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On r-hued list coloring of $K_4(7)$-minor free graphs[☆]

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ABSTRACT

For a given list assignment L of a graph G, an (L, r)-**coloring** of G is a proper coloring c such that for any vertex v with degree $d(v)$, v is adjacent to vertices of at least $\min\{d(v), r\}$ different color with $c(v) \in L(v)$. The **r-hued list chromatic number** of G, denoted as $\chi_{L,r}(G)$, is the least integer k, such that for any $v \in V(G)$ and every list assignment L with $|L(v)| = k$, G has an (L, r)-coloring. Let $K(r) = r + 3$ if $2 \le r \le 3$, $K(r) = \lfloor 3r/2 \rfloor + 1$ if $r \ge 4$. In Song et al. (2014), it is proved that if G is a K_4-minor-free graph, then $\chi_{L,r}(G) \le K(r) + 1$. Let $K_4(n)$ be the set of all subdivisions of K_4 on n vertices. Utilizing the decompositions by Chen et al for $K_4(7)$-minor free graphs in Chen et al. (2020), we prove that if G is a $K_4(7)$-minor free graph, then $\chi_{L,r}(G) \le K(r) + 1$.

1. Introduction

All graphs considered in this paper are simple and finite. Undefined terminologies and notion are referred to [4]. As in [4], $V(G)$, $E(G)$, $\Delta(G)$ and $\chi(G)$ denoted the vertex set, the edge set, the maximum degree and the chromatic number of a graph G. For $v \in V(G)$, let $N_G(v)$ denote the set of vertices adjacent to v in G, and $d_G(v) = |N_G(v)|$. A list of a graph G is an assignment $L : V(G) \to 2^{\mathbb{N}}$ that assigns every $v \in V(G)$ a list of colors $L(v)$ available at v. If L is a list of G, and H is a vertex induced subgraph of G, then L_H is a restriction of L to H. For an integer $k \in \mathbb{N}$, a list L of a graph G is a k-**list** if $|L(v)| = k$ for any $v \in V(G)$. Let r be an integer, for a given assignment $L : V(G) \to 2^{\mathbb{N}}$ in a graph G, an (L, r)-**coloring** c is a mapping $c : V(G) \to \mathbb{N}$ satisfying the following conditions.

$(C1) : c(u) \ne c(v)$ for every edge $uv \in E(G)$;

$(C2) : |c(N_G(v))| \ge \min\{d_G(v), r\}$ for any $v \in V(G)$;

$(C3) : c(v) \in L(v)$, for every $v \in V(G)$.

For a fixed integer $r > 0$, the r-**hued list chromatic number** of G, denoted by $\chi_{L,r}(G)$, is the smallest integer k, such that for any $v \in V(G)$ and every k-list L of G, G has an (L, r)-coloring. If for every $v \in V(G)$, we have $L(v) = \{1, 2, 3, \ldots, k\}$, then an (L, r)-coloring of G is a (k, r)-**coloring** of G. Accordingly, the r-**hued chromatic number** of G, denoted by $\chi_r(G)$, is the smallest integer k such that G has a (k, r)-coloring. In particular, when $r = 1$, it follows from definition that $\chi_1(G) = \chi(G)$, the chromatic number of a graph G. Thus r-hued coloring and r-hued list coloring are generalization of the vertex coloring of graphs.

The notion of r-hued coloring was first introduced in [12,15]. When $r = 2$, $\chi_2(G)$ is often called *the dynamic chromatic number* of G. In [5], Brooks proved a popular theorem on graph colorings states that a connected graph G satisfies $\chi(G) \le \Delta(G) + 1$, where the equality holds if and only if G is an odd cycle or a complete graph. Earlier Brooks type upper bounds for r-hued colorings can be found in [11,12,15], among others. Upper bounds of the r-hued list chromatic number for generic graphs have also been studied.

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Theorem 1.1. *Let G be a connected graph.*

(i) *(Kim et al. [10]) If G is a planar graph, then $\chi_{L,2}(G) \le 5$.*

(ii) *(Akbari et al. [1]) If $G \ne C_5$ and $\Delta(G) \le 3$, then $\chi_{L,2}(G) \le 4$.*

(iii) *(Akbari et al. [1]) If $\Delta(G) \ge 4$, then $\chi_{L,2}(G) \leqslant \Delta(G) + 1$.*

It is natural to consider upper bounds of the r-hued chromatic number and the r-hued list chromatic number of a planar graph G. For any planar graph G, it is proved that $\chi_2(G) \le 5$ in [6] without using the 4-Color Theorem. Utilizing the 4-Color Theorem [2,3,16], Kim et al. in [10] showed that 5-cycle is the only planar graph with 2-hued chromatic number being 5, which was conjectured in [6]. More recently, Loeb et al. in [14] proved that $\chi_3(G) \le 10$. In [18,19], Song et al. proved that any planar graph G with girth at least 6 satisfies $\chi_r(G) \le r + 5$ when $r \ge 3$, and for all planar graph G with $r \ge 8$, $\chi_r(G) \le 2r + 16$. For further literature on r-hued coloring and r-hued list coloring of planar graphs, see [8]. In 1977, Wegner [20] posed the following conjecture.

Conjecture 1.2 (*Wegner. [20]*)**.** *If G is a planar graph, then*

$$\chi_{\Delta(G)}(G) \le \begin{cases} \Delta(G) + 5, & \text{if } 4 \le \Delta(G) \le 7; \\ \lfloor 3\Delta(G)/2 \rfloor + 1, & \text{if } \Delta(G) \ge 8. \end{cases}$$

This conjecture remains open as of today. For a graph H, a graph G has an H **minor** if H can be obtained from a subgraph of G by contracting edges. A graph G is called **H-minor free** if G does not have H as a minor. For a given collection $\mathcal{K}$ of graphs, define $EX(\mathcal{K}) = \{G : G$ does not have a minor isomorphic to a member in $\mathcal{K}\}$, and let

$$K(r) = \begin{cases} r + 3, & \text{if } 1 \le r \le 3; \\ \lfloor 3r/2 \rfloor + 1, & \text{if } r \ge 4. \end{cases} \tag{1}$$

There have been quite a few efforts made towards Conjecture 1.2. Among them are the following.

Theorem 1.3. *Let $G \in EX(K_4)$ be a graph and let $r \ge 2$ be an integer. Then each of the following holds.*

(i) *(Lih et al. [13]) $\chi_{\Delta(G)}(G) \le K(\Delta(G))$.*

(ii) *(Hetherington and Woodall [9]) $\chi_{L,\Delta}(G) \le K(\Delta)$.*

(iii) *(Song et al. [17]) Both $\chi_r(G) \le K(r)$ and $\chi_{L,r}(G) \le K(r) + 1$.*

Let H be a graph. An edge $e \in E(H)$ is said to be **subdivided** when it is deleted and replaced by a path of length two connecting its end vertices. A **subdivision** of H is a graph obtained from H by a (possibly empty) sequence of edge subdivisions. If a graph contains subgraph J isomorphic to a subdivision of H, we call J an H**-subdivision**. Thus, by definition, if $\Delta(H) \le 3$, then G contains an H-minor if and only if G contains an H-subdivision. For an integer $n \ge 4$, define $K_4(n)$ to be the collection of all non-isomorphic subdivisions of K_4 on n vertices. Thus $K_4(4) = \{K_4\}$ and there is only one graph in $K_4(5)$. When it is understood in the context, we sometimes use $K_4(4)$ and $K_4(5)$ to represent the only member in the corresponding collection.

