Lai, H.-J.: Supereulerian graphs with constraints on the matching number and minimum degree. Graphs Combin. **37**, 55–64 (2021)
2. Berge, C.: Two theorems in graph theory. Proc. Nat. Acad. Sci. USA **43**, 842–844 (1957)
3. Bondy, J.A., Murty, U.S.R.: Graph Theory. Springer, New York (2008)
4. Broersma, H.J.: On some intriguing problems in hamiltonian graph theory-a survey. Discrete Math. **251**, 47–69 (2002)
5. Catlin, P.A.: Super-Eulerian graphscollapsible graphs, and four-cycles. Congr. Numer. **58**, 233–246 (1987)
6. Catlin, P.A.: A reduction method to find spanning Eulerian subgraphs. J. Graph Theory **12**, 29–44 (1988)
7. Catlin, P. A., Lai, H.-J.: Spanning trails joining two given edges. In: Alavi, Y., et al (Eds.), Graph Theory, Combin. Appl., Vol. 1, Kalamazoo, MI. 1988, Wiley, New York, 207-222 (1991)
8. Catlin, P.A., Lai, H.-J., Shao, Y.: Edge-connectivity and edge-disjoint spanning trees. Discrete Math. **309**, 1033–1040 (2009)
9. Chvátal, V., Erdös, P.: A note on Hamiltonian circuits. Discrete Math. **2**, 111–113 (1972)
10. Han, L., Lai, H.-J., Xiong, L., Yan, H.: The Chvátal-Erdös condition for supereulerian graphs and the Hamiltonian index. Discrete Math. **310**, 2082–2090 (2010)
11. On eulerian and hamiltonian graphs and line graphs: Harary, F., Nash-Williams, C. St. J. A. Canad. Math. Bull. **8**, 701–709 (1965)
12. Gould, R.J.: Advances on the Hamiltonian problem-a survey. Graphs Combin. **19**, 7–52 (2003)
13. Gould, R.J.: Recent advances on the Hamiltonian problem: survey III. Graphs Combin. **30**, 1–16 (2014)
14. Kaiser, T., Ryjáček, Z., Vrána, P.: On 1-Hamilton-connected Claw-free graphs. Discrte Math. **321**, 1–11 (2014)

15. Kužel, R., Xiong, L.: Every 4-connected Line Graph is Hamiltonian if and only if it is Hamiltonian Connected. In R. Kučzel: Hamiltonian properties of graphs, Ph.D. Thesis, U.W.B. Pilsen, (2004)
16. Lai, H.-J., Shao, Y., Yu, G., Zhan, M.: Hamiltonian connectedness in 3-connected line graphs. Discrete Appl. Math. **157**, 982–990 (2009)
17. Lai, H.-J., Shao, Y.: On s-hamiltonian line graphs. J. Graph Theory **74**, 344–358 (2013)
18. Lai, H.-J., Shao, Y., Yan, H.: An update on Supereulerian graphs. WSEAS Trans. Math. **12**, 926–940 (2013)
19. Lai, H.-J., Wang, K., Xie, X., Zhan, M.: Catlin's reduced graphs with small orders. AKCE Int. J. Graphs Combin. **17**, 679–690 (2020)
20. Lai, H.-J., Yan, H.Y.: Supereulerian graphs and matchings. Appl. Math. Lett. **24**, 1867–1869 (2011)
21. Lei, L., Li, X., Ma, X., Zhan, M., Lai, H.-J.: Chvátal-Erdős Conditions and Almost Spanning Trails, Bull. Malays. Math. Sci. Soc., (2020)
22. Li, D., Lai, H.-J., Zhan, M.Q.: Eulerian subgraphs and hamiltonian connected line graphs. Discrete Appl. Math. **145**, 422–428 (2005)
23. Li, P., Wang, K., Zhan, M., Lai, H.-J.: Strongly spanning trailable graphs with short longest paths. Ars Combinatoria **137**, 3–39 (2018)
24. Luo, W., Chen, Z.-H., Chen, W.-G.: Spanning trails containing given edges. Discrete Math. **306**, 87–98 (2006)
25. Ma, X., Wu, Y., Zhan, M., Lai, H.-J.: On s-hamiltonicity of net-free line graphs, Discrete Math, accepted
26. Matthews, M.M., Sumner, D.P.: Hamiltonian results in $K_{1,3}$-free graphs. J. Graph Theory **8**, 139–146 (1984)
27. Ryjáček, Z.: On a Closure Concept in Claw-free Graphs, J. Combin. Theory Ser. B, 70, 217-224 (1997)
28. Ryjáček, Z., Vrána, P.: Line graphs of multigraphs and Hamiltonian-connectedness of claw-free graphs. J. Graph Theory **66**, 152–173 (2011)
29. Saito, A.: Chvátal-Erdös Theorem: Old Theorem with New Aspects, H. Ito et al. (Eds.): KyotoCGGT 2007, LNCS 4535, 191-200 (2008)
30. Shao, Y.: Claw-free graphs and line graphs, Ph. D. Dissertation, West Virginia University, (2005)
31. Tian, R., Xiong, L.: The Chvátal-Erdös condition for a graph to have a spanning trail. Graphs Combin. **31**, 1739–1754 (2015)
32. Thomassen, C.: Reflections on graph theory. J. Graph Theory **10**, 309–324 (1986)
33. Wu, Y.: Circuits and Cycles in Graphs and Matroids, PhD dissertation, West Virginia University, (2020)
34. Xu, J., Li, P., Miao, Z., Wang, K., Lai, H.-J.: Supereulerian graphs with small matching number and 2-connected hamiltonian claw-free graphs. Int. J. Comput. Math. **91**, 1662–1672 (2014)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.