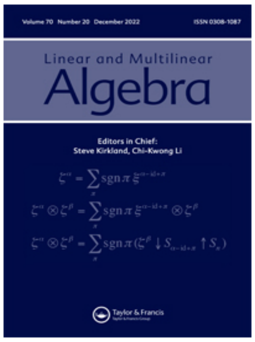


Linear and Multilinear Algebra

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/glma20

Spectral and extremal conditions for supereulerian graphs

Jia Wei, Zhifu You, Sulin Song & Hong-Jian Lai

To cite this article: Jia Wei, Zhifu You, Sulin Song & Hong-Jian Lai (2022) Spectral and extremal conditions for supereulerian graphs, Linear and Multilinear Algebra, 70:20, 5995-6017, DOI: 10.1080/03081087.2021.1942772

To link to this article: https://doi.org/10.1080/03081087.2021.1942772

Published online: 23 Jun 2021.

LINEAR AND MULTILINEAR ALGEBRA
2022, VOL. 70, NO. 20, 5995–6017
https://doi.org/10.1080/03081087.2021.1942772

Taylor & Francis
Taylor & Francis Group

Check for updates

Spectral and extremal conditions for supereulerian graphs

Jia Wei[a,b], Zhifu You[c], Sulin Song[d] and Hong-Jian Lai [d]

[a]Department of Mathematics, South China University of Technology, Guangzhou, People's Republic of China; [b]School of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an, Shaanxi, People's Republic of China; [c]School of Mathematics and Systems Sciences, Guangdong Polytechnic Normal University, Guangzhou, People's Republic of China; [d]Department of Mathematics, West Virginia University, Morgantown, WV, USA

ABSTRACT

A graph is supereulerian if it contains a spanning closed trail. We prove several Erdős-type extremal size conditions with a lower bounded minimum degree for a graph to be supereulerian and with different edge-connectivity, with the corresponding extremal graphs characterized. These results are then applied to prove sufficient conditions involving adjacency spectral radius and signless Laplacian spectral radius for a graph to be supereulerian.

ARTICLE HISTORY
Received 30 August 2020
Accepted 5 June 2021

COMMUNICATED BY
J. Y. Shao

KEYWORDS
Supereulerian; extremal graphs; reduction; spectral radius; signless Laplacian spectral radius

2010 AMS SUBJECT CLASSIFICATIONS
05C35; 05C45; 05C50

1. Introduction

Only finite and simple graphs are considered in this paper. Unless otherwise stated, we will follow Bondy and Murty [1] for undefined terms and notation. We normally use n, $\delta(G)$, $\kappa(G)$ and $\kappa'(G)$ to denote $|V(G)|$, the minimum degree, the connectivity and the edge connectivity of a graph G, respectively. The set of neighbours of a vertex u in a graph G is denoted by $N_G(u)$, Thus $N_G(u) = \{v \in V(G) : uv \in E(G)\}$. Define $d_G(u) = |N_G(u)|$ and $N_G[u] = N_G(u) \cup \{u\}$. The adjacency matrix of G is defined to be a $(0, 1)$-matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if and only if v_i and v_j are adjacent in G. Let $D(G)$ be the degree diagonal matrix of G, and $Q(G) = D(G) + A(G)$ be the signless Laplacian matrix of G. Let $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ and $q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G)$ be the eigenvalues of $A(G)$ and $Q(G)$, respectively. The largest eigenvalue of $A(G)$ and $Q(G)$, denoted by $\lambda(G) := \lambda_1(G)$ and $q(G) := q_1(G)$, are called *the spectral radius* and *the signless Laplacian spectral radius* of G, respectively. Let I_n be the identity matrix of order n.

The disjoint union of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is the graph with the vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The disjoint union of k copies of

CONTACT Hong-Jian Lai hjlai@math.wvu.edu, hjlai2015@hotmail.com Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA

a graph G is denoted by kG. *The join* of G_1 and G_2, denoted by $G_1 \vee G_2$, has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1), y \in V(G_2)\}$.

A graph is called *nontrivial* if it includes at least one edge. A graph G is called *eulerian graph* if it is connected and $O(G) = \emptyset$ where $O(G)$ is the set of all vertices of odd degree in G. A graph is called *supereulerian* if it has a spanning eulerian subgraph.

Let H be a connected subgraph of a graph G, the *contraction* G/H is the multigraph obtained from G by replacing H by a vertex v_H in G/H such that the number of edges joining any $v \in V(G) - V(H)$ to v_H in G/H equals the number of edges joining v to $V(H)$ in G. In this case, H is called the *preimage* of v_H. A graph G is called *collapsible* if for any subset R of $V(G)$ of even cardinality, there is a spanning connected subgraph H_R of G with $O(H_R) = R$. Then by the definition of collapsible, it is obvious that if G is collapsible, then G is supereulerian.

The *reduction* of a graph G, denoted by G', is obtained from G by contracting all maximal collapsible subgraphs. A graph G is called *reduced* if it is the reduction of itself. In [2], Catlin demonstrated that every vertex of G belongs to a unique maximal collapsible graph of G.

In 1977, Boesch et al. [3] put forward the supereulerian graph problem: when a graph has a spanning eulerian subgraph? Pulleyblank [4] verified that judging a graph is supereulerian or not is NP-complete in 1979. Since then, many researchers are keen on this topic. As a major step forward in the study of supereulerian graphs and the related problems, a reduction method proposed by Catlin [2] has become an important tool in this area, as can be found in surveys [5–7] and the references therein, among others. The main goals of this research are to investigate extremal conditions and spectral conditions to assure a graph to be supereulerian. As Jaeger [8] and Catlin [2] indicated that every supereulerian graph must be 2-edge-connected and every 4-edge-connected graph is supereulerian. Our studies focus on simple graphs with edge-connectivity 2 or 3.

1.1. The extremal conditions for supereulerianicity of a graph

Cai [9] first provided an Ore type condition [10] for a 2-edge-connected graph to be supereulerian.

Theorem 1.1 (Cai [9]): *Let G be a simple graph with $\kappa'(G) \geq 2$ and $n = |V(G)|$. If*

$$|E(G)| \geq \binom{n-4}{2} + 6, \tag{1}$$

then exactly one of the following holds:

(i) *G is supereulerian;*
(ii) *$G = K_{2,5}$;*
(iii) *Equality holds in (1), and either G is the cube minus a vertex, or G contains a complete subgraph $H = K_{n-4}$ such that $G/H = K_{2,3}$.*

Motivated by this result, Catlin and Chen [11] later extended Theorem 1.1 by ruling out some graphs with a small order.

Theorem 1.2 (Catlin and Chen [11]): *Let G be a simple graph with $\kappa'(G) \geq 2$ and $n = |V(G)| \geq 10$. If*

$$|E(G)| \geq \binom{n-6}{2} + 10, \tag{2}$$

then exactly one of the following holds:

(i) *G is supereulerian;*
(ii) *G is contractible to K_2 or $K_{2,3}$;*
(iii) *Equality holds in (2), and G contains a complete subgraph $H = K_{n-6}$ such that $G/H = K_{2,5}$.*

Let $\mathcal{K}_{2,3}(n)$ be the collection of graphs obtained from the $K_{2,3}$ by replacing one of the three vertices of degree 2 by a complete graph K_{n-10}, and replacing the remain two vertices of degree 2 by a complete graph K_4, respectively. For any graph $G \in \mathcal{K}_{2,3}(n)$, let E_1 be the edge set of $E(G)$ whose endpoints are both in X with $X = \{v \in V(G) : d(v) \geq n - 11\}$. Then we define

$$\mathcal{K}'_{2,3}(n) = \{G' : G' = G - E' \text{ where } G \in \mathcal{K}_{2,3}(n) \text{ and } E' \subseteq E_1 \text{ with } |E'| = 1\}. \tag{3}$$

Let P be the Petersen graph and let $\mathcal{P}(n)$ be the collection of graphs obtained from the Petersen graph P by replacing one of the ten vertices by a complete graph K_{n-9}. For any graph $G \in \mathcal{P}(n)$, let E_2 be the edge set of $E(G)$ whose endpoints are both in X with $X = \{v \in V(G) : d(v) \geq n - 10\}$. Then we define

$$\begin{aligned} \mathcal{P}_1(n) &= \{G_1 : G_1 = G - E'' \text{ where } G \in \mathcal{P}(n), E'' \subseteq E_2 \text{ with } |E''| \leq n - 12, G' = P\}, \\ \mathcal{P}'(n) &= \{G' : G' = G - E'' \text{ for any } G \in \mathcal{P}(n), \text{ and } E'' \subseteq E_2 \text{ with } |E''| = 1\}. \end{aligned} \tag{4}$$

By definition, members in $\mathcal{P}(n)$ are all in $\mathcal{P}_1(n)$ with $|E''| = 0$. Then we obtain the following result which sharpens Theorems 1.1 and 1.2 among the graphs with $\delta(G) \geq 3$.