By definition, for each $n \ge 4$, we have

$$EX(K_4) \subseteq \cdots \subseteq EX(K_4(n)) \subseteq EX(K_4(n + 1)) \subseteq \cdots$$

and for each fixed integer $n \ge 4$, $EX(K_4(n))$ contains all graphs with order less than n. Hence $\cup_{n=4}^{\infty} EX(K_4(n))$ contains all graphs. Chen et al. in [7] initiated the study of upper bounds of $\chi_r(G)$ for graphs $G \in EX(K_4(7))$ and prove the following theorem.

Theorem 1.4 (*Chen et al. Theorem 1.5 of [7]*)**.** *Let G be a graph and $r \ge 2$ be an integer. If $G \in EX(K_4(7))$, and if G has no block isomorphic to K_6, then $\chi_r(G) \le K(r)$.*

Our current study is motivated by Theorems 1.3 and 1.4. We investigate the upper bound of $\chi_{L,r}(G)$ for $K_4(7)$-minor free graphs G. The following is the main result obtained in this research.

Theorem 1.5. *Let G be a 2-connected graph and $r \ge 2$ be an integer. If $G \in EX(K_4(7))$, then $\chi_{L,r}(G) \le K(r) + 1$.*

2. Preliminaries

For an integer $n \in \{5, 6, 7\}$, a number of results on the decompositions of $K_4(n)$-minor free graphs have been developed in [7]. We present the related definitions, and results here for a complete understanding of the article.

Throughout the rest of this paper, by $H \subseteq G$ we mean that G contains a subgraph isomorphic to H, and when there is no confusion arises, we also view that H is a subgraph of G. For a graph G and a collection $\mathcal{F}$ of subgraphs of G, we define

$$[\mathcal{F}, G] = \{H : \text{ for some } F \in \mathcal{F}, F \subseteq H \subseteq G\}.$$

As in [4], K_n and $K_{m,n}$ denote the complete graph of order n and complete bipartite graph with partite set sizes m and n, respectively. Throughout this paper, we take the convention to also use K_n or $K_{m,n}$ to denote a specified copy of K_n or $K_{m,n}$, respectively. We follow [4] to define the union of graphs. Let G and G' be two graphs. The **union** of G and G', denoted by $G \cup G'$, has vertex set $V(G) \cup V(G')$ and edge set $E(G) \cup E(G')$.

Definition 2.1. Let $k \geq 1$ be an integer, $G, G', H_1, H_2, \dots, H_k$ be vertex disjoint simple graphs.

($O1$) Suppose that $u \in V(G)$ and $u' \in V(G')$. Define $G \oplus_1 G'$ to be the simple graph obtained from $G \cup G'$ by identifying u with u' to form a new vertex, which is still denoted by u. We sometimes write $G \oplus_u G'$ for $G \oplus_1 G'$ to emphasize the vertex u.

($O2$) Suppose that $u, v \in V(G)$ and $u', v' \in V(G')$. Define $G \oplus_{u,v} G'$ to be the simple graph obtained from $G \cup G'$ by identifying u with u' to form a new vertex (again denoted by u), and v with v' to form a new vertex (again denoted by v), respectively. The vertices u, v are called the **base vertices** of $G \oplus_{u,v} G'$. Thus if either $uv \in E(G)$ or $u'v' \in E(G')$, then the edge $uv \in E(G \oplus_{u,v} G')$. If u, v are understood or not to be emphasized, we often use $G \oplus_2 G'$ for $G \oplus_{u,v} G'$.

($O3$) For each j with $1 \leq j \leq k$, assuming that $G \oplus_2 (\cup_{i=1}^{j} H_i)$ is obtained, we define $G \oplus_2 (\cup_{i=1}^{j+1} H_i) = (G \oplus_2 (\cup_{i=1}^{j} H_i)) \oplus_2 H_{j+1}$ in such a way that the base vertices of $G \oplus_2 (\cup_{i=1}^{j+1} H_i)$ are in $V(G)$, and for each H_i, the base vertices may be different.

We often also take the convention to assume that in ($O1$), $V(G) \cap V(G') = \{u\}$, and in ($O2$), $V(G) \cap V(G') = \{u, v\}$. For an integer $j \geq 1$, define $D_j(G) = \{v \in V(G) : d_G(v) = j\}$. We now can use the operations in Definition 2.1 to define some related constructions.

Definition 2.2. Let $t \geq 1$ be an integer, and $k_i \geq 0$ be an integer for $1 \leq i \leq t$.

(i) Let $K_{2,t}$ be a complete bipartite graph with w_1, w_2 being the two nonadjacent vertices in $K_{2,t}$ of degree t, and $V(K_{2,t}) - \{w_1, w_2\} = \{u_1, u_2, \dots, u_t\}$. Hence if $t \neq 2$, then $D_2(K_{2,t}) = \{u_1, u_2, \dots, u_t\}$. The vertices w_1, w_2 are called the **special vertices** of $K_{2,t}$. Define $K'_{2,t}$ to be the graph obtained by adding a matching $u_1u_2, u_3u_4, \dots, u_{t-2}u_{t-1}$ if t is odd, $u_1u_2, u_3u_4, \dots, u_{t-1}u_t$ if t is even, among the non special vertices in $V(K_{2,t}) - \{w_1, w_2\}$. The special vertices of $K'_{2,t}$ are the special vertices of the related $K_{2,t}$.

(ii) Let $T = (k_1, k_2, \dots, k_t)$ be a t-tuple of non-negative integers. Let $J, J_1, \dots, J_t$ be graphs such that $J \cong K_{2,t}$ with special vertices w_1 and w_2, and for $1 \leq i \leq t$, $J_i \cong K_{2,k_i}$. Define $SK_{2,t,T}$ to be the family of graphs each of which is isomorphic to $J \oplus_2 (\cup_{i=1}^{t} J_i)$ in such a way that the special vertices of $SK_{2,t,T}$ are special vertices of J, for each j with $1 \leq j \leq t$, the base vertices x_j, y_j in $J \oplus_2 (\cup_{i=1}^{j} J_i)$ are special vertices of each J_j, and $e_j = x_jy_j$ is an edge $e_j \in E(J)$ such that all the edges $e_1, e_2, \dots, e_t$ are mutually distinct and such that for distinct i and j, any vertex incident with both e_i and e_j must be in $\{w_1, w_2\}$.

As in Definition 2.2(ii), each $e_j = x_jy_j$ can be any one of the two edges in a path joining the two special vertices of $J \cong K_{2,t}$, $SK_{2,t,T}$ in general contains more than one graph. For notational convention, we often also use $SK_{2,t,T}$ to denote a typical number of the family.

Let t denoted a positive integer, $K_{2,t}$ be given with v_1, v_2 being the special vertices of $K_{2,t}$, and K_4 be given with $V(K_4) = \{v_1, v_2, v_3, v_4\}$ such that $V(K_4) \cap V(K_{2,t}) = \{v_1, v_2\}$. Define

$$\mathcal{L} = \cup_{t\geq 1}\{K_4 \oplus_{v_1v_2} K_{2,t} - v_1v_2, K_4 \oplus_{v_1v_2} K_{2,t}\}. \tag{2}$$

Thus by definition, $K_4 \oplus_{v_1v_2} K_{2,1} - v_1v_2$ is the only graph in $K_4(5)$.

