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On hamiltonian line graphs of hypergraphs

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Abstract

A graph is *supereulerian* if it has a spanning eulerian subgraph. Harary and Nash-Williams in 1968 proved that the line graph of a graph G is hamiltonian if and only if G has a dominating eulerian subgraph, Jaeger in 1979 showed that every 4-edge-connected graph is supereulerian, and Catlin in 1988 proved that every graph with two edge-disjoint spanning trees is a contractible configuration for supereulerianicity. Utilizing the notion of partition-connectedness of hypergraphs introduced by Frank, Király, and Kriesell in 2003, we generalize the above-mentioned results of Harary and Nash-Williams, of Jaeger and of Catlin to hypergraphs by characterizing hypergraphs whose line graphs are hamiltonian, and showing that every 2-partition-connected hypergraph is a contractible configuration for supereulerianicity.

KEYWORDS

eulerian hypergraph, hamiltonian line graph, hypertree, partition-connected hypergraph, supereulerian hypergraph

1 | INTRODUCTION

We study finite graphs and hypergraphs. Undefined terms would follow [2] for graphs and [1] for hypergraphs. For two integers m and n with $m < n$, denote $[m, n] = \{m, m+1, ..., n\}$. Throughout the paper, for a positive integer m, we use $\mathbb{Z}_m$ to denote the additive group of integers modulo m.

A *hypergraph* H is an ordered pair $(V(H), \mathcal{E}(H))$, where $V(H)$ is the vertex set of H and $\mathcal{E}(H)$ is a collection of not necessarily distinct nonempty subsets of $V(H)$, called *hyperedges* or simply *edges* of H. For edges E_1 and E_2 in H, we write $E_1 \neq E_2$ to mean that they are distinct edges in H, even if they could be the same as subsets of $V(H)$; and we write $E_1 = E_2$ to mean that they are the same edge in H. A single element edge is referred to as a *loop*. We consider loopless hypergraphs. The *rank* of a hypergraph H is $r(H) = \max_{E\in\mathcal{E}(H)}\{|E|\}$. Thus if $r(H) = 2$, then H is a loopless graph with possible parallel edges. Following [2], a graph is simple if it is loopless and contains no parallel edges.

A hypergraph J is called a *sub-hypergraph* of H if $V(J) \subseteq V(H)$ and $\mathcal{E}(J)$ is a subcollection of $\mathcal{E}(H)$. If $V(J) = V(H)$, then J is called a *spanning* sub-hypergraph of H. The *line graph* $L(H)$ of a hypergraph H, is a simple graph with vertex set $V(L(H)) = \mathcal{E}(H)$, where two vertices E_i and E_j are adjacent if and only if $E_i \cap E_j \neq \emptyset$ in H.

A *trail* of a hypergraph H is an alternating sequence

$$\Gamma = (v_0 E_0 v_1 E_1 \cdots v_{s-1} E_{s-1} v_s) \tag{1}$$

of vertices and edges such that

(T1) E_i and E_j are two distinct edges for each $0 \le i < j < s$;
(T2) $v_i, v_{i+1} \in E_i$ and $v_i \neq v_{i+1}$ for each $0 \le i < s$.

We also view the trail Γ in (1) as a sub-hypergraph (also denoted by Γ) with $V(\Gamma)$ being the vertices occurring in the trail and with $\mathcal{E}(\Gamma) = \{E_0, E_1, ..., E_{s-1}\}$. We also write the trail in (1) as $\Gamma = (E_0 E_1 \cdots E_{s-1})$ in an edge sequence notation. Moreover, if $r(\Gamma) = 2$, then we can write the trail in (1) as $\Gamma = (v_0 v_1 \cdots v_s)$ in a vertex sequence notation. The trail Γ in (1) is a *closed trail* if $v_0 = v_s$.

Definition 1.1. Let Γ be the trail in (1). We define the *pivot set* $PV(\Gamma)$ of Γ as follows.

(i) If $v_0 \neq v_s$, then for each $i \in [1, s-2]$, $PV_\Gamma(E_i) = (E_{i-1} \cap E_i) \cup (E_i \cap E_{i+1})$, and define $PV(\Gamma) = \bigcup_{i\in[0,s-2]}(E_i \cap E_{i+1})$.
(ii) If $v_0 = v_s$, then for each $i \in \mathbb{Z}_s, PV_\Gamma(E_i) = (E_{i-1} \cap E_i) \cup (E_i \cap E_{i+1})$, and define $PV(\Gamma) = \bigcup_{i\in\mathbb{Z}_s}(E_i \cap E_{i+1}) = \bigcup_{i\in\mathbb{Z}_s} PV_\Gamma(E_i)$,

To describe a closed trail in an edge sequence $(E_0 E_1 \cdots E_{s-1})$, we make the following observations, which are immediate consequences of the definition.

Observation 1.1. Let the edge sequence $\Gamma = (E_0 E_1 \cdots E_{s-1})$ denote the trail in (1). Then, Γ is a closed trail if and only if for each $i, j \in \mathbb{Z}_s$, each of the following holds.

(CT1) E_i and E_j are two distinct edges for each $j \neq i$;
(CT2) $E_i \cap E_j \neq \emptyset$ for each j with $|i - j| = 1$;
(CT3) $|\bigcup_{|i-j|=1} E_i \cap E_j| \ge 2$.

A hypergraph H is *eulerian* if it has a closed trail Γ with $\mathcal{E}(H) = \mathcal{E}(\Gamma)$. Thus, an eulerian sub-hypergraph of H is a closed trail of H. If a vertex $v \in PV_\Gamma(E_i)$, then v is called a *pivot* of edge E_i with respect to the closed trail Γ. A closed trail Γ in H is *pivot-spanning* if $PV(\Gamma) = V(H)$. A hypergraph H is *pivot-supereulerian* if H has a pivot-spanning eulerian sub-hypergraph. A closed trail Γ in H is *dominating* if for any $E \in \mathcal{E}(H), E \cap PV(\Gamma) \neq \emptyset$. We define a hypergraph H to be *supereulerian* if H has a dominating spanning eulerian sub-hypergraph.

For a vertex $v \in V(H)$ and a sub-hypergraph J of H, define

$$\mathcal{E}_J(v) = \{E \in \mathcal{E}(J) : v \in E\}.$$

A hypergraph H is *heavy supereulerian* if H has a dominating spanning eulerian sub-hypergraph Γ such that $|\mathcal{E}_\Gamma(v)| \geq 2$ for each $v \in V(H)$. In Figure 1, an example is presented to indicate that a heavy supereulerian hypergraph may not always be pivot-supereulerian. Nevertheless, we have the following observations from their definitions.

Observation 1.2. Each of the following holds.

(i) Every pivot-supereulerian hypergraph is heavy supereulerian.
(ii) Every heavy supereulerian hypergraph is supereulerian.
(iii) If $r(H) = 2$, then a hypergraph H is pivot-supereulerian if and only if H is heavy supereulerian, which is also equivalent to that H is supereulerian.

Harary and Nash-Williams [8] discovered a nice relationship between dominating eulerian subgraphs in a graph G and Hamilton cycles in the line graph $L(G)$.