Theorem 1.3: *Let G be a simple graph with $\kappa'(G) \geq 2$, $n = |V(G)| \geq 15$ and $\delta(G) \geq 3$. If*

$$|E(G)| \geq \binom{n-10}{2} + 17, \tag{5}$$

then G is supereulerian if and only if $G \notin \mathcal{K}_{2,3}(n) \cup \mathcal{P}_1(n) \cup \mathcal{K}'_{2,3}(n)$.

Erdős [12] initiated an extremal sufficient condition involving the lower bound of the minimum degree for a graph G to be Hamiltonian.

Theorem 1.4 (Erdős [12]): *Let G be a graph of order n and the minimum degree $\delta(G)$ and k be an integer with $1 \leq k \leq \delta(G) \leq \frac{n-1}{2}$. If*

$$|E(G)| > \max\left\{\binom{n-k}{2} + k^2, \binom{\lceil \frac{n+1}{2} \rceil}{2} + \left\lfloor \frac{n+1}{2} \right\rfloor^2\right\},$$

then G is Hamiltonian.

This motivates our current study to seek an extremal sufficient condition involving the lower bound of the minimum degree for a graph G to be supereulerian. Let $\mathcal{K}'_{2,3}(n,k)$ be the collection of graphs obtained from the $K_{2,3}$ by replacing one of the three vertices of degree 2 by a complete graph K_{n-4k-4}, and replacing the remain vertices by a complete graph K_{k+1}, respectively; and let $\mathcal{K}''_{2,3}(n,k)$ be the collection of graphs obtained from the $K_{2,3}$ by replacing one of the two vertices of degree 3 by a complete graph K_{n-4k-4}, and replacing the remain vertices by a complete graph K_{k+1}, respectively. Following Erdős' footsteps, we prove the following theorem, which can be viewed as a generalization of the theorems above.

Theorem 1.5: *Let G be a simple graph with $\kappa'(G) \geq 2$, $n = |V(G)| \geq 5k+5$ and $\delta(G) \geq k \geq 4$. If*

$$|E(G)| \geq \binom{n-4k-4}{2} + 2k(k+1) + 6, \tag{6}$$

then G is supereulerian if and only if $G \notin \mathcal{K}'_{2,3}(n,k) \cup \mathcal{K}''_{2,3}(n,k)$.

Cai [9] conjectured that any 3-edge-connected graph G with order n is supereulerian if $|E(G)| \geq \binom{n-9}{2} + 16$. In 1991, Catlin and Chen [11] proved this conjecture and obtained a stronger conclusion as stated below. As collapsible graphs are supereulerian, this proves Cai's conjecture. It is shown in [11] that the edge lower bound is best possible both in Cai's conjecture and in Theorem 1.6.

Theorem 1.6 (Catlin and Chen [11]): *Let G be a 3-edge-connected simple graph on n vertices. If*

$$|E(G)| \geq \binom{n-9}{2} + 16,$$

then G is collapsible.

For a sufficiently large n, let $\mathcal{P}(n,k)$ be the collection of graphs obtained from the P by replacing one of the vertices by a complete graph K_{n-9k-9}, and replacing each of the remain vertices by a complete graph K_{k+1}, respectively. Once again we follow the idea of Erdős [12] to obtain a generalization of Theorem 1.6.

Theorem 1.7: *Let G be a 3-edge-connected graph of order $n \geq 10k+10$ and $\delta(G) \geq k \geq 4$. If*

$$|E(G)| \geq \binom{n-9k-9}{2} + \frac{9}{2}k(k+1) + 15, \tag{7}$$

then G is supereulerian if and only if $G \notin \mathcal{P}(n,k)$.

1.2. The spectral conditions for supereulerianicity of a graph

There have been many studies investigating the relationship between spectral radii of a graph G and the Hamiltonian properties of G, as seen in [13–18] and the references therein, among others. These results also motivate the current research. We investigate sufficient spectral conditions for a 2-edge-connected graph to be supereulerian with different minimum degree constraints and prove the following Theorems 1.8 and 1.9.

Theorem 1.8: *Let G be a 2-edge-connected graph of order n and $\delta(G) \geq 3$.*

(1) *If $n \geq 15$ and $\lambda(G) \geq n-10$, then G is supereulerian if and only if $G \notin \mathcal{P}(n)$.*
(2) *If $n \geq 63$ and $q(G) \geq 2(n-10)$, then G is supereulerian if and only if $G \notin \mathcal{P}(n)$.*

Theorem 1.9: *Let G be a 2-edge-connected graph of order $n \geq 5k+5$ and $\delta(G) \geq k \geq 4$.*

(1) *If $\lambda(G) > \frac{k-1+\sqrt{4n^2-36(k+1)n+81(k+1)^2+48}}{2}$, then G is supereulerian.*
(2) *If $q(G) > 2n-8k-10+\frac{20k^2+32k+24}{n-1}$, then G is supereulerian.*

Among simple graphs with edge-connectivity 3, the sufficient spectral radii conditions have smaller lower bounds, as expected.

Theorem 1.10: *Let G be a 3-edge-connected graph of order $n \geq 10k+10$ and $\delta(G) \geq k \geq 4$.*

(1) *If $\lambda(G) > \frac{k-1+\sqrt{4n^2-76(k+1)n+361(k+1)^2+120}}{2}$, then G is supereulerian.*
(2) *If $q(G) > 2n-20-18k+\frac{90k^2+162k+102}{n-1}$, then G is supereulerian.*

In the next section, we present the preliminaries, including some of tools that will be used in the proofs of the main results.

2. Extremal size of supereulerian graphs

The main purpose of this section is to prove Theorems 1.3, 1.5 and 1.7, determining the optimal sizes to assure a graph to be supereulerian under different edge connectivity and minimum degree conditions. We start with some terms and notation that will be used in our arguments. Let $H_1, H_2, \dots, H_c$ denote the list of all maximal collapsible subgraphs of G, and let G' be the reduction of G. Thus G' is obtained from G by contracting the $H_1, H_2, \dots, H_c$ to distinct vertices $v_1, v_2, \dots, v_c$, respectively. We assume that the vertices are so labelled that

$$d_{G'}(v_1) \leq d_{G'}(v_2) \leq \cdots \leq d_{G'}(v_c). \tag{8}$$

We first state some former results which play important roles in this section. Define $D_i(G) = \{v \in V(G) : d_G(v) = i\}$ and $d_i(G) = |D_i(G)|$.

Theorem 2.1 (Catlin [2]): *Let G be a graph and G' be the reduction of G. Then*

(i) *G is supereulerian if and only if G' is supereulerian.*
(ii) *If G' is nontrivial and 2-edge-connected, then $d_2(G') + d_3(G') \geq 4$, and if $d_2(G') + d_3(G') = 4$, then G' is eulerian.*

(iii) *If $G' \notin \{K_1, K_2\}$, then G' is simple and K_3-free with $\delta(G') \leq 3$ and*

$$|E(G')| \leq 2|V(G')| - 4.$$

(iv) *G is reduced if and only if G has no nontrivial collapsible subgraphs.*
(v) *If G is supereulerian, then any contraction of G is also supereulerian.*

Theorem 2.2 (Catlin [2]): *Let H_1 and H_2 be two subgraphs of H such that $H_1 \cup H_2 = H$ and $H_1 \cap H_2 \neq \emptyset$. If H_1 and H_2 are collapsible, then H is collapsible.*

Let $F(G)$ denote the minimum number of extra edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. Then Catlin obtained the following Theorems 2.3 and 2.4 in 1987 and in 1996, respectively.

Theorem 2.3 (Catlin [19]): *If G is a connected reduced graph, then $F(G) = 2|V(G)| - |E(G)| - 2$.*

Theorem 2.4 (Catlin et al. [20]): *If G is a connected graph and $F(G) \leq 2$, then either G is collapsible, or the reduction of G is a K_2 or a $K_{2,t}$ for some integer $t \geq 1$.*

Then we present two lemmas, namely Lemmas 2.5 and 2.6 which are useful tools to prove Theorems 1.3, 1.5 and 1.7 in this section.