Definition 2.3. Let n, t_1, t_2, t_3 be non-negative integers with $n \geq 4$, $T = (k_1, k_2, \dots, k_{t_3})$ be a t_3-tuple of positive integers. In the definitions below, we always assume that $F_1 \cong K_{2,t_1}$, $F_2 \cong K'_{2,t_1}$, and $F_3 \cong SK_{2,t_3,T}$ are graphs with the special vertices of F_1, F_2, F_3 being $\{v_1, v_2\}$. Let $F'_1 \cong K_{2,t_2}$ be a graph with special vertices $\{v_3, v_l\}$ with $l \in \{1, 4\}$, and any graph K_n here with $V(K_n) = \{v_1, v_2, \dots, v_n\}$.

(i) Define $L_1 := L_1(t_1, t_2) = K_4 \oplus_{v_1,v_2} F_1 \oplus_{v_3,v_l} F'_1$, $N_1 := N_1(t_1, t_2) = L_1(t_1, t_2) - \{v_1v_2, v_3v_l\}$, and $\mathcal{L}_1 = \{G \in [N_1, L_1] : |V(G)| \geq 6\}$.

(ii) Define $L_2 := L_2(t_1, t_3, T) = K_4 \oplus_{v_1,v_2} F_2 \oplus_{v_1,v_2} F_3$, where $T = (k_1, k_2, \dots, k_{t_3})$ is a t_3-tuple with $k_1 \geq k_2 \geq \cdots \geq k_{t_3} \geq 0$, $N_2 := N_2(t_1, t_3, T) = K_4 \oplus_{v_1,v_2} F_1 \oplus_{v_1,v_2} F_3 - \{v_1v_2\} - \cup_{i=1}^{t_3} e_i$, where the e_i's defined in Definition 2.2. Define $\mathcal{L}_2 = \{G \in [N_2, L_2] : |V(G)| \geq 6\}$.

(iii) Define $L_3 := L_3(t_1) = K_5 \oplus_{v_1,v_2} F_1$, $N_3 := N_3(t_1) = L_3 - \{v_1v_2, v_1v_3, v_2v_5\}$, and $\mathcal{L}_3 = \{G \in [N_3, L_3] : |V(G)| \geq 6\}$.

Theorem 2.4 (*Chen et al. [7]*)**.** *Let G be a 2-connected simple graph. Then each of the following holds.*

(i) *$G \in EX(K_4(5))$ if and only if $G \in \{K_4\} \cup EX(K_4)$.*

(ii) *$G \in EX(K_4(6))$ if and only if $G \in EX(K_4(5)) \cup \mathcal{L} \cup [K_4(5), K_5]$.*

(iii) *$G \in EX(K_4(7))$ if and only if $G \in EX(K_4(6)) \cup \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \cup [K_4(6), K_6]$.*

3. Proof of Theorem 1.3

Throughout this section, let r be an integer with $r \geq 2$. Recall that $K(r)$ is defined in (1). Next, we shall show that if $G \in EX(K_4(7))$ and be a 2-connected graph, then for any $r \geq 2$, $\chi_{L,r}(G) \leq K(r) + 1$. In the argument below, we often adopt the notation in Definitions 2.1–2.3 for convenience.

Lemma 3.1. *Let $s \le r-2$ be an integer and let $L_1, L_2, \ldots, L_s$ be list of color such that each i with $1 \le i \le s$, $|L_i| \ge r-2$. Then there exists an injective mapping $\phi : \{1, 2, \ldots, s\} \to \cup_{i=1}^{s} L_i$ such that each j with $1 \le j \le s$, $\phi(j) \in L_j$.*

Proof. Let $G[L, C]$ be a bipartite graph with bipartition $L = \{L_1, L_2, \ldots, L_s\}$ and $C = \cup_{i=1}^{s} L_i = \{c_1, c_2, \ldots, c_t\}$. Thus $t \ge |L_1| \ge r-2 \ge s$. Define an edge (L_i, c_j) in G if and only if $c_j \in L_i$. We need to show that G has a matching covering all vertices in L. For any nonempty subset $S \subseteq L$, without loss of generality, we assume that $L_i \subseteq S$, then $|N(S)| \ge |L_i| \ge r-2 \ge s = |L| \ge |S|$, and so by Hall's Theorem (see Theorem 16.4 of [4]), $G[L, C]$ contains a matching covering all vertices in L. This completes the proof of Lemma 3.1. □

Lemma 3.2. *Let v_1, v_2 be the special vertices of $K_{2,t}$ with $t > 0$. For each graph G given below, let L be an $(r+3)$-list of G. Then each of the following holds.*

(*i*) *Let $G = K_4 \oplus_{v_1,v_2} K_{2,t} - v_1v_2$. There is an (L, r)-coloring of G.*
(*ii*) *Let $G = K_4 \oplus_{v_1,v_2} F_2$. There is an (L, r)-coloring of G.*
(*iii*) *Let $G = K_5 \oplus_{v_1,v_2} K_{2,t}$. There is an (L, r)-coloring of G.*

Proof. As (*i*), (*ii*) and (*iii*) can be proved in a similar way, we only prove (*i*). Let $\{v_1, v_2, v_3, v_4\}$ denote the vertices of the K_4 as in Definition 2.3, and let $D_2(G) = \{u_1, u_2, \ldots, u_t\}$. For $r = 2$, we define $c : V(G) \to \mathbb{N}$ in the following steps. As L is a 5-list, for $i \in \{1, 2, 3, 4\}$, we assign $c(v_i) \in L(v_i)$ such that $c(v_1)$, $c(v_2)$, $c(v_3)$ and $c(v_4)$ are distinct colors. To color the vertices $\{u_1, u_2, \ldots, u_t\}$, for each $u \in D_2(G)$ choose $c(u) \in L(u) - c(N_G(u))$. Since $|L(u)| = 5$, and $|c(N_G(u))| \le 2$, such a $c(u)$ can always be found. By definition, the coloring c defined above is an $(L, 2)$-coloring of G.

Suppose that $r \ge 3$. We construct a coloring $c : V(G) \to \mathbb{N}$ as following. If $t \le r-2$, by Lemma 3.1, an (L, r)-coloring $c : V(G) \to \mathbb{N}$ of G exists. We assume that $t \ge r-1$. Applying Lemma 3.1 to $G - \{u_{r-1}, \ldots, u_t\}$, $G - \{u_{r-1}, \ldots, u_t\}$ has an $(L_{G-\{u_{r-1},\ldots,u_t\}}, r)$-coloring c. For each i with $r-1 \le i \le t$, as $|L(u_i)| = r+3$ and $|c(N_G(u_i))| \le 2$. We can always define $c(u_i) \in L(u_i) - c(N_G(u_i))$. Thus, the extended c is a proper coloring. Since $G - \{u_{r-1}, \ldots, u_t\}$ has an $(L_{G-\{u_{r-1},\ldots,u_t\}}, r)$-coloring c, and the choice of $c(u_i)$ for $r-1 \le i \le t$, the extended c is an (L, r)-coloring of G. □

Proposition 3.3. *Let $r \ge 2$ be an integer and let G be a 2-connected graph. Each of the following holds.*

(*i*) *If $G \in EX(K_4(6))$, then $\chi_{L,r}(G) \le K(r) + 1$.*
(*ii*) *If $G \in [K_4(6), K_6]$, then $\chi_{L,r}(G) \le K(r) + 1$.*

Proof. By Theorem 2.4, $EX(K_4(5)) = \{K_4\} \cup EX(K_4)$. By Theorem 1.3, $\chi_{L,r}(K_4) = 4 \le K(r) + 1$. It follows that for any $G \in EX(K_4(5))$, $\chi_{L,r}(G) \le K(r) + 1$. By Theorem 2.4, $EX(K_4(6)) - EX(K_4(5)) \subseteq \mathcal{L} \cup [K_4(5), K_5]$, where $\mathcal{L}$ is defined in (2). Since for any $r \ge 2$, $K(r) \ge 5$, to prove (i), it suffices to show that $\chi_{L,r}(G) \le K(r) + 1$ for any $G \in \mathcal{L}$ with $|V(G)| \ge 6$.