Theorem 1.1 (Harary and Nash-Williams [8, Proposition 8])**.** *Let G be a graph with at least three edges. Then $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

Li et al. [11, Corollary 7] charaterized the correspondent relationship between hamiltonicity of a line graph of a hypergraph with rank 3 and the dominating structure in the root hypergraph. Thus, one of the purposes of this study is to extend Theorem 1.1 to hypergraphs.

Theorem 1.2. *Let H be a hypergraph with at least three edges. Then $L(H)$ is hamiltonian if and only if H has a dominating eulerian sub-hypergraph.*

For a graph G, if G is supereulerian, then G has a spanning eulerian subgraph, which is dominating. Theorem 1.1 indicates that every supereulerian graph with at least three edges has a hamiltonian line graph. As indicated in Catlin's resourceful survey [4], supereulerian graphs play an important role in the investigation of hamiltonian line graphs. Another purpose of this study is to generalize certain supereulerian graph results to hypergraphs. Theorem 1.3 was first obtained by Jaeger [9], and extended by Catlin in [3].

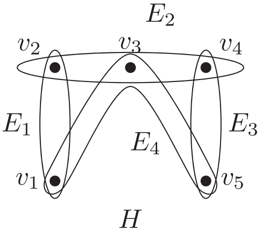

FIGURE 1 A heavy supereulerian but not pivot-supereulerian hypergraph

Theorem 1.3 (Jaeger [9] and Catlin [3, Theorem 2]). *If a graph G has two edge-disjoint spanning trees, then G is supereulerian. In particular, every 4-edge-connected graph is supereulerian.*

For a proper subset $U \subset V(H)$, $\partial(U)$ is the set of all the edges of H which intersect both U and $V(H) - U$. For an integer $k > 0$, a hypergraph H is *k-edge-connected* if for every nonempty proper subset U of $V(H)$, $|\partial(U)| \geq k$.

Definition 1.2 (Frank, Király, and Kriesell [6]). Let $\mathcal{P}(H)$ be the collection of all partitions of $V(H)$ such that a partition $P = (V_1, V_2, \ldots, V_t) \in \mathcal{P}(H)$ if and only if P satisfies each of the following:

(P1) $V(H) = \cup_{i=1}^{t} V_i$,
(P2) $V_i \neq \emptyset$, $1 \leq i \leq t$, and
(P3) for each $1 \leq i < j \leq t$, $V_i \cap V_j = \emptyset$.

For a partition $P = (V_1, V_2, \ldots, V_t) \in \mathcal{P}(H)$, each V_i is a partition class of P. Let $|P| = t$ denote the number of classes of P, and let $e(P)$ be the number of edges intersecting at least two classes of P. A hypergraph H is *k-partition-connected* if $e(P) \geq k(|P| - 1)$ for every partition $P \in \mathcal{P}(H)$.

We extend Theorem 1.3 to hypergraphs as follows.

Theorem 1.4. *If H is a 2-partition-connected hypergraph, then H is supereulerian. In particular, every $2r$-edge-connected hypergraph with rank r is supereulerian.*

Thus Corollary 1.5 follows from Theorems 1.2 and 1.4.

Corollary 1.5. *If H is a 2-partition-connected hypergraph, then the line graph $L(H)$ is hamiltonian. In particular, if H is $2r$-edge-connected with rank r, then $L(H)$ is hamiltonian.*

In [3], Catlin introduced a powerful reduction method to study supereulerian graphs. Let H be a hypergraph. For an edge subset $X \subseteq \mathcal{E}(H)$, the *contraction* H/X is a hypergraph obtained from H by identifying all vertices of each edge in X and then by deleting the resulting loops. If J is a sub-hypergraph of H, then we write H/J for $H/\mathcal{E}(J)$. Moreover, if J is connected, then we denote the new vertex by v_J that all vertices in $V(J)$ are contracted into in H/J.

Theorem 1.6 (Catlin [3, Theorem 2]). *Let G be a graph and L be a subgraph of G with two edge-disjoint spanning trees. Then, G is supereulerian if and only if G/L is supereulerian.*

In the current research, we prove the following, as an attempt to extend Theorem 1.6 to hypergraphs.

Theorem 1.7. *Let J be a 2-partition-connected sub-hypergraph of a hypergraph H. If H/J has a dominating spanning closed trail Γ with $v_J \in PV(\Gamma)$, then H is supereulerian. In particular, if H/J is pivot-supereulerian, then H is pivot-supereulerian.*

Corollary 1.8. *Let H be a hypergraph and J be a 2-partition-connected sub-hypergraph of H. Then, H is pivot-supereulerian if and only if H/J is pivot-supereulerian.*

We will cover the necessary preliminaries in Section 2. In Section 3, we discuss the properties of partition-connectedness of hypergraphs and hypertrees. These are powerful tools in the proof of the main results in Section 4. Some conjectures and discussions are displayed in Section 5.

2 | PRELIMINARIES AND NOTATION

Let H be a hypergraph. We denote the number of connected components of H by $\omega(H)$. If $W \subseteq V(H)$, then the hypergraph $(W, \mathcal{E}_W)$, where $\mathcal{E}_W = \{F \in \mathcal{E}(H) : F \subseteq W\}$, is the *sub-hypergraph induced by the vertex subset* W, denoted by $H[W]$. If $X \subseteq \mathcal{E}(H)$ and $V_X = \bigcup_{F \in X} F$, then (V_X, X) is defined as *the sub-hypergraph induced by the edge subset* X, denoted by $H[X]$. For notational convenience, we often also use X to denote the hypergraph (V_X, X).

For a subset $X \subseteq \mathcal{E}(H)$, let $H - X = (V(H), \mathcal{E}(H) - X)$. Let H_1 and H_2 be two hypergraphs. The *intersection* of H_1 and H_2, denoted by $H_1 \cap H_2$, has $V(H_1 \cap H_2) = V(H_1) \cap V(H_2)$ and $\mathcal{E}(H_1 \cap H_2) = \mathcal{E}(H_1) \cap \mathcal{E}(H_2)$; and the *union* of H_1 and H_2, denoted by $H_1 \cup H_2$, has $V(H_1 \cup H_2) = V(H_1) \cup V(H_2)$ and $\mathcal{E}(H_1 \cup H_2) = \mathcal{E}(H_1) \cup \mathcal{E}(H_2)$. If $H_2 = \{E\}$ for some edge $E \in \mathcal{E}(H_2)$, then we write $H_1 \cup E$ for $H_1 \cup H_2$.

2.1 | Contractions

We start with a formal definition of hypergraph contractions.

Definition 2.1. Let J be a sub-hypergraph of H with components labeled by $J_1, J_2, \ldots, J_s$, and let $U_J = \{v_{J_1}, v_{J_2}, \ldots, v_{J_s}\}$. Define a mapping $c : V(H) \to V(H) \cup U_J$ by

$$c(v) = \begin{cases} v_{J_i}, & v \in V(J_i), \\ v, & \text{otherwise}. \end{cases} \tag{2}$$

For a given mapping c, we denote the *images* of vertex $v \in V(H)$ and $E \in \mathcal{E}(H)$ by

$$im(v) = c(v) \quad \text{and} \quad im(E) = \{c(v) : v \in E\},$$

respectively. Conversely, the vertex v and the edge E are called *preimages* of $im(v)$ and $im(E)$, respectively. Let $U \subseteq V(H)$ and $X \subseteq \mathcal{E}(H)$. Then, $im(U) = \{im(v) : v \in U\}$ and $im(X) = \{im(E) : E \in X\}$ are called the *images* of U and X, respectively.