Lemma 2.5 (Liu et al., Lemma 3.1 of [21]): *Let G' be the reduction of G. If there exists an $i \in \{1, 2, \ldots, c\}$ such that $d_{G'}(v_i) \leq \delta(G) - 1$, then $|V(H_i)| \geq \delta(G) + 1$.*

Lemma 2.6: *Let $l, a_1, a_2, \ldots, a_l$ be integers with $l \geq 2$ and $0 \leq a_1 \leq a_2 \leq \cdots \leq a_l$, and let $f(x_1, x_2, \ldots, x_l) = \sum_{i=1}^{l} \binom{x_i}{2}$ be an integral function with $\sum_{i=1}^{l} x_i = n$, such that for any $i \in \{1, 2, \ldots, l\}$, $a_i \leq x_i$. Then*

$$f(x_1, x_2, \ldots, x_l) \leq \binom{n - \sum_{i=1}^{l-1} a_i}{2} + \sum_{i=1}^{l-1} \binom{a_i}{2}.$$

Moreover, the equality holds if and only if $x_i = a_i$ for any $i \in \{1, 2, \ldots, l-1\}$ and $x_l = n - \sum_{i=1}^{l} a_i$.

Proof: As $a_i \leq x_i$ for any $i \in \{1, 2, \ldots, l\}$ and $a_1 \leq a_2 \leq \cdots \leq a_l$, we observe that $x_l \geq a_i$ for any $i \in \{1, 2, \ldots, l\}$. Thus for any $i \in \{1, 2, \ldots, l-1\}$,

$$\begin{aligned}\sum_{j=1}^{l-1} (x_j - a_j) + 2x_l - x_i - a_i &= \sum_{j=1, j\neq i}^{l-1} (x_j - a_j) + x_i - a_i + 2x_l - x_i - a_i \\ &= \sum_{j=1, j\neq i}^{l-1} (x_j - a_j) + 2x_l - 2a_i \geq 0.\end{aligned}$$

This, together with $\sum_{i=1}^{l} x_i = n$, yields

$$\begin{aligned}
&\binom{n-\sum_{i=1}^{l-1} a_i}{2} + \sum_{i=1}^{l-1}\binom{a_i}{2} - \sum_{i=1}^{l}\binom{x_i}{2} \\
&= \binom{n-\sum_{i=1}^{l-1} a_i}{2} - \binom{x_l}{2} + \sum_{i=1}^{l-1}\binom{a_i}{2} - \sum_{i=1}^{l-1}\binom{x_i}{2} \\
&= \frac{\left(n-\sum_{i=1}^{l-1} a_i - x_l\right)\left(n-\sum_{i=1}^{l-1} a_i - 1 + x_l\right)}{2} - \sum_{i=1}^{l-1}\left(\binom{x_i}{2} - \binom{a_i}{2}\right) \\
&= \frac{\sum_{i=1}^{l-1}(x_i - a_i)\left(n-\sum_{j=1}^{l-1} a_j - 1 + x_l\right)}{2} - \frac{\sum_{i=1}^{l-1}(x_i-a_i)(x_i-1+a_i)}{2} \\
&= \frac{\sum_{i=1}^{l-1}(x_i - a_i)\left(\sum_{j=1}^{l-1}(x_j-a_j) + 2x_l - x_i - a_i\right)}{2} \geq 0.
\end{aligned}$$

Observe that the equality holds if and only if $x_i = a_i$ for any $i \in \{1, 2, \ldots, l-1\}$ and $x_l = n - \sum_{i=1}^{l-1} a_i$. This completes the proof of the Lemma 2.6. ■

Catlin and Chen [11] showed a sufficient condition to judge a class of graphs to be 3-edge-connected.

Lemma 2.7 (Catlin and Chen [11]): *Let n be the smallest natural number such that there is a 2-edge-connected reduced graph G of order n and size $2n-4$, such that G is not $K_{2,n-2}$. Then $n \geq 14$ and G is 3-edge-connected.*

We next define a family of graphs which will be used to summarize the results of reduced graphs with small orders. And we will show some graphs in $\mathcal{F}$ with small parameters in Figure 1.

Definition 2.8 (Chen and Lai [22]): Let s_1, s_2, s_3, m, l, t be natural numbers with $t \geq 2$ and $m, l \geq 1$. Let $K \cong K_{1,3}$ with centre a and ends a_1, a_2, a_3. Define $K_{1,3}(s_1, s_2, s_3)$ to be the graph obtained from K by adding s_i vertices with neighbours $\{a_i, a_{i+1}\}$, where $i \equiv 1, 2, 3 (\mathrm{mod}\, 3)$. Let $K_{2,t}(u, u')$ be a $K_{2,t}$ with u, u' being the nonadjacent vertices of degree t. Let $K'_{2,t}(u, u', u'')$ be the graph obtained from a $K_{2,t}(u, u')$ by adding a new vertex u'' that joins to u' only. Hence u'' has degree 1 and u has degree t in $K'_{2,t}(u, u', u'')$. Let $K''_{2,t}(u, u', u'')$ be the graph obtained from a $K_{2,t}(u, u')$ by adding a new vertex u'' that joins to a vertex of degree 2 of $K_{2,t}$. Hence u'' has degree 1 and both u and u' have degree t in $K''_{2,t}(u, u', u'')$. We shall use $K'_{2,t}$ and $K''_{2,t}$ for a $K'_{2,t}(u, u', u'')$ and a $K''_{2,t}(u, u', u'')$, respectively. Let $S(m, l)$ be the graph obtained from a $K_{2,m}(u, u')$ and a $K'_{2,l}(w, w', w'')$ by identifying u with w, and w'' with u'; let $J(m, l)$ denote the graph obtained from a $K_{2,m+1}$ and a $K'_{2,l}(w, w', w'')$ by identifying w, w'' with the two ends of an edge in $K_{2,m+1}$, respectively; let $J'(m, l)$ denote the graph obtained from a $K_{2,m+2}$ and a $K'_{2,l}(w, w', w'')$ by identifying w, w'' with two vertices of degree 2 in $K_{2,m+2}$, respectively. Define $\mathcal{F} = \{K_1, K_2, K_{2,t}, K'_{2,t}, K''_{2,t}, K_{1,3}(s, s', s''), S(m, l), J(m, l), J'(m, l), P\}$, where t, s, s', s'', m, l are both nonnegative integers.

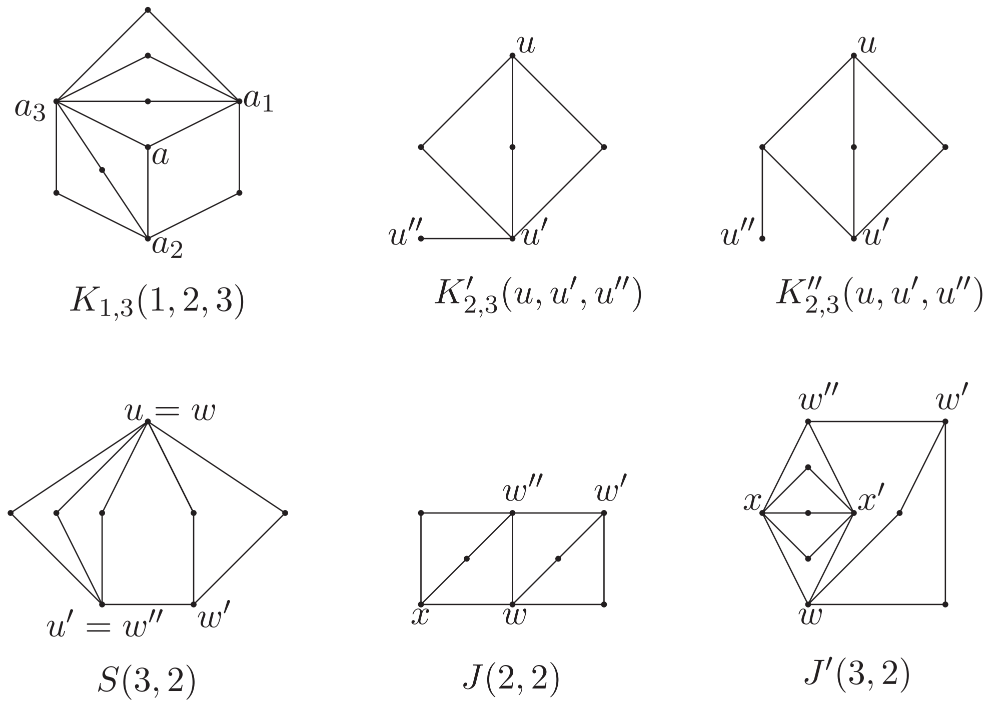


Figure 1. Some graphs in $\mathcal{F}$ with small parameters.