Let $G \in \mathcal{L}$ be a graph with $|V(G)| \ge 6$. By (2), there exists an integer s such that $G \in \{K_4 \oplus_{v_1,v_2} K_{2,s} - v_1v_2, K_4 \oplus_{v_1,v_2} K_{2,s}\}$. $D_2(G) = \{u_1, u_2, \ldots, u_s\}$. Let L be an $(r+3)$-list of G. By Lemma 3.2(*i*), the graph G has an (L, r)-coloring $c : V(G) \to \mathbb{N}$, and so $\chi_{L,r}(G) \le r + 3 \le K(r) + 1$. This proves Proposition 3.3(*i*).

To justify (ii), we observe that for any $G \in [K_4(6), K_6]$, $\chi_{L,r}(G) \le |V(G)| = 6 \le K(r) + 1$. This completes the proof of the proposition. □

By Theorem 2.4, $EX(K_4(7)) - (EX(K_4(6)) \cup [K_4(6), K_6]) \subseteq \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$. Thus by Proposition 3.3, it suffices to assume that $G \in \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$ to prove that $\chi_{L,r}(G) \le K(r) + 1$.

Lemma 3.4. *Let t_1, t_2 be non-negative integers. If $G = K_4 \oplus_2 K_{2,t_1} \oplus_2 K_{2,t_2}$, then $\chi_{L,r}(G) \le r + 3$.*

Proof. It suffices to show that for any $(r+3)$-list L of G, we can always find an (L, r)-coloring of G. Let $\{v_1, v_2, v_3, v_4\}$ denote the vertices of the K_4 as in Definition 2.3. We may assume that $G = K_4 \oplus_{v_1,v_2} K_{2,t_1} \oplus_{v_3,v_1} K_{2,t_2}$ with v_1 and v_2 being the special vertices of K_{2,t_1}, and with v_1 and v_3 being the special vertices of K_{2,t_2}. Thus by definition, $v_1, v_2 \in N_G(v_3)$. Denoted $D_2(K_{2,t_1}) = \{u_1, u_2, \ldots, u_{t_1}\}$ and $D_2(K_{2,t_2}) = \{u_{t_1+1}, u_{t_1+2}, \ldots, u_{t_1+t_2}\}$.

Since L is an $(r+3)$-list of G, for any $v \in V(G)$, $|L(v)| = r+3$. Let $G' = G - \{u_{t_1+1}, u_{t_1+2}, \ldots, u_{t_1+t_2}\}$. Then by Lemma 3.2(*i*), there exists a coloring $c' : V(G') \to \mathbb{N}$ such that c' is an $(L_{G'}, r)$-coloring of G'. Let

$$C(v_3) = c'(N_{G'}(v_3)), d = |C(v_3)|. \tag{3}$$

Since $d_{G'}(v_3) = 3$ and since c' is an $(L_{G'}, r)$-coloring, we have $d \ge \min\{r, 3\}$. To extend c' to an (L, r)-coloring c of G, for each vertex $z \in V(G')$, we set $c(z) = c'(z)$. Then we need to color the vertices in $\{u_{t_1+1}, u_{t_1+2}, \ldots, u_{t_1+t_2}\}$ so that c satisfies (C1), (C2) and (C3).

Case 1. $t_1 \ge r-1$. Then as v_1 and v_2 are the special vertices of K_{2,t_1} and as $t_1 \ge r-1$, we have $|c'(N_G(v_1))| \ge r$ and $|c'(N_G(v_2))| \ge r$. By (3), if $d \ge r$, then we also have $|c'(N_G(v_3))| \ge r$, and so we can pick a color subset $C' \subseteq C(v_3)$ with $|C'| = r$. Since L is an $(r+3)$-list of G, we have $|L(u_{t_1+i}) - (C' \cup \{c'(v_1), c'(v_3)\})| \ge (r+3) - (r+2) > 0$. Hence for each i with $1 \le i \le t_2$, it is possible to choose

$$c(u_{t_1+i}) \in L(u_{t_1+i}) - (C' \cup \{c'(v_1), c'(v_3)\}). \tag{4}$$

It follows by (4) and $\min\{|c'(N_G(v_1))|, |c'(N_G(v_2))|, |c'(N_G(v_3))|\} \geq r$ that c is an (L, r)-coloring of G. Therefore, we may assume that $d < r$. As L is an $(r+3)$-list, for any i with $1 \leq i \leq t_2$,

$$|L(u_{t_1+i}) - (C(v_3) \cup \{c'(v_3)\})| \geq (r+3) - (d+1) > 0.$$

We define $c(u_{t_1+1})$ by choosing $c(u_{t_1+1}) \in L(u_{t_1+1}) - (C(v_3) \cup \{c'(v_3)\})$. Assume that inductively, we have defined $c(u_{t_1+i})$ with $1 \leq i \leq s$ for some $s < \min\{r-d, t_2\}$ in such a way that

$$c(u_{t_1+i}) \in L(u_{t_1+i}) - (C(v_3) \cup \{c'(v_3)\} \cup \{c(u_{t_1+j}) : 1 \leq j \leq i-1\}). \tag{5}$$

Then as L is an $(r+3)$-list, we have

$$|L(u_{t_1+s+1}) - (C(v_3) \cup \{c'(v_3)\} \cup \{c(u_{t_1+i}) : 1 \leq i \leq s\})| \geq (r+3) - (d+s+1) > 0,$$

and so it is possible to define $c(u_{t_1+s+1})$ by choosing

$$c(u_{t_1+s+1}) \in L(u_{t_1+s+1}) - (C(v_3) \cup \{c'(v_3)\} \cup \{c(u_{t_1+i}) : 1 \leq i \leq s\}).$$

Thus after we have colored all vertices in $\{u_{t_1+i} : 1 \leq i \leq \min\{r-d, t_2\}\}$, then (5) hold for all i with $1 \leq i \leq \min\{r-d, t_2\}$. We conclude by (5) that there are $\min\{r, d_G(v_3)\}$ differently colored vertices in $N_G(v_3)$. If $t_2 \leq r-d$, then the mapping c is already an (L, r)-coloring of G.

Assume that $t_2 > r-d$. For i with $r-d < i \leq t_2$, set $c(u_{t_1+i}) \in L(u_{t_1+i}) - \{c'(v_1), c'(v_3)\}$. By (5), the extended coloring c satisfies (C1), (C2) and (C3) as $|c(N_G(v_3))| \geq \min\{d_G(v_3), r\}$ and for each $u \in \{u_{t_1+1}, u_{t_1+2}, \ldots, u_{t_1+t_2}\}$, $|c(N_G(u))| = d_G(u) = 2$. Thus in Case 1, c' can be extended to c, which is an (L, r)-coloring of G.