The terms and notation of hypergraph contraction in Definition 2.1 allow us to make the following observation.

Observation 2.1. Let J be a sub-hypergraph of a hypergraph H such that J has components $J_1, J_2, \ldots, J_s$, and let $U_J = \{v_{J_1}, v_{J_2}, \ldots, v_{J_s}\}$. Define a mapping c as in (2). Then the contraction H/J is the hypergraph with vertex set $V(H/J) = im(V(H))$ and edge set $\mathcal{E}(H/J) = im(\mathcal{E}(H))$. By definition, $V(H/J) = (V(H) - V(J)) \cup U_J$.

Given a sub-hypergraph Γ of H, $im(\Gamma) = (H/J)[im(V(\Gamma))]$ is called the *image* of Γ. Thus, if every vertex $v \in V(\Gamma)$ lies in an edge $E \in \mathcal{E}(\Gamma)$, then $im(\Gamma) = (H/J)[im(\mathcal{E}(\Gamma))]$. In particular, if $\Gamma = H[X]$ is a sub-hypergraph induced by the edge subset X, then $im(\Gamma) = (H/J)[im(X)]$.

Conversely, given $W \subseteq V(H/J)$, $Y \subseteq \mathcal{E}(H/J)$, and a sub-hypergraph Γ_1 of H/J. The pre-images of W, Y, and Γ_1 are $pre(W) = \{v \in V(H) : im(v) \in W\}$, $pre(Y) = \{E \in \mathcal{E}(H): im(E) \in Y\}$ and $pre(\Gamma_1) = H[pre(V(\Gamma_1))]$, respectively. The following lemma will be used in our arguments. As a number of other lemmas will need to be developed to justify Lemma 2.1, we postpone the validation of Lemma 2.1 to Section 4.

Lemma 2.1. *Let H be a hypergraph and J be a sub-hypergraph of H. Each of the following holds.*

(i) *If H is supereulerian, then H/J has a dominating closed trail.*
(ii) *If H is pivot-supereulerian, then H/J is pivot-supereulerian.*
(iii) *If H is heavy supereulerian and J is connected, then H/J is supereulerian.*

2.2 | Collapsible graphs

Let G be a graph and $O(G)$ be the set of all odd degree vertices in G. In [3], Catlin defined a graph G to be *collapsible* if for every subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a subgraph Γ_R such that $O(\Gamma_R) = R$ and $G - \mathcal{E}(\Gamma_R)$ is connected. By definition, all complete graphs K_n except K_2 are collapsible. As shown in Proposition 1 of [10], a graph G is collapsible if and only if for every subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph L_R with $O(L_R) = R$. As $L_\varnothing$ is a spanning eulerian subgraph, every collapsible graph is supereulerian.

Theorem 2.2 (Catlin [3, Theorem 2]). *Let G be a graph. If G has two edge-disjoint spanning trees, then G is collapsible, and so G is supereulerian.*

3 | PARTITION-CONNECTED HYPERGRAPHS AND HYPERTREES

Frank, Király, and Kriesell in [6] indicated the following proposition that k-partition-connected hypergraphs can be characterized in a different form, which is often used in applications.

Theorem 3.1 (Frank, Király, and Kriesell [6]). *Let H be a hypergraph and $k > 0$ be an integer. The following are equivalent.*

(i) *H is k-partition-connected;*
(ii) *for each partition $P \in \mathcal{P}(H)$, $e(P) \geq k(|P| - 1)$;*
(iii) *for each subset $X \subseteq \mathcal{E}(H)$, $|X| \geq k(\omega(H - X) - 1)$.*

By definition, every k-partition-connected hypergraph must be k-edge-connected. Following [6], a hypergraph is *partition-connected* if it is 1-partition-connected. In general, partition-connected hypergraphs must be connected, but a connected hypergraph may not be partition-connected.

Theorem 3.2. *Let H be a hypergraph with a sub-hypergraph J and $k > 0$ be an integer. Each of the following holds.*

(i) (*Frank, Király, and Kriesell* [6, *Corollary* 2.9]) *If H is kr-edge-connected where $r = r(H)$, then H is k-partition-connected.*
(ii) (*Gu and Lai* [7, *Proposition* 4.1]) *If H is k-partition-connected, then for any $E \in \mathcal{E}(H)$, H/E is k-partition-connected. Furthermore, if J and H/J are k-partition-connected, then H is k-partition-connected.*

A hypergraph H is a *hyperforest* if for every nonempty subset $U \subseteq V(H)$, $|\mathcal{E}(H[U])| \leq |U| - 1$. A hyperforest T is a *hypertree* if $|\mathcal{E}(T)| = |V(T)| - 1$. For a partition $P = (V_1, V_2, \ldots, V_t)$ of $V(T)$,

$$e(P) = |\mathcal{E}(T)| - \sum_{i=1}^{t} |\mathcal{E}(T[V_i])| \geq (|V(T)| - 1) - \sum_{i=1}^{t} (|V_i| - 1) = t - 1.$$

It shows that every hypertree is partition-connected.

Theorem 3.3 (Frank, Király, and Kriesell [6, Corollary 2.6]). *Each of the following statements holds.*

(i) *For each partition-connected hypergraph H, $|\mathcal{E}(H)| \geq |V(H)| - 1$ with equality if and only if H is a hypertree.*
(ii) *Each partition-connected hypergraph contains a spanning hypertree.*

Theorem 3.4 (Frank, Király, and Kriesell [6, Theorem 2.8]). *A hypergraph H is k-partition-connected if and only if H has k edge-disjoint spanning partition-connected sub-hypergraphs.*

Lemma 3.5. *Suppose that H is a partition-connected hypergraph and $E \in \mathcal{E}(H)$ with $|E| \geq 3$. Then there exists a vertex $v \in E$ such that with $E' = E - \{v\}$, $(H - E) \cup E'$ is partition-connected.*

Proof. For a vertex $u \in E$, let $E_u = E - \{u\}$ and $H_u = (H - E) \cup E_u$. By Theorem 3.3(ii), H contains a spanning hypertree. If E is not contained in this hypertree, then for each vertex $u \in E$, H_u is partition-connected. Thus we assume that E lies in every spanning hypertree of H. Let T be a hypertree of H such that

$$T \text{ contains } E \text{ as an edge with } |V(T)| \text{ minimized.} \tag{3}$$

As T is partition-connected, by Theorem 3.1, $1 = |\{E\}| \geq \omega(T - E) - 1$, which implies that $\omega(T - E) \leq 2$. As $|E| \geq 3$, it follows that there exist two vertices $u, v \in E$ such that both u and v are in the same component of $T - E$.

Claim 1: $T' = (T - E) \cup E_v$ is a hypertree.