The following theorem which researched the reduced graphs with at most 11 vertices is obtained by Chen and Lai in 1998, which will be the main tool to demonstrate Corollary 2.10.

Theorem 2.9 (Chen and Lai [22]): *If G is a connected reduced graph with $|V(G)| \leq 11$ and $F(G) \leq 3$, then $G \in \mathcal{F}$.*

By Theorem 2.9, we can obtain the following corollary which will play an important role to prove Theorem 1.3.

Corollary 2.10: *Let G be a 2-edge-connected reduced graph with $|V(G)| \leq 11$ and $F(G) \leq 3$. If $d_2(G) \leq 2$, then $G \in \{K_1, P\}$.*

Proof: By Theorem 2.9, $G \in \mathcal{F}$. By Definition 2.8, if $G \in \{K_2, K'_{2,t}, K''_{2,t}, K_{1,3}(s,s',s'')\}$ with $ss's'' = 0$, then $d_1(G) \geq 1$, which contrary to G is 2-edge-connected. Thus $G \in \{K_1, K_{2,t}, K_{1,3}(s,s',s''), S(m,l), J(m,l), J'(m,l), P\}$ with $ss's'' \neq 0$. By Definition 2.8, it is a routine matter to prove that if $G \in \{K_{2,t}, K_{1,3}(s,s',s''), S(m,l), J(m,l), J'(m,l)\}$ with $t \geq 3$ and $ss's'' \neq 0$, then $d_2(G) \geq 3$, contrary to the fact that $d_2(G) \leq 2$. Thus $G \in \{K_1, P\}$. This proves Corollary 2.10. ■

We next present a theorem by Chen for reduced 3-edge-connected graphs with at most 11 vertices. This will be applied to prove Theorem 1.7.

Theorem 2.11 (Chen [23]): *Let G be a reduced graph of order at most 11 with $\kappa'(G) \geq 3$, then $G \in \{K_1, P\}$.*

2.1. Proof of Theorem 1.3

As any collapsible graphs are supereulerian, we prove the following slightly stronger result, which implies Theorem 1.3.

Theorem 2.12: *Let G be a 2-edge-connected graph of order $n \geq 15$ and $\delta(G) \geq 3$. Suppose that (5) holds. Then G is collapsible if and only if $G \notin \mathcal{K}_{2,3}(n) \cup \mathcal{P}_1(n) \cup \mathcal{K}'_{2,3}(n)$.*

Proof: Suppose first that $G \in \mathcal{K}_{2,3}(n) \cup \mathcal{P}_1(n) \cup \mathcal{K}'_{2,3}(n)$. Then by definition, G can be contracted to a non-collapsible graph $K_{2,3}$ or P, and so by Theorem 2.1, G cannot be collapsible. It remains to assume (5) and $G \notin \mathcal{K}_{2,3}(n) \cup \mathcal{P}_1(n) \cup \mathcal{K}'_{2,3}(n)$ to show that G is collapsible.

Let G' be the reduction of G with $c = |V(G')|$. Then by Theorem 2.3,

$$F(G') = 2|V(G')| - |E(G')| - 2. \tag{9}$$

Let $h = d_2(G')$. By (8), for any $h+1 \leq i \leq c$, both $d_{G'}(v_i) \geq 3$ and $|V(H_i)| \geq 1$. If $h \geq 1$, then for any $1 \leq i \leq h$, $d_{G'}(v_i) = 2$; and as $\delta(G) \geq 3$, by Lemma 2.5, $|V(H_i)| \geq \delta(G) + 1 \geq 4$. It follows that

$$n = \sum_{i=1}^{c} |V(H_i)| = \sum_{i=1}^{h} |V(H_i)| + \sum_{i=h+1}^{c} |V(H_i)| \geq 4h + (c-h) = 3h + c$$

and $|E(G)| \leq \sum_{i=1}^{c} \binom{|V(H_i)|}{2} + |E(G')|$. Then by (9) and $|V(G')| = c$, we conclude that $|E(G)| \leq \sum_{i=1}^{c} \binom{|V(H_i)|}{2} + 2|V(G')| - |F(G')| - 2$. For $x \geq 1$, define $f(x) = \binom{n-(c-x)-4(x-1)}{2} + (x-1)\binom{4}{2} + 2c - 2 = \binom{n-3x+4-c}{2} + 6x + 2c - 8$. Thus by Lemma 2.6,

$$|E(G)| \leq \begin{cases} f(h) - F(G'), & \text{if } h \geq 1; \\ f(1) - F(G'), & \text{if } h = 0. \end{cases} \tag{10}$$

Claim 1: $F(G') \leq 3$.

We argue by contradiction and assume that $F(G') \geq 4$. As $|V(G')| = \sum_{i \geq 1} d_i(G')$, $2|E(G')| = \sum_{i \geq 1} id_i(G')$ and by (9), we have

$$2d_2(G') + d_3(G') \geq 12 + \sum_{i \geq 5} (i-4)d_i(G') \geq 12 + \sum_{i \geq 5} d_i(G'). \tag{11}$$

If $h \geq 1$, then (11) implies that $n \geq 3h + c \geq 4d_2(G') + d_3(G') \geq 14$. As $F(G') \geq 4$,

$$\begin{aligned} &\binom{n-10}{2} + 17 - [f(h) - F(G')] \\ &\geq \frac{(3h+c-14)(2n-7-3h-c)}{2} + 29 - 6h - 2c \end{aligned}$$

$$\geq \frac{1}{2}(3h+c)^2 - \frac{25}{2}(3h+c) + 78 > 0. \tag{12}$$

If $h = 0$, then by (11), $n \geq c \geq d_3(G') \geq 12$. This, together with facts that $n \geq 15$ and $F(G') \geq 4$, implies

$$\binom{n-10}{2} + 17 - [f(1) - F(G')] \geq \frac{1}{2}(c-11)(2n-14-c) + 1 > 0. \tag{13}$$

It follows from (10), (12) and (13), and by algebraic manipulations that for all values of $h \geq 0$, $|E(G)| < \binom{n-10}{2} + 17$, contrary to (5). This proves Claim 1.

Claim 2: $c \leq 11$.

Again by contradiction, we assume that $c = |V(G')| \geq 12$, and so $G' \notin \{K_1, K_2\}$. By (iii) of Theorem 2.1,

$$|E(G')| \leq 2|V(G')| - 4. \tag{14}$$

Then by (9), (10) and (14),

$$|E(G)| \leq \begin{cases} f(h) - 2, & \text{if } h \geq 1; \\ f(1) - 2, & \text{if } h = 0. \end{cases} \tag{15}$$

If $h \geq 1$, then $n \geq 3h + c \geq 15$. Thus

$$\begin{aligned} &\binom{n-10}{2} + 17 - [f(h) - 2] \\ &= \frac{(3h+c-14)(2n-7-3h-c)}{2} + 27 - 6h - 2c \\ &\geq \frac{1}{2}(3h+c)^2 - \frac{25}{2}(3h+c) + 76 > 0. \end{aligned} \tag{16}$$

If $h = 0$, by $c \geq 12$ and $n \geq 15$,

$$\binom{n-10}{2} + 17 - [f(1) - 2] \geq \frac{(c-11)(2n-14-c)}{2} - 1 > 0. \tag{17}$$

Hence by (15), (16) and (17), for $h \geq 0$, we have $|E(G)| < \binom{n-10}{2} + 17$, contrary to (5). This proves Claim 2.

Claim 3: *Each of the following holds:*

(i) $d_2(G') \leq 3$.
(ii) *If* $d_2(G') \leq 2$, *then* $G' \in \{K_1, P\}$.
(iii) *If* $d_2(G') = 3$, *then* $G' = K_{2,3}$.

By contradiction. Assume that $h = d_2(G') \geq 4$. As $|V(G')| = \sum_{i\geq 1} d_i(G')$, $2|E(G')| = \sum_{i\geq 1} id_i(G')$ and by (14),

$$2d_2(G') + d_3(G') \geq 8 + \sum_{i\geq 5}(i-4)d_i(G') \geq 8 + \sum_{i\geq 5} d_i(G'). \tag{18}$$

By (18) and $h \geq 4$, we have $n \geq 3h + c \geq 4d_2(G') + d_3(G') + d_4(G') \geq 16$. Thus using the algebraic manipulations similar to those in the proof of (16) above, we conclude that when $h \geq 4$, $|E(G)| < \binom{n-10}{2} + 17$, contrary to (5). This proves Claim 3(i).