Case 2. $t_1 < r-1$. Then as c' is an $(L_{G'}, r)$-coloring of G' and $t_1 < r-1$. We have $|c'(N_G(v_2))| \geq \min\{d_G(v_2), r\}$. Let

$$M(v_1) = c'(N_{G'}(v_1)) \text{ and } m = |M(v_1)|.$$

If $m \geq r$, then we also have $|c'(N_G(v_1))| \geq r$, and so we can color each vertex $u \in \{u_{t_1+1}, u_{t_1+2}, \ldots, u_{t_1+t_2}\}$ the same way that we did in Case 1. Therefore, we may assume that $m < r$. As L is an $(r+3)$-list, for any i with $1 \leq i \leq t_2$,

$$|L(u_{t_1+i}) - (M(v_1) \cup \{c'(v_1)\})| \geq (r+3) - (m+1) > 0.$$

We define $c(u_{t_1+1})$ by choosing $c(u_{t_1+1}) \in L(u_{t_1+1}) - (M(v_1) \cup \{c'(v_1)\})$. Assume that inductively, we have defined $c(u_{t_1+i})$ with $1 \leq i \leq h$ for some $h < \min\{r-m, t_2\}$ in such a way that

$$c(u_{t_1+i}) \in L(u_{t_1+i}) - (M(v_1) \cup \{c'(v_1)\} \cup \{c(u_{t_1+j}) : 1 \leq j \leq i-1\}). \tag{6}$$

Then as L is an $(r+3)$-list, we have

$$|L(u_{t_1+h+1}) - (M(v_1) \cup \{c'(v_1)\} \cup \{c(u_{t_1+i}) : 1 \leq i \leq h\})| \geq (r+3) - (m+h+1) > 0,$$

and so it is possible to define $c(u_{t_1+h+1})$ by choosing

$$c(u_{t_1+h+1}) \in L(u_{t_1+h+1}) - (M(v_1) \cup \{c'(v_1)\} \cup \{c(u_{t_1+i}) : 1 \leq i \leq h\}).$$

Thus after we have colored all vertices in $\{u_{t_1+i} : 1 \leq i \leq \min\{r-m, t_2\}\}$, then for any i with $1 \leq i \leq \min\{r-m, t_2\}$ (6) holds. We conclude by (6) that there are $\min\{r, d_G(v_1)\}$ differently colored vertices in $N_G(v_1)$. If $t_2 \leq r-m$, then the mapping c is already an (L, r)-coloring of G.

Assume that $t_2 > r-m$. We have defined $c(u_{t_1+j})$ with $1 \leq j \leq i-1$ and $r-m < i \leq s$ for some $s < \min\{r-d, t_2\}$, then as L is an $(r+3)$-list, we have

$$|L(u_{t_1+i}) - (C(v_3) \cup \{c'(v_3)\} \cup \{c(u_{t_1+j}) : 1 \leq j \leq i-1\})| \geq (r+3) - (d+1+s-r+m) > 0,$$

and so it is possible to define $c(u_{t_1+i})$ by choosing

$$c(u_{t_1+i}) \in L(u_{t_1+i}) - (C(v_3) \cup \{c'(v_3)\} \cup \{c(u_{t_1+j}) : 1 \leq j \leq i-1\}). \tag{7}$$

Thus after we have colored all vertices in $\{u_{t_1+i} : r-m < i \leq \min\{r-d, t_2\}\}$, we conclude by (7) that there are $\min\{r, d_G(v_3)\}$ differently colored vertices in $N_G(v_3)$. If $t_2 \leq r-d$, then the mapping c is already an (L, r)-coloring of G.

Assume that $t_2 > r-d$. For i with $r-d < i \leq t_2$, set $c(u_{t_1+i}) \in L(u_{t_1+i}) - \{c'(v_1), c'(v_3)\}$. By (6) and (7), the extended coloring c satisfies (C1), (C2) and (C3) as $|c(N_G(v_1))| \geq \min\{d_G(v_1), r\}$, $|c(N_G(v_3))| \geq \min\{d_G(v_3), r\}$ and for each $u \in \{u_{t_1+1}, u_{t_1+2}, \ldots, u_{t_1+t_2}\}$, $|c(N_G(u))| = d_G(u) = 2$. Thus in Case 2, c' can be extended to c, which is an (L, r)-coloring of G.

Similarly, if $G = K_4 \oplus_{v_1,v_2} K_{2,t_1} \oplus_{v_3,v_4} K_{2,t_2}$ with v_1 and v_2 being the special vertices of K_{2,t_1}, and with v_3 and v_4 being the special vertices of K_{2,t_2}. We also have $\chi_{L,r}(G) \leq r+3$. $\square$

Proposition 3.5. *Let $r \geq 2$ be an integer and $G \in \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$ be a 2-connected graph. Then $\chi_{L,r}(G) \leq K(r)$.*

Proof. We continue adopting the notation in Definition 2.3 in the arguments. Thus for some $n \in \{4, 5\}$, the construction of G involves a complete graph K on n vertices. As in Definition 2.3, we let $V(K) = \{v_1, v_2, \ldots, v_n\}$.

Claim 1. *If $G \in \mathcal{L}_1$, then $\chi_{L,r}(G) \le r+3 \le K(r)$.*

As $G \in \mathcal{L}_1$, $K = K_4$. By Definition 2.3, there exist non-negative integers t_1 and t_2 such that G is spanned by $N_1(t_1, t_2)$ with possibly $v_1v_2, v_3v_l \in E(G)$. Denote $D_2(F_1) = \{u_1, u_2, \ldots, u_{t_1}\}$ and $D_2(F_1') = \{u_{t_1+1}, u_{t_1+2}, \ldots, u_{t_1+t_2}\}$. Let L be an $(r+3)$-list of G. By Lemma 3.4, there exists a coloring $c_1 : V(G) \to \mathbb{N}$ is an (L, r)-coloring of G, independent of whether the edge v_1v_2, v_3v_l are in $E(G)$ or not. Thus $\chi_{L,r}(G) \le r+3 \le K(r)$.

Claim 2. *If $G \in \mathcal{L}_3$, then $\chi_{L,r}(G) \le r+3 \le K(r)$.*

As $G \in \mathcal{L}_3$, $K = K_5$. By Definition 2.3, for an integer $t_1 \ge 1$, G is a spanning subgraph of $L_3 := L_3(t_1) = K_5 \oplus_{v_1,v_2} F_1$ where $F_1 = K_{2,t_1}$. Let $D_2(F) = \{u_1, u_2, \ldots, u_{t_1}\}$. Let L be an $(r+3)$-list of G. By Lemma 3.2(*iii*), there exists a coloring $c_2 : V(G) \to \mathbb{N}$ is an (L, r)-coloring of G, Thus $\chi_{L,r}(G) \le r+3 \le K(r)$.