Suppose to the contrary that T' is not a hypertree. Since $V(T') = V(T)$ and $|\mathcal{E}(T')| = |\mathcal{E}(T)|$, by definition, there exists a nonempty subset $U \subseteq V(T')$ such that $|\mathcal{E}(T'[U])| > |U| - 1$. Since T is a hypertree, $|\mathcal{E}(T[U])| \leq |U| - 1 < |\mathcal{E}(T'[U])|$. It follows that $|\mathcal{E}(T[U])| = |\mathcal{E}(T'[U])| - 1$ and $E - U = \{v\}$. Then, $|U| - 1 < |\mathcal{E}(T'[U])| \leq |U|$, which leads to $|\mathcal{E}(T'[U])| = |U|$ and $|\mathcal{E}(T[U])| = |U| - 1$. Let $U' = U \cup \{v\}$. Then $|\mathcal{E}(T[U'])| \geq |\mathcal{E}(T[U])| + 1 = |U| = |U'| - 1$. As T is a hypertree, $|\mathcal{E}(T[U'])| \leq |U'| - 1$, and then $|\mathcal{E}(T[U'])| = |U'| - 1$, which means $T[U']$ is also a hypertree. By (3), we have $T = T[U']$. Since $|\mathcal{E}(T[U])| = |U| - 1 = |\mathcal{E}(T[U'])| - 1$, E is the only one edge satisfying both $E \cap U \neq \emptyset$ and $v \in E$. It follows that v is an isolated vertex in $T - E$, which contradicts the fact that vertices u and v are in one component of $T - E$. This contradiction implies that T' must be a hypertree. This proves Claim 1.

By Claim 1 and Theorem 3.2(ii), both T' and $H_v/T' = H/T$ are partition-connected. Hence by Theorem 3.2(ii), $H_v = (H - E) \cup E_v$ is also partition-connected. □

Lemma 3.5 motivates the concept of partition-connected mappings on hypergraphs when studying partition-connectedness of hypergraphs. Let $2^X = \{F : F \subseteq E \in X\}$ be the power set of a multiset X. An injective mapping $g : \mathcal{E}(H) \to 2^{\mathcal{E}(H)}$ is a *partition-connected mapping* (or *pc-mapping*) of a hypergraph H if each of the following holds:

(PC1) For any $E \in \mathcal{E}(H)$, $g(E) \subseteq E$, and $(H - E) \cup g(E)$ is partition-connected, and
(PC2) $g(H) := g(\mathcal{E}(H))$ is a connected (multi)graph with $V(g(H)) = V(H)$.

Corollary 3.6. *Let H be a partition-connected hypergraph. Each of the following holds.*

(i) *H has a pc-mapping.*
(ii) *If $g(H)$ is supereulerian where g is a pc-mapping of H, then H is pivot-supereulerian.*

Proof. Suppose that H is a partition-connected hypergraph. We shall argue by induction on $\theta(H) = \sum_{E \in \mathcal{E}(H), |E| \geq 3}(|E| - 2)$ to prove (i). If $\theta(H) = 0$, then as H is a (multi)graph, the identity mapping is a pc-mapping of H, and so we are done. Thus we assume that $\theta(H) \geq 1$ and that (i) holds for partition-connected hypergraphs with smaller values of θ. Since $\theta(H) \geq 1$, there exists an edge $E_0 \in \mathcal{E}(H)$ with $|E_0| \geq 3$. By Lemma 3.5, there exists a vertex $v \in E_0$ such that with $E_0' = E_0 - \{v\}$, $H' = (H - E_0) \cup E_0'$ is partition-connected. By definition, we have $\theta(H') < \theta(H)$ and $V(H') = V(H)$, and so by induction, H' has a pc-mapping g'. Set $g : \mathcal{E}(H) \to 2^{\mathcal{E}(H)}$ with $g(E) = g'(E_0')$ if $E = E_0$, and $g(E) = g'(E)$ if $E \neq E_0$. Since g' is injective, g is injective as well. Note that $g(H) = g'(H')$ is a connected graph and $V(H) = V(H') = V(g'(H')) = V(g(H))$. This means that g satisfies (PC2). Note that $g(E_0) = g'(E_0') \subseteq E_0' \subseteq E_0$ and $(H - E_0) \cup g(E_0) \cong (H' - E_0') \cup g'(E_0')$ is partition-connected. For each edge $E \in \mathcal{E}(H) - E_0$, we have $g(E) = g'(E) \subseteq E$ and

$(H - E) \cup g(E) \cong (H' - E) \cup g'(E)$ is partition-connected. Thus, g satisfies (PC1) and then it is a pc-mapping of H, and so (i) follows by induction.

To prove (ii), we assume that $g(H)$ has a dominating spanning closed trail $\Gamma' = (F_1 F_2 \cdots F_m)$ where each $F_i \in \mathcal{E}(g(H))$. Then $\Gamma = H[g^{-1}(\mathcal{E}(\Gamma'))] = (g^{-1}(F_1) g^{-1}(F_2) \cdots g^{-1}(F_m))$ is a closed trail. As $V(H) \supseteq PV(\Gamma) \supseteq PV(\Gamma') = V(g(H)) = V(H)$, we have Γ is a pivot-spanning closed trail in H. □

Corollary 3.7. *Let H be a hypergraph and $J_1, J_2, \ldots, J_q$ be a list of pairwise edge-disjoint partition-connected sub-hypergraphs of H. Then, there exists an injection $g : \mathcal{E}(H) \to 2^{\mathcal{E}(H)}$ such that:*

(i) *$g|_{\mathcal{E}(J_i)}$ is a pc-mapping of J_i for each i;*
(ii) *$V(g(H)) = V(H)$ where $g(H) = g(\mathcal{E}(H))$, and for each $E \in \mathcal{E}(H), g(E) \subseteq E$.*

Furthermore, if $g(H)$ is supereulerian, then H is supereulerian.

Proof. By Corollary 3.6, let $g_1, g_2, \ldots, g_q$ be the corresponding pc-mappings of $J_1, J_2, \ldots, J_q$. Take $g : \mathcal{E}(H) \to 2^{\mathcal{E}(H)}$ with $g(E) = g_i(E)$ if $E \in \mathcal{E}(J_i)$; otherwise, $g(E) = E$. Then, g is an injection satisfying (i) and (ii).