As $d_2(G') \leq 2$, then by Claims 1, 2 and Corollary 2.10, we have $G' \in \{K_1, P\}$. This proves Claim 3(ii).

If $2d_2(G') + d_3(G') + d_4(G') \geq 9$, then by $d_2(G') = 3$, we have $n \geq 3h + c \geq 4d_2(G') + d_3(G') + d_4(G') \geq 15$. Thus with the same proof of (16) in Claim 2, $|E(G)| < \binom{n-10}{2} + 17$ when $h \geq 3$, which is contrary to the condition that $|E(G)| \geq \binom{n-10}{2} + 17$. Thus $2d_2(G') + d_3(G') + d_4(G') \leq 8$, and so by (18), we have $2d_2(G') + d_3(G') = 8$ and $\sum_{i\geq 4} d_i(G') = 0$, which imply that $d_2(G') = 3$, $d_3(G') = 2$ and for any $i \geq 4$, $d_i(G') = 0$. Thus $|E(G')| = 2|V(G')| - 4$. As $d_2(G') = 3$, G' cannot be 3-edge-connected. Thus by Lemma 2.7, $G' = K_{2,3}$. This proves Claim 3(iii), and so Claim 3 holds.

By Claims 1, 2 and 3,

$$G' \in \{K_1, K_{2,3}, P\}. \tag{19}$$

If $G' \in \{K_{2,3}, P\}$, then by $\delta(G) \geq 3$ and $|E(G)| \geq \binom{n-10}{2} + 17$, it is routine to verify that $G \in \mathcal{K}_{2,3}(n) \cup \mathcal{P}_1(n) \cup \mathcal{K}'_{2,3}(n)$, contrary to the assumption that $G \notin \mathcal{K}_{2,3}(n) \cup \mathcal{P}_1(n) \cup \mathcal{K}'_{2,3}(n)$. Thus by (19), we have $G' = K_1$, and so G is collapsible. This proves Theorem 2.12. ■

2.2. Proof of Theorem 1.5

Let G' be the reduction of G with $c = |V(G')|$. And let $f_1(n,k) = \binom{n-4k-4}{2} + 2k(k+1) + 6$. If $G \in \mathcal{K}'_{2,3}(n,k) \cup \mathcal{K}''_{2,3}(n,k)$, then $G' = K_{2,3}$. As $K_{2,3}$ is not supereulerian, then by (i) of Theorem 2.1, G is not supereulerian. Thus if G is supereulerian, then $G \notin \mathcal{K}'_{2,3}(n,k) \cup \mathcal{K}''_{2,3}(n,k)$.

To prove the converse, we may assume that $G \notin \mathcal{K}'_{2,3}(n,k) \cup \mathcal{K}''_{2,3}(n,k)$ to prove G is supereulerian. We argue by contradiction and assume that G is not supereulerian. By (i) of Theorem 2.1, G' is not supereulerian.

Claim 4: $F(G') \leq 2$.

Suppose, to the contrary, that $F(G') \geq 3$. By Theorem 2.3,

$$2|V(G')| - |E(G')| \geq 5. \tag{20}$$

As $|V(G')| = \sum_{i\geq 1} d_i(G')$ and $2|E(G')| = \sum_{i\geq 1} id_i(G')$,

$$2d_2(G') + d_3(G') \geq 10 + \sum_{i\geq 5}(i-4)d_i(G') \geq 10 + \sum_{i\geq 5} d_i(G'). \tag{21}$$

By (25), algebraic manipulation leads to $d_2(G') + d_3(G') \geq 5$. Thus $c \geq 5$ and by (8), we have $d_{G'}(v_5) \leq 3$. Since $\delta(G) \geq k \geq 4$, by Lemma 2.5, $|V(H_i)| \geq \delta(G) + 1 \geq k + 1$ for any

$1 \le i \le 5$. Thus $n = \sum_{i=1}^{c} |V(H_i)| = \sum_{i=1}^{5} |V(H_i)| + \sum_{i=6}^{c} |V(H_i)| \ge 5(k+1) + (c-5) = 5k + c$. Let

$$g(c) = f_1(n,k) - \left[\binom{n-4k-c+1}{2} + 2k(k+1) + 2c - 5\right].$$

Since $k \ge 4$, $n \ge 5k + c$ and $c \ge 5$, we have

$$\begin{aligned} g(c) &= f_1(n,k) - \left[\binom{n-4k-c+1}{2} + 2k(k+1) + 2c - 5\right] \\ &= \frac{(c-5)(2n-8k-c-4)}{2} + 11 - 2c \\ &\ge \frac{(c-5)(2k+c-4)}{2} + 11 - 2c \\ &= \frac{c^2}{2} + \left(k - \frac{13}{2}\right)c - 5k + 21 \ge 1 > 0. \end{aligned} \tag{22}$$

Thus by Lemma 2.6, (20) and (22), we observe that

$$\begin{aligned} |E(G)| &\le \sum_{i=1}^{c} \binom{|V(H_i)|}{2} + |E(G')| \\ &\le \binom{n-(c-5)-4(k+1)}{2} + 4\binom{k+1}{2} + 2|V(G')| - 5 \\ &\le \binom{n-4k-c+1}{2} + 2k(k+1) + 2c - 5 \\ &< f_1(n,k) = \binom{n-4k-4}{2} + 2k(k+1) + 6, \end{aligned}$$

contrary to (6). This proves Claim 1.

By Theorem 2.4 and Claim 1, either G' is collapsible, or G' is a K_2 or a $K_{2,t}$ for some integer $t \ge 1$. Since G' is reduced, nonsupereulerian and $\kappa'(G') \ge \kappa'(G) \ge 2$, it follows that G' is a $K_{2,t}$ for some odd integer $t \ge 3$. As $G' = K_{2,t}$, we have $c = t + 2$. Then by (8), $d_{G'}(v_i) = 2$ for any $1 \le i \le t$ and $d_{G'}(v_{t+1}) = d_{G'}(v_{t+2}) = t$. Since $\delta(G) \ge k \ge 4$, by Lemma 2.5, $|V(H_i)| \ge \delta(G) + 1 \ge k + 1$ for any $1 \le i \le t$ and $|V(H_i)| \ge 1$ for any $i \in \{t+1, t+2\}$. It follows that $n \ge t(k+1) + 2$. Thus by Lemma 2.6,

$$|E(G)| \le \sum_{i=1}^{c} \binom{|V(H_i)|}{2} + |E(G')| \le \binom{n-2-(t-1)(k+1)}{2} + (t-1)\binom{k+1}{2} + 2t. \tag{23}$$

Claim 5: *$t = 3$.*

By contradiction, we assume that $t \ne 3$. Since $t \ge 3$ and t is an odd number, we have $t \ge 5$. Let $\theta(t) = f_1(n,k) - [\binom{n-2-(t-1)(k+1)}{2} + (t-1)\binom{k+1}{2} + 2t]$. As $n \ge t(k+1) + 2$ and $k \ge 4$, we have

$$\theta(t) = f_1(n,k) - \left[\binom{n-2-(t-1)(k+1)}{2} + (t-1)\binom{k+1}{2} + 2t\right]$$

$$
\begin{aligned}
&= \frac{1}{2}(t(k+1)-5k-3)(2n-3k-6-t(k+1)) + \frac{1}{2}k(k+1)(5-t)+6-2t \\
&\geq \frac{1}{2}(t(k+1)-5k-3)(t(k+1)-3k-2) + \frac{1}{2}k(k+1)(5-t)+6-2t \\
&\geq \frac{1}{2}(t(k+1)-5k-3)(2k+3) + \frac{1}{2}k(k+1)(5-t)+6-2t \\
&= \frac{1}{2}(t-5)k^2 + \left(2k-\frac{1}{2}\right)t - 8k + \frac{3}{2} \geq 2k-1 > 0. \qquad (24)
\end{aligned}
$$

Thus by (23) and (24), $|E(G)| < f_1(n,k) = \binom{n-4k-4}{2} + 2k(k+1) + 6$, contrary to the condition $|E(G)| \geq \binom{n-4k-4}{2} + 2k(k+1) + 6$. Thus $G' = K_{2,3}$. As $\delta(G) \geq k \geq 4$ and $|E(G)| \geq \binom{n-4k-4}{2} + 2k(k+1) + 6$, it is routine to prove that $G \in \mathcal{K}'_{2,3}(n,k) \cup \mathcal{K}''_{2,3}(n,k)$, contrary to the assumption that $G \notin \mathcal{K}'_{2,3}(n,k) \cup \mathcal{K}''_{2,3}(n,k)$. Thus we have G' is supereulerian. The proof of the theorem is completed. ■

2.3. *Proof of Theorem 1.7*

Let G' be the reduction of G with $c = |V(G')|$. As P is not supereulerian, and P is the reduction for any $G \in \mathcal{P}(n,k)$, by (i) of Theorem 2.1, for any $G \in \mathcal{P}(n,k)$, G is not supereulerian. Thus if G is supereulerian, then $G \notin \mathcal{P}(n,k)$.