Claim 3. *If $G \in \mathcal{L}_2$, then $\chi_{L,r}(G) \le r+3 \le K(r)$.*

As $G \in \mathcal{L}_2$, $K = K_4$. By Definition 2.3, every graph in $\mathcal{L}_2$ is a planar graph. Thus by Theorem 1.1, if $G \in \mathcal{L}_2$, then $\chi_{L,2}(G) \le 5 = K(2)$. Therefore, we assume that $r \ge 3$ and $|V(G)| \ge K(3)+1 = 7$, and continue using the notation in Definition 2.3. Let $N_2 := N_2(t_1, t_3, T) = K_4 \oplus_{v_1,v_2} F_1 \oplus_{v_1,v_2} F_3 - \{v_1v_2\} - \cup_{i=1}^{t_3} e_i$, and $L_2 := L_2(t_1, t_3, T) = K_4 \oplus_{v_1,v_2} F_2 \oplus_{v_1,v_2} F_3$ with $t_1 \ge 0$ and $t_3 \ge 0$, where $T = (k_1, k_2, \ldots, k_{t_3})$ with $k_1 \ge k_2 \ge \cdots \ge k_{t_3} \ge 0$. Let $G \in [N_2, L_2]$. As in Definition 2.3 we have $F_1 \cong K_{2,t_1}$, $F_2 \cong K'_{2,t_1}$, and $F_3 \cong SK_{2,t_3,T}$. For each $j \in \{1, 2, \ldots, t_3\}$, let x_j, y_j denote the special vertices of $J_j \cong K_{2,k_j}$ in Definition 2.2(*ii*) with $x_1 = x_2 = \cdots = x_f = v_1$, $y_{f+1} = y_{f+2} = \cdots = y_{t_3} = v_2$, and $D_2(J_j) = \{w_1^j, w_2^j, \ldots, w_{k_j}^j\}$. As $G \in [N_2, L_2]$, v_1v_2 may or may not be in $E(G)$. By Definition 2.3(ii), we may view F_1 as a spanning subgraph of F_2, and so some of the edges in $E(F_2) - E(F_1)$ may not be in G as well. Denote $V(K) = \{v_1, v_2, v_3, v_4\}$ and $V(F_1') = \{u_1, u_2, \ldots, u_{t_1}\}$, such that for some $t_1' \ge 0$ with $2t_1' \le t_1$, we have $\{u_{2i-1}u_{2i} : 1 \le i \le t_1'\} \subseteq E(G)$, and such that $2t_1'+1 \le j \le t_1$, $\{u_{2t'+1}, \ldots, u_{t_1}\}$ is an independent set.

Let L be an $(r+3)$-list of G. We shall construct an (L, r)-coloring $c_3 : V(G) \to \mathbb{N}$ of G in the following steps. Before the coloring, we let

$$W_1 = w_1^1, w_2^1, \ldots, w_{k_1}^1, w_1^2, w_2^2, \ldots, w_{k_2}^2, \ldots, w_1^f, w_2^f, \ldots, w_{k_f}^f,$$

$$W_2 = w_1^{f+1}, w_2^{f+1}, \ldots, w_{k_{f+1}}^{f+1}, w_1^{f+2}, w_2^{f+2}, \ldots, w_{k_{f+2}}^{f+2}, \ldots, w_1^{t_3}, w_2^{t_3}, \ldots, w_{k_{t_3}}^{t_3}. \tag{8}$$

be two sequences of vertices of G.

Step 1. Let $G_1 = G[V(K) \cup V(F_1')]$, and $L_1 = L|_{V(G_1)}$ be the restriction of L to $V(G_1)$. (See Fig. 1, where G_1 is spanned by a subgraph in the graph depicted in Fig. 1.) Since L is an $(r+3)$-list with $r \ge 2$, the vertices in $V(K)$ can be so colored that $c_3(v_i) \in L(v_i)$ with $|c_3(V(K))| = 4$. Thus $|c_3(N_G(v_i))| \ge 2$, for any $v_i \in V(K)$. As $r \ge 2$ and L is an $(r+3)$-list of G, we have $|L(u_i) - c_3(V(K))| = r-1 > 0$, and so the color $c_3(u_1) \in L(u_1) - c_3(V(K))$ can be chosen.

Let $N_0 = \min\{r-2, t_1\}$. Color $c_3(u_1) \in L(u_1) - c_3(V(K))$. For each $i = 2, 3, \ldots, N_0$, set

$$c_3(u_i) \in L(u_i) - (\{c_3(v_1), c_3(v_2)\} \cup \{c_3(u_j) : 1 \le j \le i-1\}). \tag{9}$$

If $t_1 \le r-2$, then the coloring of this step is done. If $t_1 > r-2$, then for any $i = N_0+1, \ldots, t_1$, set

$$c_3(u_i) \in L(u_i) - \{c_3(v_1), c_3(v_2), c_3(u_{i-1})\}. \tag{10}$$

By (9) and (10), c_3 is an (L_1, r)-coloring of G_1.

Step 2. In Step 2, we are to color the vertices in $\{y_1, y_2, \ldots, y_f\} \cup \{x_{f+1}, x_{f+2}, \ldots, x_{t_3}\}$. By Definition 2.3, there exists an index f' with $0 \le f' \le f$ such that for all i with $1 \le i \le f'$, $v_1y_i \in E(G)$, and for any j with $f'+1 \le j \le f$, $v_1y_j \notin E(G)$. Thus if $f' = 0$, then for any j with $1 \le j \le f$, $v_1y_j \notin E(G)$. Similarly, there exists an index f'' with $0 \le f'' \le t_3 - f$ such that for all i with $1 \le i \le f''$, $v_2x_{f+i} \in E(G)$, and for any j with $f''+1 \le j \le t_3$, $v_2x_{f+j} \notin E(G)$. Thus if $f'' = 0$, then for any j with $1 \le j \le t_3 - f$, $v_2x_{f+j} \notin E(G)$.

We shall use the following notation in the arguments throughout the rest of the proof. For all i with $0 \le i \le f'$ (whence $v_1y_i \in E(G)$), let $u_{t_1+i} = y_i$; and for all j with $0 \le j \le f''$ (whence $v_2x_{f+j} \in E(G)$), let $u_{t_1+f'+j} = x_{f+j}$. Define $G_2' = G[V(G_1) \cup \{u_{t_1+1}, u_{t_1+2}, \ldots, u_{t_1+f'+f''}\}]$, and $L_2' = L|_{V(G_2')}$ be the restriction of L to $V(G_2')$. (See Fig. 1, where G_2' is spanned by a subgraph in the graph depicted in Fig. 1.)

Let $C_1 = c_3(N_{G_1}(v_2) - \{v_1\}) = c_3(N_{G_1}(v_1) - \{v_2\})$, $d_1 = |C_1|$, and define

$$N_1 = \begin{cases} 0 & \text{if } d_1 \ge r, \\ \min\{r - d_1, f' + f''\} & \text{if } d_1 < r. \end{cases}$$

Suppose first that $N_1 = 0$ and so $d_1 \ge r$. Then each of v_1 and v_2 has already r differently colored neighbors. In this case, for each i with $1 \le i \le f'+f''$, we choose

$$c_3(u_{t_1+i}) \in L(u_{t_1+i}) - \{c_3(v_1), c_3(v_2)\}.$$

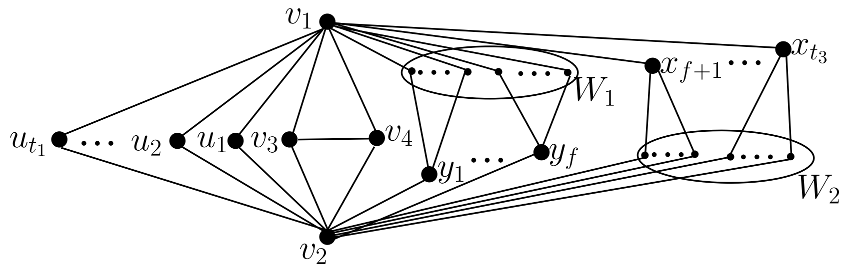

Fig. 1. Graph G, where $G_1 = G - (W_1 \cup W_2 \cup \{y_1, y_2, \ldots, y_f\} \cup \{x_{f+1}, x_{f+2}, \ldots, x_{t_3}\})$.