Suppose that $g(H)$ has a dominating spanning closed trail $\Gamma' = (F_1 F_2 \cdots F_m)$ where each $F_i \in \mathcal{E}(g(H))$. Let $\Gamma = H[g^{-1}(\mathcal{E}(\Gamma'))] = (g^{-1}(F_1) g^{-1}(F_2) \cdots g^{-1}(F_m))$. Then, Γ is a closed trail in H with $PV(\Gamma) \supseteq PV(\Gamma')$. As $V(H) \supseteq V(\Gamma) \supseteq V(\Gamma') = V(g(H)) = V(H)$, Γ is spanning. Pick an edge $E \in \mathcal{E}(H)$. Since Γ' is dominating and $g(E) \subseteq E, \emptyset \neq g(E) \cap PV(\Gamma') \subseteq E \cap PV(\Gamma') \subseteq E \cap PV(\Gamma)$, which implies that Γ is dominating. Hence, H is supereulerian. □

Proposition 3.8. *Let H be a hypergraph and T be a partition-connected sub-hypergraph of H. Then the following are equivalent.*

(a) *T is a spanning hypertree;*
(b) *T has a pc-mapping and for every pc-mapping g of $T, g(T)$ is a tree with $V(g(T)) = V(H)$;*
(c) *T is an edge-minimum spanning partition-connected sub-hypergraph of H.*

Proof. Suppose that T is an edge-minimum spanning partition-connected sub-hypergraph of H. By Theorem 3.3(i), $|\mathcal{E}(T)| \geq |V(T)| - 1 = |V(H)| - 1$. By Theorem 3.3(ii), T has a spanning hypertree T_0. It follows that $|\mathcal{E}(T_0)| = |V(T_0)| - 1 = |V(T)| - 1 \leq |\mathcal{E}(T)|$. If $|\mathcal{E}(T_0)| < |\mathcal{E}(T)|$, then it contradicts the assumption that T is an edge-minimum spanning partition-connected sub-hypergraph of H. Then, $|V(T)| - 1 = |\mathcal{E}(T_0)| = |\mathcal{E}(T)|$ and then T is a hypertree by Theorem 3.3(i). Thus, (c) implies (a).

Now, we show that (a) implies (b). As T is a spanning partition-connected sub-hypergraph of H, by Corollary 3.6, T has a pc-mapping g and $V(g(T)) = V(T) = V(H)$. Since T is a hypertree, we have $|\mathcal{E}(g(T))| = |\mathcal{E}(T)| = |V(T)| - 1 = |V(g(T))| - 1$, which implies that $g(T)$ is a tree as $g(T)$ is connected.

Then, we claim that (b) implies (c). Suppose T_1 is a spanning partition-connected sub-hypergraph of H. By Corollary 3.6, T_1 has a pc-mapping g_1. Then, $g_1(T_1)$ is a connected graph with $|\mathcal{E}(g_1(T_1))| = |\mathcal{E}(T_1)|$ and $V(g_1(T_1)) = V(T_1) = V(H)$. It follows that

$$|\mathcal{E}(g_1(T_1))| \geq |V(g_1(T_1))| - 1 = |V(H)| - 1 = |V(g(T))| - 1 = |\mathcal{E}(g(T))|,$$

which shows that T is an edge-minimum spanning partition-connected sub-hypergraph of H. □

4 | PROOFS OF THE MAIN RESULTS

For notational convenience, we allow an empty sequence to denote an empty trail (or path) in a hypergraph. If $\Gamma_1 = (v_0E_0v_1E_1 \cdots v_{j-1}E_{j-1}v_j)$ and $\Gamma_2 = (v_jE_jv_{j+1}E_{j+1} \cdots v_{n-1}E_{n-1}v_n)$ are two edge-disjoint trails, then we use $\Gamma_1\Gamma_2$ or, to emphasize the termini of the trails, $v_0\Gamma_1v_j\Gamma_2v_n$, to denote the trail $\Gamma = (v_0E_0v_1E_1 \cdots v_{n-1}E_{n-1}v_n)$ obtained by amalgamating the trails Γ_1 and Γ_2. Thus if Γ_2 is an empty trail, then $\Gamma_1\Gamma_2 = \Gamma_1$. As $\Gamma' = (E_iv_{i+1}E_{i+1} \cdots E_j)$ is a subtrail of Γ, this trail amalgamating notation allows us to rewrite Γ as $(v_0E_0v_1E_1 \cdots v_i\Gamma'v_{j+1} \cdots v_{n-1}E_{n-1}v_n)$. If some vertex $v \in V(\Gamma)$ and some indices i and j with $j > i$, we have $v_i = v_{i+1} = \cdots = v_{j+1} = v$, then we define a *$v$-subsequence* of Γ to be $(v_iE_iv_{i+1}E_{i+1} \cdots v_jE_jv_{j+1})$. If $v_{i-1} \neq v$ and $v_{j+2} \neq v$, then the v-subsequence is a *maximal v-subsequence*. A maximal v sequence of Γ is denoted by Γ_v.

4.1 | The proofs of Theorems 1.2 and 1.4

Proof of Theorem 1.2. To prove the sufficiency, we assume that H has a dominating eulerian sub-hypergraph $H' = (v_1E_1v_2E_2 \cdots v_tE_tv_1)$. Define $S_1 = \{F \in \mathcal{E}(H) - \mathcal{E}(H') : v_1 \in F\}$. Inductively, for each $i \geq 2$, assume that $S_1, \ldots, S_{i-1}$ have been defined, we set

$$S_i = \left\{F \in \mathcal{E}(H) - \left(\mathcal{E}(H') \cup \left(\bigcup_{j<i} S_j\right)\right) : v_i \in F\right\}.$$

It is possible that some of the S_i's may be empty. Since H' is dominating in H, $\mathcal{E}(H) - \mathcal{E}(H')$ can be partitioned into $S_1, S_2, \ldots, S_t$. For each $i \in [1, t]$, let $S_i = \{F_i^1, F_i^2, \ldots, F_i^{s(i)}\}$ and $P_i = (F_i^1F_i^2 \cdots F_i^{s(i)})$ denote a path from F_i^1 to $F_i^{s(i)}$ in the line graph $L(H)$ of H. Thus we obtain a Hamilton cycle in $L(H)$ by amalgamating the paths $P_1, P_2, \ldots, P_t$, as follows:

$$(E_tP_1E_1P_2 \cdots E_{t-1}P_tE_t).$$

Conversely, we assume that $L(H)$ is hamiltonian to prove the necessity. Let $(E_0E_1 \cdots E_{m-1}E_0)$ be a Hamilton cycle in $L(H)$ where each $E_i \in \mathcal{E}(H)$. By the definition of $L(H)$, for each $i \in \mathbb{Z}_m$, $E_i \cap E_{i+1} \neq \emptyset$ and then let $v_{i+1} \in E_i \cap E_{i+1}$. Then, $\Gamma = (v_0E_0v_1E_1 \cdots v_{m-1}E_{m-1}v_0)$ satisfies (CT1) and (CT2). Let $V = \{v_0, v_1, \ldots, v_{m-1}\}$. Construct a new sequence $\Gamma' = \Gamma / \bigcup_{v \in V} \Gamma_v$ by contracting every maximal v-subsequence Γ_v into the vertex v for every $v \in V$. Then each two consecutive vertices in Γ' are distinct. It follows that Γ' satisfies (CT1)–(CT3) and then Γ' is a closed trail by Observation 1.1. By the definition of Γ', for any edge $E \in \mathcal{E}(H) - \mathcal{E}(\Gamma')$, there exists a vertex $u \in V$ such that $E \in \mathcal{E}(\Gamma_u)$, and so $u \in E$. Hence, Γ' is a dominating eulerian sub-hypergraph of H. □

Proof of Theorem 1.4. If H is 2-partition-connected, then by Theorem 3.3(ii) and Theorem 3.4, H has two edge-disjoint spanning hypertrees T_1 and T_2. By Corollary 3.7, there exists an injection $g : \mathcal{E}(H) \to 2^{\mathcal{E}(H)}$ such that each of $g|_{\mathcal{E}(T_1)}$ and $g|_{\mathcal{E}(T_2)}$ is a pc-mapping, and $V(g(H)) = V(H)$ where $g(H) = g(\mathcal{E}(H))$. As each of $g|_{\mathcal{E}(T_1)}$ and $g|_{\mathcal{E}(T_2)}$ is a pc-mapping, it follows that the graph $g(H)$ has two edge-disjoint spanning trees by Proposition 3.8. This follows that $g(H)$ is supereulerian by Theorem 2.2. Applying Corollary 3.7 again, we conclude that H is supereulerian.