Therefore, in the rest of the proof, we may assume $G \notin \mathcal{P}(n,k)$ to prove that G is supereulerian. We argue by contradiction, and assume that G is not supereulerian. By (i) of Theorem 2.1, G' is not supereulerian. Then the following claim holds.

Claim 6: $F(G') \geq 3$.

Assume that $F(G') \leq 2$. By Theorem 2.4, either G' is collapsible or G' is a K_2 or a $K_{2,t}$ for some integer $t \geq 1$, contrary to the assumption G' is not supereulerian or $\delta(G') \geq \delta(G) \geq k \geq 3$. This proves Claim 1.

Then by Theorem 2.3 and Claim 1,

$$2|V(G')| - |E(G')| \geq 5. \qquad (25)$$

By (25) and by the facts $|V(G')| = \sum_{i\geq 1} d_i(G')$ and $2|E(G')| = \sum_{i\geq 1} id_i(G')$, we have

$$d_3(G') \geq 10 + \sum_{i\geq 5}(i-4)d_i(G') \geq 10 + \sum_{i\geq 5} d_i(G'). \qquad (26)$$

Claim 7: $c = 10$.

By contradiction, we assume that $c \geq 11$. Since $\delta(G) \geq k \geq 4$, by Lemma 2.5, $|V(H_i)| \geq \delta(G) + 1 \geq k+1$ for any $1 \leq i \leq 10$. Thus $n = \sum_{i=1}^{c} |V(H_i)| = \sum_{i=1}^{10} |V(H_i)| + \sum_{i=11}^{c} |V(H_i)| \geq 10(k+1) + c - 10 = 10k + c$. By Lemma 2.6 and (25),

$$|E(G)| \leq \sum_{i=1}^{c} \binom{|V(H_i)|}{2} + |E(G')| \leq \binom{n-(c-10)-9(k+1)}{2} + 9\binom{k+1}{2} + 2c - 5$$

$$= \binom{n-9k-c+1}{2} + \frac{9}{2}k(k+1) + 2c - 5.$$

As $n \geq 10k + c$, $k \geq 4$ and $c \geq 11$, we have

$$\binom{n-9k-9}{2} + \frac{9}{2}k(k+1) + 15 - \left[\binom{n-9k-c+1}{2} + \frac{9}{2}k(k+1) + 2c - 5\right]$$
$$= \frac{1}{2}(c-10)(2n-18k-13-c) \geq (c-10)(2k+c-13) > 0.$$

It follows that $|E(G)| \leq \binom{n-9k-c+1}{2} + \frac{9}{2}k(k+1) + 2c - 5 < \binom{n-9k-9}{2} + \frac{9}{2}k(k+1) + 15$, contrary to (7). This proves Claim 2.

Then by Claim 2 and Theorem 2.11, $G' = P$. As $\delta(G) \geq k \geq 4$ and $|E(G)| \geq \binom{n-9k-9}{2} + \frac{9}{2}k(k+1) + 15$, it is routine to show that $G \in \mathcal{P}(n,k)$, contrary to the fact that $G \notin \mathcal{P}(n,k)$. Thus G' is supereulerian. This completes the proof of Theorem 1.7. ■

3. Eigenvalues of supereulerian graphs

In this section, we aim to prove Theorems 1.8, 1.9 and 1.10. We start with the following two lemmas, which be utilized in this section.

Lemma 3.1 (Hong et al. [24] and Nikiforov [25]): *Let G be a graph of order n with the minimum degree $\delta(G) \geq k$. Then*

$$\lambda(G) \leq \frac{k-1+\sqrt{(k+1)^2+4(2|E(G)|-nk)}}{2}.$$

Remark: As G is a simple graph, if $n = |V(G)|$, then $2|E(G)| \leq n(n-1)$. Standard calculus arguments can be applied to show that the function $h(x) = \frac{x-1+\sqrt{(x+1)^2+4(2|E(G)|-nx)}}{2}$ is decreasing in x for $x \in [1, n-1]$.

Lemma 3.2 (Feng [26]): *Let G be a graph of order n. Then*

$$q(G) \leq \frac{2|E(G)|}{n-1} + n - 2.$$

Let $\pi = (V_1, \ldots, V_t)$ be a partition of $V(G)$. For $1 \leq i, j \leq t$, let b_{ij} denote the average number of neighbours in V_j of the vertices in V_i. The *quotient matrix* of this partition is the $t \times t$ matrix $B = G/\pi$ whose (i,j)th entry equals b_{ij}. The partition is called *equitable* if for each i, j, every vertex in V_i has same number of neighbours in V_j. The following theorems are useful.

Lemma 3.3 (Godsil and Royle [27]): *If the partition of graph G is equitable, then the spectral radius of the quotient matrix is equal to the spectral radius of G.*

Given two distinct vertices u, v in a graph G, the graph $G' = G'(u,v)$ is obtained from G by replacing all edges vw by uw for each $w \in N_G(v) \setminus (N_G(u) \cup \{u\})$. This operation is called the *Kelmans transformation* [28]. By utilizing this transformation, the following results have been proved.

Lemma 3.4: *Let G be a graph and G' be the graph obtained from G by some Kelmans transformation. Then*

(i) *(Csikvári [29])* $\lambda(G) \le \lambda(G')$.
(ii) *(Li and Ning [14])* $q(G) \le q(G')$.

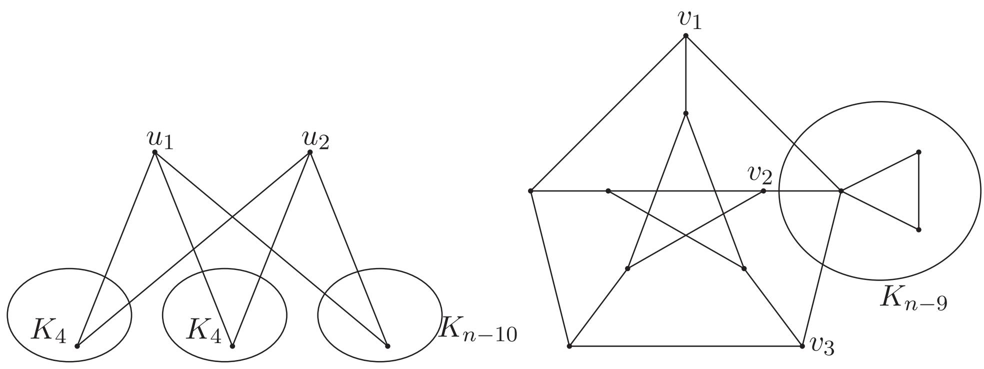
Let $u_1, u_2 \in V(G)$ denote the two vertices of degree 3 in $K_{2,3}$, and denote $V(K_{2,3}) - \{u_1, u_2\} = \{v_1, v_2, v_3\}$. Suppose that $K^1 \cong K_{n-10}$, $K^2 \cong K^3 \cong K_4$ are three vertex disjoint graphs. Define $K_{2,3}(n)$ to be the graph formed from $K_{2,3}$ by identifying v_i with a vertex in K^i, for each $1 \le i \le 3$. Equivalently, we have

$$K_{2,3}(n) = \{G : G \in \mathcal{K}_{2,3}(n) \text{ with } N_{K_{2,3}(n)}(u_1) = N_{K_{2,3}(n)}(u_2)\}.$$

Let $u \in P$ be a vertex with $N_P(u) = \{v_1, v_2, v_3\}$ and let $K \cong K_{n-9}$. Define $P(n)$ to be the graph formed from P by identifying a vertex in K with u. Equivalently, we have

$$P(n) = \{G : G \in \mathcal{P}(n) \text{ with } N_{P(n)}(v_1) \cap N_{P(n)}(v_2) \cap N_{P(n)}(v_3) \neq \emptyset\}.$$

As illustrations, both $K_{2,3}(n)$ and $P(n)$ are shown below. And they will be used to prove Lemma 3.5.



$K_{2,3}(n)$ $P(n)$

Lemma 3.5: *Each of the following holds:*

(i) *for any* $G \in \mathcal{K}_{2,3}(n) \cup \mathcal{K}'_{2,3}(n)$ *with* $n \ge 15$, $\lambda(G) < n - 10$ *and* $q(G) < 2n - 20$;
(ii) *for any* $G \in \mathcal{P}_1(n) \setminus \mathcal{P}(n)$, *if* $n \ge 15$, *then* $\lambda(G) < n - 10$. *And if* $n \ge 30$, *then* $q(G) < 2n - 20$.