Assume that $N_1 > 0$. As L is an $(r+3)$-list of G, we have $|L(u_{t_1+1}) - C_1 - \{c_3(v_1), c_3(v_2)\}| = r + 3 - d_1 > 0$, and so the color $c_3(u_{t_1+1}) \in L(u_{t_1+1}) - C_1 - \{c_3(v_1), c_3(v_2)\}$ can be chosen. For each $i = 2, 3, \ldots, N_1$, set

$$c_3(u_{t_1+i}) \in L(u_{t_1+i}) - (C_1 \cup \{c_3(u_{t_1+j}) : 1 \le j \le i-1\}) - \{c_3(v_1), c_3(v_2)\}. \tag{11}$$

If $f' + f'' > r - d_1$, then for any $i = N_1 + 1, \ldots, f' + f''$, set

$$c_3(u_{t_1+i}) \in L(u_{t_1+i}) - \{c_3(v_1), c_3(v_2)\}. \tag{12}$$

By (11) and (12), c_3 is an (L_2', r)-coloring of G_2'.

Let $G_2 = G[V(G_2') \cup \{y_{f'+1}, y_{f'+2}, \ldots, y_f\} \cup \{x_{f+f''+1}, x_{f+f''+2}, \ldots, x_{t_3}\}]$, and $L_2 = L|_{V(G_2)}$ be the restriction of L to $V(G_2)$. (See Fig. 1, where G_2 is spanned by a subgraph in the graph depicted in Fig. 1.)

Let $C_2 = c_3(N_{G_2'}(v_2) - \{v_1\})$, $d_2 = |C_2|$, and define

$$N_2 = \begin{cases} 0 & \text{if } d_2 \ge r, \\ \min\{r - d_2, f - f'\} & \text{if } d_2 < r. \end{cases}$$

Suppose first that $N_2 = 0$ and so $d_2 \ge r$. Then each of v_1 and v_2 has already r differently colored neighbors. In this case, for each i with $f' + 1 \le i \le f$, we choose

$$c_3(y_i) \in L(y_i) - \{c_3(v_1), c_3(v_2)\}.$$

Assume that $N_2 > 0$. As L is an $(r+3)$-list of G, we have $|L(y_{f'+1}) - C_2 - \{c_3(v_1), c_3(v_2)\}| = r + 3 - d_2 > 0$, and so the color $c_3(y_{f'+1}) \in L(y_{f'+1}) - C_2 - \{c_3(v_1), c_3(v_2)\}$ can be chosen. For each $i = 2, 3, \ldots, N_2$, set

$$c_3(y_{f'+i}) \in L(y_{f'+i}) - (C_2 \cup \{c_3(y_{f'+j}) : 1 \le j \le i-1\}) - \{c_3(v_1), c_3(v_2)\}. \tag{13}$$

If $f - f' > r - d_2$, then for any $i = N_2 + 1, \ldots, f - f'$, set

$$c_3(y_{f'+i}) \in L(y_{f'+i}) - \{c_3(v_1), c_3(v_2)\}. \tag{14}$$

Let $C_2' = c_3(N_{G_2'}(v_1) - \{v_2\})$, $d_2' = |C_2'|$, and define

$$N_2' = \begin{cases} 0 & \text{if } d_2' \ge r, \\ \min\{r - d_2', t_3 - (f + f'')\} & \text{if } d_2' < r. \end{cases}$$

Suppose first that $N_2' = 0$ and so $d_2' \ge r$. Then each of v_1 and v_2 has already r differently colored neighbors. In this case, for each j with $f + f'' + 1 \le j \le t_3$, we choose

$$c_3(x_j) \in L(x_j) - \{c_3(v_1), c_3(v_2)\}.$$

Assume that $N_2' > 0$. As L is an $(r+3)$-list of G, we have $|L(x_{f+f''+1}) - C_2' - \{c_3(v_1), c_3(v_2)\}| = r + 3 - d_2' > 0$, and so the color $c_3(x_{f+f''+1}) \in L(x_{f+f''+1}) - C_2' - \{c_3(v_1), c_3(v_2)\}$ can be chosen. For each $i = 2, 3, \ldots, N_2'$, set

$$c_3(x_{f+f''+i}) \in L(x_{f+f''+i}) - (C_2' \cup \{c_3(x_{f+f''+j}) : 1 \le j \le i-1\}) - \{c_3(v_1), c_3(v_2)\}. \tag{15}$$

If $t_3 - (f + f'') > r - d_2$, then for any $i = N_2' + 1, \ldots, t_3 - (f + f'')$, set

$$c_3(x_{f+f''+i}) \in L(x_{f+f''+i}) - \{c_3(v_1), c_3(v_2)\}. \tag{16}$$

After finishing coloring vertices in $\{y_{f'+1}, y_{f'+2}, \ldots, y_f\} \cup \{x_{f+f''+1}, x_{f+f''+2}, \ldots, x_{t_3}\}$, we have completed the coloring of $V(G_2)$. By (13), (14), (15) and (16), c_3 is an (L_2, r)-coloring of G_2.

Step 3. In Step 3, we are to color the vertices in $W_1 \cup W_2$ using the notation in (8) for vertices in W_1 and W_2, and so complete the coloring of $V(G)$. We first color vertices in W_1.

Let $C_3 = c_3(N_{G_2}(v_1) - \{v_2\})$, $d_3 = |C_3|$ and define

$$N_3 = \begin{cases} 0 & \text{if } d_3 \geq r, \\ \min\{r - d_3, |W_1|\} & \text{if } d_3 < r. \end{cases}$$

Suppose first that $N_3 = 0$ and so $d_3 \geq r$. Then v_1 has already r differently colored neighbors. In this case, for any ℓ with $1 \leq \ell \leq f$, we have

$$|L(w_1^\ell) - \{c_3(y_\ell), c_3(v_1), c_3(v_2)\}| \geq r.$$

Thus we can choose $c_3(w_1^\ell) \in L(w_1^\ell) - \{c_3(y_\ell), c_3(v_1), c_3(v_2)\}$, and for any i with $2 \leq i \leq p_\ell$ for some $p_\ell = \min\{r-1, k_\ell\}$, we can find a color $c_3(w_i^\ell)$ so that

$$c_3(w_i^\ell) \in L(w_i^\ell) - (\{c_3(y_\ell), c_3(v_1), c_3(v_2)\} \cup \{c_3(w_t^\ell) : 1 \leq t \leq i-1\}) \tag{17}$$