By Theorem 3.2(i), if $r(H) = r$, then every $2r$-edge-connected hypergraph H is 2-partition-connected, and so H is supereulerian. □

4.2 | The proofs of Lemma 2.1, Theorem 1.7, and Corollary 1.8

Since Corollary 1.8 is an immediate consequence of Lemma 2.1 and Theorem 1.7, it suffices to validate Lemma 2.1 and Theorem 1.7.

We adopt the terms and notation in Definition 2.1 in our discussions. Let J be a sub-hypergraph of a hypergraph H with its components labeled as $J_1, J_2, \dots, J_s$. As in Definition 2.1, we denote $U_J = \{v_{J_1}, v_{J_2}, \dots, v_{J_s}\}$, where v_{J_i} is the image of $V(J_i)$ for each $i \in [1, s]$. If H has a closed trail Γ, then we define

$$U_J(\Gamma) = \{v_{J_i} \in U_J : V(J_i) \cap PV(\Gamma) \neq \emptyset\},$$

and

$$X(J, \Gamma) = \{E \in \mathcal{E}(\Gamma) : E - V(J) \neq \emptyset, PV_\Gamma(E) \subseteq V(J_i) \text{ for some } i\}.$$

By definitions, $im(PV(\Gamma)) = (PV(\Gamma) - V(J)) \cup U_J(\Gamma) \subseteq V(H/J)$.

Lemma 4.1. *Let H be a hypergraph with a closed trail Γ and J be a sub-hypergraph of H. If $E \cap PV(\Gamma) \neq \emptyset$ where $E \in \mathcal{E}(H)$, then $im(E) \cap im(PV(\Gamma)) \neq \emptyset$.*

Proof. Pick $E \in \mathcal{E}(H)$. Suppose that there exists a vertex $v \in E \cap PV(\Gamma)$. If $v \in V(J)$, then $im(v) \in U_J(\Gamma)$; otherwise, $v \in PV(\Gamma) - V(J)$, then $im(v) \in PV(\Gamma) - V(J)$. It follows that $im(v) \in (PV(\Gamma) - V(J)) \cup U_J(\Gamma) = im(PV(\Gamma))$. As $im(v) \in im(E)$, $im(E) \cap im(PV(\Gamma)) \neq \emptyset$. □

Lemma 4.2. *Let H be a hypergraph with a closed trail Γ and J be a sub-hypergraph of H. Then, $L = im(\mathcal{E}(\Gamma) - X(J, \Gamma))$ is a closed trail of H/J with $PV(L) = im(PV(\Gamma))$.*

Proof. Let $\Gamma = (E_0 E_1 \cdots E_{s-1})$ be an edge sequence satisfying (CT1)–(CT3) and let $Y = \mathcal{E}(\Gamma) - X(J, \Gamma)$. For each $i \in [0, s-1]$, let $F_i = im(E_i)$ if $E_i \in Y$, and let $L = im(Y) = \{F_{y(0)}, F_{y(1)}, \dots, F_{y(t-1)}\}$ where $y(0) < y(1) < \cdots < y(t-1)$.

Claim 2: $\bigcup_{i \in \mathbb{Z}_t}(F_{y(i)} \cap F_{y(i+1)}) = im(PV(\Gamma))$.

Note that a vertex $v \in PV(\Gamma) - V(J)$, if and only if $v \in (E_r \cap E_{r+1}) - V(J)$ for some $E_r, E_{r+1} \in \mathcal{E}(\Gamma)$, if and only if $v \in (F_r \cap F_{r+1}) - U_J(\Gamma) \subseteq \bigcup_{i \in \mathbb{Z}_t}(F_{y(i)} \cap F_{y(i+1)}) - U_J(\Gamma)$. Then, $PV(\Gamma) - V(J) = \bigcup_{i \in \mathbb{Z}_t}(F_{y(i)} \cap F_{y(i+1)}) - U_J(\Gamma)$. As $im(PV(\Gamma)) = (PV(\Gamma) - V(J)) \cup U_J(\Gamma)$, it suffices to show that $U_J(\Gamma) \subseteq \bigcup_{i \in \mathbb{Z}_t}(F_{y(i)} \cap F_{y(i+1)})$. Pick $u \in U_J(\Gamma)$. By the definition of $U_J(\Gamma)$, there exists $v \in V(J_i) \cap PV(\Gamma)$ such that $u = im(v)$ and J_i is a component of J. It follows that $v \in E_k \cap E_{k+1}$ for some edges $E_k, E_{k+1} \in \mathcal{E}(\Gamma)$. Let $k_1 \leq k$ be the largest integer with $E_{k_1} \in Y$ and let $k_2 > k$ be the smallest integer with $E_{k_2} \in Y$. It means that $u \in F_{k_1} \cap F_{k_2} \subseteq \bigcup_{i \in \mathbb{Z}_t} (F_{y(i)} \cap F_{y(i+1)})$.

Claim 3: L is a closed trail.

We can view $L = im(Y) = (F_{y(0)}F_{y(1)} \cdots F_{y(t-1)})$ as an edge sequence. By Observation 1.1, it suffices to show that L satisfies (CT1)–(CT3).

Pick $F_{y(i)}, F_{y(j)} \in L$. Since $E_{y(i)}$ and $E_{y(j)}$ are distinct edges, $F_{y(i)} = im(E_{y(i)})$ and $F_{y(j)} = im(E_{y(i)})$ are distinct edges as well, which means that L satisfies (CT1).

To show that L satisfies (CT2), by symmetry, it suffices to show that $F_{y(i)} \cap F_{y(i+1)} \neq \varnothing$. If $y(i+1) - y(i) = 1$, then $E_{y(i)+1} \in Y$ and $F_{y(i+1)} = F_{y(i)+1}$. Since $E_{y(i)} \cap E_{y(i)+1} \neq \varnothing$, $F_{y(i)} \cap F_{y(i+1)} \neq \varnothing$. If $y(i+1) - y(i) = q \geq 2$, then $\{E_{y(i)+1}, E_{y(i)+2}, ..., E_{y(i)+q-1}\} \subseteq X(J, \Gamma)$. It follows that for each $k \in [1, q-1]$, $PV_\Gamma(E_{y(i)+k}) \subseteq V(J_k)$ for some component J_k of J. As $PV_\Gamma(E_{y(i)+k}) \cap PV_\Gamma(E_{y(i)+k+1}) \neq \varnothing$ for each $k \in [1, q-2]$, $\bigcup_{k \in [1,q-1]} PV_\Gamma(E_{y(i)+k}) \subseteq V(J_r)$ for some component J_r of J. This implies that $v_{J_r} \in F_{y(i)} \cap F_{y(i+1)}$. Hence, L satisfies (CT2).