Proof of Lemma 3.5(i): Adopting the notation in the definition of $K_{2,3}(n)$ and $\mathcal{K}_{2,3}(n)$, we justify the following claim.

Claim 8: *For any* $G \in \mathcal{K}_{2,3}(n)$ *with* $n \ge 15$, $\lambda(G) \le \lambda(K_{2,3}(n))$ *and* $q(G) \le q(K_{2,3}(n))$.

For any $G \in \mathcal{K}_{2,3}(n)$ with $G \neq K_{2,3}(n)$, by the definition of graphs in $\mathcal{K}_{2,3}(n)$, there exists an edge $ab \in E(G)$ with $a \in N_G(u_1)$ and $b \in N_G(u_2)$. For such an $ab \in E(G)$, we let $u = a$, $v = b$ in G. Thus $N_G(v)\backslash(N_G(u) \cup \{u\}) = \{u_2\}$. Let $G_1' = G_1'(u, v)$ be a Kelmans transformation of G. Then by Lemma 3.4, $\lambda(G) \leq \lambda(G_1')$ and $q(G) \leq q(G_1')$. If there exists such an edge $ab \in E(G_1')$, we repeat the above step until there is no such ab. And let G_2' be the final graph we obtained, then it is obvious that $G_2' = K_{2,3}(n)$. Thus we have $\lambda(G) \leq \lambda(G_2') = \lambda(K_{2,3}(n))$ and $q(G) \leq q(G_2') = q(K_{2,3}(n))$. This proves Claim 1.

For graph $K_{2,3}(n)$, define $X = \{v \in V(K_{2,3}(n)) : d_{K_{2,3}(n)}(v) = n - 9\}$, $Y = \{v \in V(K_{2,3}(n)) : d_{K_{2,3}(n)}(v) = n - 11 \text{ and } v \in N_{K_{2,3}(n)}(X)\}$, $U = \{v \in V(K_{2,3}(n)) : d_{K_{2,3}(n)}(v) = 3 \text{ and } v \in N_{K_{2,3}(n)}(X)\}$, $V = \{v \in V(K_{2,3}(n)) : d_{K_{2,3}(n)}(v) = 5 \text{ and } v \in N_{K_{2,3}(n)}(U)\}$, and $W = \{v \in V(K_{2,3}(n)) : d_{K_{2,3}(n)}(v) = 3 \text{and} v \in N_{K_{2,3}(n)}(V)\}$. Then the following claim holds.

Claim 9: *For any* $G \in \mathcal{K}_{2,3}(n) \cup \mathcal{K}_{2,3}'(n)$, $\lambda(G) < n - 10$.

Let $K_{2,3}(n)/\pi$ be the quotient matrix of $K_{2,3}(n)$ of the partition $\pi = (U, V, W, X, Y)$. Then by definition, this partition is equitable and

$$K_{2,3}(n)/\pi = \begin{pmatrix} 0 & 2 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 & n-11 \\ 0 & 0 & 0 & 1 & n-12 \end{pmatrix}.$$

Thus the characteristic polynomial $\det(\lambda I_n - K_{2,3}(n)/\pi)$ of matrix $K_{2,3}(n)/\pi$ is equal to

$$f(\lambda) = \lambda^5 + (10 - n)\lambda^4 + (n - 22)\lambda^3 + (11n - 118)\lambda^2 + (73 - 5n)\lambda + 160 - 14n.$$

As $n \geq 15$, by algebraic manipulations, we have the following inequalities holds:

$$\begin{aligned}
f(n-10) &= (n^2 - 21n + 90)(n-10)^2 + (7n - 61)(n-10) + 20 > 0, \\
f'(n-10) &= 5\lambda^4 + 4(10-n)\lambda^3 + 3(n-22)\lambda^2 + 2(11n-118)\lambda + 73 - 5n\big|_{\lambda=n-10} \\
&= (n^2 - 17n + 34)(n-10)^2 + (22n - 241)(n-10) + 23 > 0, \\
f''(n-10) &= 20\lambda^3 + 12(10-n)\lambda^2 + 6(n-22)\lambda + 2(11n-118)\big|_{\lambda=n-10} \\
&= (8n^2 - 154n + 668)(n-10) + 2(11n - 118) > 0, \\
f'''(n-10) &= 60\lambda^2 + 24(10-n)\lambda + 6(n-22)\big|_{\lambda=n-10} = 6(6n^2 - 119n + 578) > 0, \\
f^{(4)}(n-10) &= 120\lambda + 24(10-n)\big|_{\lambda=n-10} = 96(n-10) > 0, \\
f^{(5)}(n-10) &= 120 > 0.
\end{aligned}$$

Thus by the Fourier–Budan theorem [30], there is no root of the polynomial $f(\lambda)$ in the interval $[n-10, +\infty)$. Then $\lambda(K_{2,3}(n)/\pi) < n - 10$. As the partition π is equitable, by Lemma 3.3, we have $\lambda(K_{2,3}(n)) < n - 10$. And by the definition of $\mathcal{K}_{2,3}(n)$ and $\mathcal{K}_{2,3}'(n)$, it is obvious that for any $G_1 \in \mathcal{K}_{2,3}'(n)$, there exists a graph $G_2 \in \mathcal{K}_{2,3}(n)$ such that $G_1 \subseteq G_2$. Thus by Claim 1, for any $G \in \mathcal{K}_{2,3}(n) \cup \mathcal{K}_{2,3}'(n)$, $\lambda(G) < n - 10$. This proves Claim 2.

Claim 10: *For any* $G \in \mathcal{K}_{2,3}(n) \cup \mathcal{K}'_{2,3}(n)$, $q(G) < 2n - 20$.

Let $Q(K_{2,3}(n))/\pi$ be the quotient matrix of $Q(K_{2,3}(n))$ of the partition $\pi = (U, V, W, X, Y)$. Then by definition, this partition is equitable and

$$Q(K_{2,3}(n))/\pi = \begin{pmatrix} 3 & 2 & 0 & 1 & 0 \\ 2 & 5 & 3 & 0 & 0 \\ 0 & 1 & 5 & 0 & 0 \\ 2 & 0 & 0 & n-9 & n-11 \\ 0 & 0 & 0 & 1 & 2n-23 \end{pmatrix}.$$

Thus the characteristic polynomial $det(\lambda I_n - Q(K_{2,3}(n))/\pi)$ of matrix $Q(K_{2,3}(n))/\pi$ is equal to

$$\begin{aligned} g(q) = q^5 &+ (19-3n)q^4 + (2n^2 - 3n - 152)q^3 - (26n^2 - 406n + 1370)q^2 \\ &+ (96n^2 - 1918n + 9408)q - 92n^2 + 2020n - 11040. \end{aligned}$$

As $n \geq 15$, by algebraic manipulations, we have the following inequalities holds:

$$\begin{aligned} g(2n-20) &= (4n^2 - 110n + 750)(2n-20)^2 + (36n - 532)(2n-20) \\ &\quad + 180n - 1840 > 0, \\ g'(2n-20) &= 5q^4 + 4(19-3n)q^3 + 3(2n^2 - 3n - 152)q^2 - 2(26n^2 - 406n + 1370)q \\ &\quad + 96n^2 - 1918n + 9408\big|_{q=2n-20} \\ &= (n^2 - 16n + 16)(2n-20)^2 + (2n^2 - 44n + 210)(n-15)(2n-20) \\ &\quad + 52n^2 - 978n + 4408 > 0, \\ g''(2n-20) &= 20q^3 + 12(19-3n)q^2 + 6(2n^2 - 3n - 152)q - 52n^2 \\ &\quad + 812n - 2740\big|_{q=2n-20} \\ &= (20n^2 - 468n + 2665)(2n-20) + 18n > 0, \\ g'''(2n-20) &= 60q^2 + 24(19-3n)q + 6(2n^2 - 3n - 152)\big|_{q=2n-20} \\ &= 18(6n^2 - 137n + 776) > 0, \\ g^{(4)}(2n-20) &= 120q + 24(19-3n)\big|_{q=2n-20} = 24(7n - 81) > 0, \\ g^{(5)}(2n-20) &= 120 > 0. \end{aligned}$$

Thus by the Fourier–Budan theorem [30], there is no root of the polynomial $g(q)$ in the interval $[2n-20, +\infty)$. Then $q(Q(K_{2,3}(n))/\pi) < 2n - 20$. As the partition π is equitable, by Lemma 3.3(ii), we have $q(K_{2,3}(n)) < 2n - 20$. And by the definition of $\mathcal{K}_{2,3}(n)$ and $\mathcal{K}'_{2,3}(n)$, it is obvious that for any $G_1 \in \mathcal{K}'_{2,3}(n)$, there exists a graph $G_2 \in \mathcal{K}_{2,3}(n)$ such that $G_1 \subseteq G_2$. Thus by Claim 1, for any $G \in \mathcal{K}_{2,3}(n) \cup \mathcal{K}'_{2,3}(n)$, $q(G) < 2n - 20$. This proves Claim 3. Thus Lemma 3.5(i) follows from Claims 2 and 3. ■

Proof of Lemma 3.5(ii): Adopting the notation in the definition of $P(n)$, we let $x \in V(P) \cap V(K)$, and $y_1, y_2 \in V(K) - V(P)$. Then up to isomorphism, $G_1 = P(n) - xy_1$ and

$G_2 = P(n) - y_1y_2$ are the only two graphs obtained from $P(n)$ by deleting one edge not in $E(P)$.