If $k_\ell > r-1$, then for any $i = p_\ell + 1, \ldots, k_\ell$, set

$$c_3(w_i^\ell) \in L(w_i^\ell) - \{c_3(y_\ell), c_3(v_1)\}.$$

Assume that $N_3 = |W_1| > 0$. Then as L is an $(r+3)$-list of G, we have $|L(w_1^1) - (C_3 \cup \{c_3(y_1), c_3(v_1), c_3(v_2)\})| = r + 3 - (d_3 + 3) > 0$, and so the color $c_3(w_1^1) \in L(w_1^1) - (C_3 \cup \{c_3(y_1), c_3(v_1), c_3(v_2)\})$ can be chosen. For any ℓ with $1 \leq \ell \leq f$ and i with $1 \leq i \leq k_\ell$,

$$c_3(w_i^\ell) \in L(w_i^\ell) - (C_3 \cup \{c_3(y_\ell), c_3(v_1), c_3(v_2)\} \cup \{c_3(w_t^s) : 1 \leq t \leq i-1, 1 \leq s \leq \ell\}). \tag{18}$$

Now we assume that $N_3 = r - d_3$. Then there exist $r - d_3$ distinct vertices $w_1, w_2, \ldots, w_{r-d_3}$ in W_1. As L is an $(r+3)$-list of G, it is possible to set $c_3(w_1) \in L(w_1) - (C_3 \cup c_3(N_G(w_1)) \cup \{c_3(v_2)\})$, and for any i with $1 \leq i \leq r - d_3 - 1$ we have

$$c_3(w_{i+1}) \in L(w_{i+1}) - (C_3 \cup c_3(N_G(w_{i+1})) \cup \{c_3(v_2)\} \cup \{c_3(w_j) : 1 \leq j \leq i\}). \tag{19}$$

Thus after we have colored all vertices in $\{w_i : 1 \leq i \leq r - d_3\}$, then (18) hold for all such vertex. For other vertices in $w_i^\ell \in W_1 - \{w_1, w_2, \ldots, w_{r-d_3}\}$ with $1 \leq \ell \leq f$, $1 \leq i \leq p_\ell$ for some $p_\ell = \min\{r-1, k_\ell\}$, set

$$c_3(w_i^\ell) \in L(w_i^\ell) - (\{c_3(y_\ell), c_3(v_1), c_3(v_2)\} \cup \{c_3(w_t^\ell) : 1 \leq t \leq i-1\}$$

$$\cup \{c_3(w_q) : 1 \leq q \leq r - d_3, w_q \in N_G(y_\ell)\}). \tag{20}$$

If $k_\ell > r-1$, then for any $i = p_\ell + 1, \ldots, k_\ell$, set

$$c_3(w_i^\ell) \in L(w_i^\ell) - \{c_3(y_\ell), c_3(v_1)\}.$$

We then extend c_2 to color vertices in W_2 using similar strategy. Let $C_3' = c_3(N_{G_2}(v_2) - \{v_1\})$, $d_3' = |C_3'|$ and define

$$N_3' = \begin{cases} 0 & \text{if } d_3' \geq r, \\ \min\{r - d_3', |W_2|\} & \text{if } d_3' < r. \end{cases}$$

Suppose first that $N_3' = 0$ and so $d_3' \geq r$. Then v_2 has already r differently colored neighbors. In this case, for any ℓ with $f+1 \leq \ell \leq t_3$, we have

$$|L(w_1^\ell) - \{c_3(x_\ell), c_3(v_1), c_3(v_2)\}| \geq r.$$

Thus we can choose $c_3(w_1^\ell) \in L(w_1^\ell) - \{c_3(x_\ell), c_3(v_1), c_3(v_2)\}$, and for any i with $2 \leq i \leq q_\ell$ for some $q_\ell = \min\{r-1, k_\ell\}$, we can find a color $c_3(w_i^\ell)$ so that

$$c_3(w_i^\ell) \in L(w_i^\ell) - (\{c_3(x_\ell), c_3(v_1), c_3(v_2)\} \cup \{c_3(w_t^\ell) : 1 \leq t \leq i-1\}) \tag{21}$$

If $k_\ell > r-1$, then for any $i = q_\ell + 1, \ldots, k_\ell$, set

$$c_3(w_i^\ell) \in L(w_i^\ell) - \{c_3(x_\ell), c_3(v_2)\}.$$

Assume that $N_3' = |W_2| > 0$. Then as L is an $(r+3)$-list of G, we have $|L(w_1^{f+1}) - (C_3' \cup \{c_3(x_{f+1}), c_3(v_1), c_3(v_2)\})| = r + 3 - (d_3' + 3) > 0$, and so the color $c_3(w_1^{f+1}) \in L(w_1^{f+1}) - (C_3' \cup \{c_3(x_{f+1}), c_3(v_1), c_3(v_2)\})$ can be chosen. For any ℓ with $f+1 \leq \ell \leq t_3$ and i with $1 \leq i \leq k_\ell$,

$$c_3(w_i^\ell) \in L(w_i^\ell) - (C_3' \cup \{c_3(x_\ell), c_3(v_1), c_3(v_2)\} \cup \{c_3(w_t^s) : 1 \leq t \leq i-1, f+1 \leq s \leq \ell\}). \tag{22}$$

Now assume that $N_3' = r - d_3'$. Then there exist $r - d_3'$ distinct vertices $w_1', w_2', \ldots, w_{r-d_3'}'$ in W_2. As L is an $(r+3)$-list of G, it is possible to set $c_3(w_1') \in L(w_1') - (C_3' \cup c_3(N_G(w_1')) \cup \{c_3(v_1)\})$, and for any i with $1 \leq i \leq r - d_3' - 1$ we have

$$c_3(w_{i+1}') \in L(w_{i+1}') - (C_3' \cup c_3(N_G(w_{i+1}')) \cup \{c_3(v_1)\} \cup \{c_3(w_j') : 1 \leq j \leq i\}). \tag{23}$$

Thus after we have colored all vertices in $\{w_i' : 1 \le i \le r - d_3'\}$, then (22) hold for all such vertex. For other vertices in $w_i^\ell \in W_2 - \{w_1', w_2', \ldots, w_{r-d_3'}'\}$ with $f + 1 \le \ell \le t_3$, $1 \le i \le q_\ell$ for some $q_\ell = \min\{r - 1, k_\ell\}$, set

$$\begin{aligned} c_3(w_i^\ell) \in L(w_i^\ell) - (\{c_3(x_\ell), c_3(v_1), c_3(v_2)\} \cup \{c_3(w_t^\ell) : 1 \le t \le i - 1\} \\ \cup \{c_3(w_q) : 1 \le q \le r - d_3', w_q \in N_G(x_\ell)\}). \end{aligned} \tag{24}$$

If $k_\ell > r - 1$, then for any $i = q_\ell + 1, \ldots, k_\ell$, set

$$c_3(w_i^\ell) \in L(w_i^\ell) - \{c_3(x_\ell), c_3(v_2)\}.$$

After finishing coloring vertices in $W_1 \cup W_2$, we have completed the coloring of $V(G)$. By (17), (18), (19),(20), (21), (22), (23), and (24), c_3 is an (L, r)-coloring of G. □

Proof of Theorem 1.5. By Proposition 3.3, $G \in EX(K_4(6)) \cup [K_4(6), K_6]$ is a 2-connected graph, we have $\chi_{L,r}(G) \le K(r) + 1$. By Proposition 3.5, $G \in \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$ is a 2-connected graph, we have $\chi_{L,r}(G) \le K(r)$. By Theorem 2.4, the validity of these Propositions completes the proof of Theorem 1.5.

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