We are to show that L satisfies (CT3). By contradiction, and by the fact that L satisfies (CT2), we assume that $|\bigcup_{|i-j|=1} F_{y(i)} \cap F_{y(j)}| = 1$ for some i, say $\{u\} = F_{y(i)} \cap F_{y(i-1)} = F_{y(i)} \cap F_{y(i+1)}$. By Claim 2, either $u \in PV(\Gamma) - V(J)$ or $u \in U_J(\Gamma)$. If $u \in U_J(\Gamma)$, then $E_{y(i)} \in X(J, \Gamma)$ contradicting with $E_{y(i)} \in Y = \mathcal{E}(\Gamma) - X(J, \Gamma)$; otherwise, $u \in PV(\Gamma) - V(J)$, then $y(i-1) = y(i) - 1, y(i+1) = y(i) + 1$, and $\{u\} = E_{y(i)} \cap E_{y(i)-1} = E_{y(i)} \cap E_{y(i)+1}$, which contradicts that Γ satisfies (CT3).

By Claims 2 and 3, L is a closed trail with $PV(L) = im(PV(\Gamma))$. □

Proof of Lemma 2.1. Let Γ be a closed trail of H and $X = X(J, \Gamma)$. By Lemma 4.2, $L = im(\mathcal{E}(\Gamma) - X)$ is a closed trail of H/J with $PV(L) = im(PV(\Gamma))$.

(i) Suppose that Γ is dominating and spanning in H. Pick an edge $E' \in \mathcal{E}(H/J)$. Let E be the preimage of E' in H. Since Γ is dominating in H, $E \cap PV(\Gamma) \neq \varnothing$, and then, by Lemma 4.1, $\varnothing \neq E' \cap im(PV(\Gamma)) = E' \cap PV(L)$. It shows that L is dominating in H/J.

(ii) Suppose $PV(\Gamma) = V(H)$. Then, $U_J(\Gamma) = U_J$. This follows that $PV(L) = im(PV(\Gamma)) = (PV(\Gamma) - V(J)) \cup U_J(\Gamma) = (V(H) - V(J)) \cup U_J = V(H/J)$, and then H/J is pivot-supereulerian.

(iii) Suppose that for each vertex $v \in V(H)$, $|\mathcal{E}_\Gamma(v)| \geq 2$. Since J is connected and Γ is dominating, $|U_J| = |U_J(\Gamma)| = 1$. Let $\{v_J\} = U_J = U_J(\Gamma)$. Then, $v_J \in im(PV(\Gamma)) = PV(L)$. For each edge $E \in \mathcal{E}(H)$, we denote $E' = im(E)$ to be the image of E in H/J. We shall verify (iii) by showing the following claims.

Claim 4: For each vertex $u \in V(H/J) - V(L)$, there exists a pair of edges $\{E_u, F_u\} \subseteq X$ such that $C_u = (v_J E'_u u F'_u v_J)$ is a closed trail.

As $V(H/J) - V(L) = V(im(X)) \cap V(H)$, for each vertex $u \in V(H/J) - V(L)$, there exists $E_u \in X$ such that $u \in E'_u \cap E_u$. Then, there exists $F_u \in \mathcal{E}(\Gamma)$ such that $u \in F_u \neq E_u$ as $|\mathcal{E}_\Gamma(u)| \geq 2$. If $F_u \notin X$, then $u \in V(L)$, which contradicts with $u \in V(H/J) - V(L)$. Thus, $F_u \in X$ and then $u \in F'_u$. As $\{E_u, F_u\} \subseteq X$, we have $\{v_J, u\} \subseteq E'_u \cap F'_u$. Hence, $C_u = (v_J E'_u u F'_u v_J)$ is a closed trail.

Claim 5: There exists a subset $W \subseteq V(H/J) - V(L)$ such that $\mathcal{C}_W = \bigcup_{u \in W} C_u$ is a closed trail with $W \cup \{v_J\} \subseteq PV(\mathcal{C}_W)$ and $V(\mathcal{C}_W) \cup V(L) = V(H/J)$.

By Claim 4, we assume that $W_1 \subseteq V(H/J) - V(L)$ such that $\mathcal{C}_{W_1} = \bigcup_{u \in W_1} C_u$ is a closed trail with $W_1 \cup \{v_J\} \subseteq PV(\mathcal{C}_{W_1})$ and $|V(\mathcal{C}_{W_1})|$ maximized. If $V(\mathcal{C}_{W_1}) - \{v_J\} = V(H/J) - V(L)$, then $V(\mathcal{C}_{W_1}) \cup V(L) = V(H/J)$ and so we are done by taking $W = W_1$. Now, we consider that there exists a vertex $w \in V(H/J) - V(L) - V(\mathcal{C}_{W_1})$. By Claim 4, there exists a pair of edges $\{E_w, F_w\} \subseteq X$ such that $C_w = (v_J E'_w w F'_w v_J)$ is a closed trail. If $\{E'_w, F'_w\} \cap \mathcal{E}(\mathcal{C}_{W_1}) \neq \emptyset$, then $w \in V(\mathcal{C}_{W_1})$, which contradicts with $w \in V(H/J) - V(L) - V(\mathcal{C}_{W_1})$. Then $\{E'_w, F'_w\} \cap \mathcal{E}(\mathcal{C}_{W_1}) = \emptyset$. Set $W_2 = W_1 \cup \{w\}$. Then, $\mathcal{C}_{W_2} = \bigcup_{u \in W_2} C_u = \mathcal{C}_{W_1} \cup C_w = (v_J \mathcal{C}_{W_1} v_J C_w v_J)$ is a closed trail with $PV(\mathcal{C}_{W_2}) \supseteq PV(\mathcal{C}_{W_1}) \cup \{w\} \supseteq W_1 \cup \{v_J, w\} = W_2 \cup \{v_J\}$ and $|V(\mathcal{C}_{W_2})| > |V(\mathcal{C}_{W_1})|$, which contradicts the maximality of $|V(\mathcal{C}_{W_1})|$.

Claim 6: $L \cup \mathcal{C}_W$ is a spanning closed trail of H/J.

As Γ is a closed trail and by the definition of contraction, every pair of edges in $L \cup \mathcal{C}_W$ are distinct. Then, as $v_J \in PV(L) \cap \mathcal{C}_W$, $L \cup \mathcal{C}_W = (v_J L v_J \mathcal{C}_W v_J)$ is a closed trail of H/J. By Claim 5, $V(L \cup \mathcal{C}_W) = V(\mathcal{C}_W) \cup V(L) = V(H/J)$.

Claim 7: $L \cup \mathcal{C}_W$ is dominating.

Pick $F' \in \mathcal{E}(H/J) - \mathcal{E}(L \cup \mathcal{C}_W)$. Suppose $F' \cap PV(L \cup \mathcal{C}_W) = \emptyset$. Since $PV(L \cup \mathcal{C}_W) \supseteq PV(L) \cup W$, $\emptyset = F' \cap (PV(L) \cup W) = F' \cap (im(PV(\Gamma)) \cup W)$, which implies that $F' \cap im(PV(\Gamma)) = \emptyset$. Then, by Lemma 4.1, $F \cap PV(\Gamma) = \emptyset$ where F is the preimage of F'. It contradicts that Γ is dominating in H.

Combine Claims 6 and 7, H/J is supereulerian. □

Proof of Theorem 1.7. Suppose that J is 2-partition-connected and H/J has a dominating spanning closed trail Γ with $v_J \in PV(\Gamma)$. Let $X = \{E \in \mathcal{E}(\Gamma) : v_J \in PV_\Gamma(E)\}$. Then $|X| \equiv 0 \,(mod\ 2)$. For each $F \in pre(X)$, since $F \cap V(J) \neq \emptyset$, we choose a vertex $v \in F \cap V(J)$. Let R be the collection of all these vertices. Note that there may be a pair of vertices v_1 and v_2 in R such that $v_1 = v_2$. Remove this pair of vertices and repeat this operation such that the rest of vertices form a set of vertices R'. Then $R' \subseteq V(J)$ and $|R'| \equiv 0 \,(mod\ 2)$.

Case 1. $r(J) = 2$.

Since J is 2-partition-connected and $r(J) = 2$, by Theorems 2.2 and 3.4, J is collapsible. It follows that J has a spanning connected subgraph L with $O(L) = R'$ as $|R'| \equiv 0 \,(mod\ 2)$. Then, $\Gamma_1 = L \cup pre(\Gamma)$ is a closed trail of H with $PV(\Gamma_1) = V(J) \cup (PV(\Gamma) - \{v_J\})$. Since $V(\Gamma_1) = V(L) \cup V(pre(\Gamma)) = V(J) \cup (V(pre(\Gamma)) - V(J)) = V(J) \cup (V(\Gamma) - \{v_J\}) = V(J) \cup (V(H/J) - \{v_J\}) = V(H)$, Γ_1 is spanning. Pick an edge $E \in \mathcal{E}(H)$. If $E \cap V(J) \neq \emptyset$, then $E \cap PV(\Gamma_1) \neq \emptyset$; otherwise, $im(E) = E$, then $E \cap (PV(\Gamma_1) = E \cap [V(J) \cup (PV(\Gamma) - \{v_J\})] = E \cap (PV(\Gamma) - \{v_J\}) = im(E) \cap (PV(\Gamma) - \{v_J\}) \neq \emptyset$ as Γ is dominating. Thus, Γ_1 is dominating spanning a closed trail of H and then H is supereulerian.

In particular, if $PV(\Gamma) = V(H/J)$, then $PV(\Gamma_1) = V(J) \cup (PV(\Gamma) - \{v_J\}) = V(J) \cup (V(H/J) - \{v_J\}) = V(H)$. This implies that H is pivot-supereulerian.

Case 2. $r(J) \geq 3$.

As J is 2-partition-connected, by Theorem 3.3(ii) and Theorem 3.4, J has 2 edge-disjoint spanning hypertrees T_1 and T_2. By Corollary 3.7, there exists an injection $g : \mathcal{E}(H) \to 2^{\mathcal{E}(H)}$ satisfying that $g|_{\mathcal{E}(T_i)}$ is a pc-mapping of T_i for each i, $V(g(H)) = V(H)$ where $g(H) = g(\mathcal{E}(H))$, and for each $E \in \mathcal{E}(H), g(E) \subseteq E$. By Proposition 3.8, $g(T_i)$ is a tree with $V(g(T_i)) = V(J)$ for each $i = 1, 2$. Let $H_1 = g(T_1) \cup g(T_2) \cup (H - \mathcal{E}(J))$. Then, $V(H_1) = V(H)$. Since $H_1/(g(T_1) \cup g(T_2)) \cong H/J$ and $r(g(T_1) \cup g(T_2)) = 2, H_1$ is supereulerian by Case 1. Let L be a dominating spanning closed trail of H_1. Then $V(J) \subset PV(L)$. As H_1 is a spanning sub-hypergraph of $g(H)$, to show that $g(H)$ is supereulerian, it suffices to prove that for each edge $E \in \mathcal{E}(g(H)) - \mathcal{E}(H_1), E \cap PV(L) \neq \emptyset$. Pick $E \in \mathcal{E}(g(H)) - \mathcal{E}(H_1)$. Then $E \subseteq V(J)$, and so $E \cap PV(L) \neq \emptyset$ since $V(J) \subset PV(L)$. Therefore, $g(H)$ is supereulerian and so, by Corollary 3.7, H is supereulerian.

In particular, if H/J is pivot-supereulerian, H_1 is pivot-supereulerian by Case 1. Let L_1 be a pivot-spanning closed trail of H_1. As $PV(L_1) = V(H_1) = V(H)$, H is pivot-supereulerian. □

5 | REMARKS

By Theorem 1.2, the line graph of a supereulerian hypergraph is always hamiltonian. Let J be a graph. A graph G is *J-free* if G does not have an induced subgraph isomorphic to J. Thomassen [14] conjectured that every 4-connected line graph is hamiltonian. Matthews and Sumner [12] also conjectured that every 4-connected $K_{1,3}$-free graph is hamiltonian. Chen and Schelp extended the conjecture of Matthews and Sumner in the following.

Conjecture 5.1 (Chen and Schelp [5, Conjecture 2]). *Let $r \geq 2$ be an integer. Every $2r$-connected $K_{1,r+1}$-free graph of order $n \geq 3$ is hamiltonian.*

When $r = 2$, Conjecture 5.1 is exactly Matthews–Sumner Conjecture, and Ryjáček in [13] proved that it is equivalent to Thomassen Conjecture. It is known that if H is a hypergraph with rank r, then $L(H)$ is a $K_{1,r+1}$-free graph. The following is a weaker form of Conjecture 5.1 which is also of interest on its own.

Conjecture 5.2. *Let $r \geq 2$ be an integer.*

(i) *There is an integer $\varphi(r)$ such that every $\varphi(r)$-connected line graph of a rank r hypergraph is hamiltonian.*
(ii) *Furthermore, we conjecture that $\varphi(r) = 2r$.*

Thomassen [14, Conjecture 2] conjectured that $\varphi(2) = 4$, which motivates Conjecture 5.2(ii). While Ryjáček [13] indicated that Conjectures 5.1 and 5.2 are equivalent when $r = 2$, it is currently not known whether such equivalence exists for large values of r.

Recently, the class of line graphs of hypergraphs of rank 3 has been investigated by Li et al. in [11]. They obtained the equivalent versions of Thomassen conjecture in [14] for line graphs of hypergraphs of rank 3. A graph G is *Hamilton-connected* if G has a hamiltonian (u, v)-path for any $u, v \in V(G)$. A cycle C in a graph G is called a *Tutte cycle* if each component of $G - E(C)$ has at most three neighbors on C.

Conjecture 5.3 (Li et al. [11, Conjectures 1–4]).

(i) *every* 2*-connected line graph of a rank* 3 *hypergraph has a Tutte maximal cycle containing any two prescribed vertices.*
(ii) *every* 3*-connected line graph of a rank* 3 *hypergraph has a Tutte maximal cycle containing any three prescribed vertices.*
(iii) *every connected line graph of a rank* 3 *hypergraph has a Tutte maximal* (u, v)*-path two vertices* u, v.
(iv) *every* 4*-connected line graph of a rank* 3 *hypergraph is Hamilton-connected.*

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