Using the notation in Figure 2, let $u = x, v = y_2$ in G_1. Thus $N_G(v)\backslash(N_G(u) \cup \{u\}) = \{y_1\}$. Let $G_1' = G_1'(u, v)$ be a Kelmans transformation of G_1. Then $G_1' = G_2$. By Lemma 3.4, $\lambda(G_1) \leq \lambda(G_1') = \lambda(G_2)$ and $q(G_1) \leq q(G_1') = q(G_2)$. Then we obtain the following Claim.

Claim 11: *For any $G \in \mathcal{P}_1(n) \setminus \mathcal{P}(n)$ with $n \geq 15$, $\lambda(G) \leq \lambda(G_2)$ and $q(G) \leq q(G_2)$.*

For any $G \in \mathcal{P}'(n)$ and $G \neq G_2$ with $n \geq 15$, let $e = ab \notin E(G)$ and G is obtained from one graph in $\mathcal{P}(n)$ by deleting e. If $N_G(a) \neq N_G(b)$ (equivalently, $(\bigcup_{i=1}^{3} N_G(v_i)) \cap \{a, b\} \neq \emptyset$), by $n \geq 15$, there exists such an edge $cd \in E(G)$ such that $N_G(c) = N_G(d)$ (equivalently, $(\bigcup_{i=1}^{3} N_G(v_i)) \cap \{c, d\} = \emptyset$) and $\{a, b\} \subseteq N_G(c) = N_G(d)$. let $u = a$, $v = c$ in G. Thus $N_G(v)\backslash(N_G(u) \cup \{u\}) = \{b\}$. Let $G_1' = G_1'(u, v)$ be a Kelmans transformation of G_1. By Lemma 3.4, $\lambda(G) \leq \lambda(G_1')$ and $q(G) \leq q(G_1')$. Let $u = b$, $v = d$ in G_1'. Thus $N_{G_1'}(v)\backslash(N_{G_1'}(u) \cup \{u\}) = \{c\}$. Let $G_2' = G_2'(u, v)$ be a Kelmans transformation of G_1'. By Lemma 3.4, $\lambda(G) \leq \lambda(G_1') \leq \lambda(G_2')$ and $q(G) \leq q(G_1') \leq q(G_2')$.

For graph G_2', if $\bigcap_{i=1,2,3} N_{G_2'}(v_i) = \emptyset$, without loss of generality, we may assume that $a_1a_2 \in E(G_2')$ such that $a_1 \in N_{G_2'}(v_1) \setminus N_{G_2'}(v_2)$ and $a_2 \in N_{G_2'}(v_2) \setminus N_{G_2'}(v_1)$. Let $u = a_1, v = a_2$ in G_2'. Thus $N_{G_2'}(v)\backslash(N_{G_2'}(u) \cup \{u\}) = \{v_1\}$. Let $G_3' = G_3'(u, v)$ be a Kelmans transformation of G_2'. By Lemma 3.4, $\lambda(G_2') \leq \lambda(G_3')$ and $q(G_2') \leq q(G_3')$. For graph G_3', if $\bigcap_{i=1,2,3} N_{G_3'}(v_i) = \emptyset$, then we repeat the step above until $\bigcap_{i=1,2,3} N_{G_3'}(v_i) \neq \emptyset$. And let G_4' be the final graph we obtained, then it's routine to obtain that $G_4' = G_2$. By Lemma 3.4, $\lambda(G_2') \leq \lambda(G_3') \leq \lambda(G_2)$ and $q(G_2') \leq q(G_3') \leq q(G_2)$. Thus for any $G \in \mathcal{P}'(n)$, we have $\lambda(G) \leq \lambda(G_2)$ and $q(G) \leq q(G_2)$.

As all complete graphs of order $n \geq 3$ are collapsible, and so by Theorem 2.2, for any $G \in \mathcal{P}'(n)$, $G' = P$. Thus by the definition of $\mathcal{P}_1(n)$ and $\mathcal{P}'(n)$, we have for any $H_1 \in \mathcal{P}_1(n) \setminus \mathcal{P}(n)$, there exists a graph $H_2 \in \mathcal{P}'(n)$ such that $H_1 \subseteq H_2$. Thus for any $G \in \mathcal{P}_1(n) \setminus \mathcal{P}(n)$, we have $\lambda(G) \leq \lambda(G_2)$ and $q(G) \leq q(G_2)$. This proves Claim 1.

For graph G_2, define $X = \{v \in V(G_2) : d_{P(n)}(v) = n - 7\}$, $Y = \{v \in V(G_2) : d_{G_2}(v) = n - 11\}$, $W = \{v \in V(G_2) : d_{G_2}(v) = n - 10\}$, $U = \{v \in V(G_2) : d_{G_2}(v) = 3$ and $v \notin$

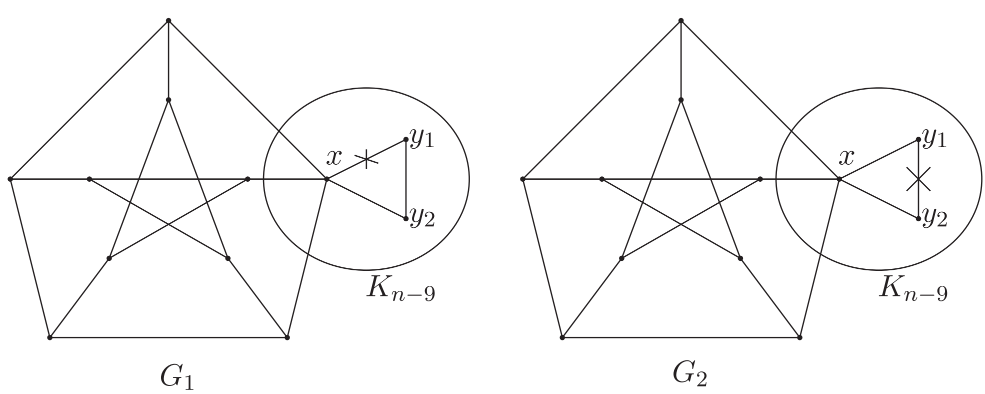


Figure 2. The graphs obtained from $P(n)$ by deleting one edge not in $E(P)$.

$N_{G_2}(X)\}$, and $V = \{v \in V(G_2) : d_{G_2}(v) = 3 \text{ and } v \in N_{G_2}(X)\}$. Then we obtain the following claim.

Claim 12: *For any $G \in \mathcal{P}_1(n) \setminus \mathcal{P}(n)$ with $n \geq 15$, $\lambda(G) < n - 10$.*

Let G_2/π be the quotient matrix of G_2 of the partition $\pi' = (U, V, X, W, Y)$. Then by definition, we get this partition is equitable and

$$G_2/\pi' = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & n-12 & 2 \\ 0 & 0 & 1 & n-13 & 2 \\ 0 & 0 & 1 & n-12 & 0 \end{pmatrix}.$$

Thus the characteristic polynomial $det(\lambda I_n - G_2/\pi')$ of matrix G_2/π' is equal to

$$g(\lambda) = \lambda^5 + (11-n)\lambda^4 + (3-n)\lambda^3 + (9n-105)\lambda^2 + (10n-106)\lambda + 100 - 8n.$$

As $n \geq 15$, by algebraic manipulations, we obtain the following evaluations of g and its higher order of derivatives.

$$\begin{aligned} g(n-10) &= (2n-35)(n-10)^2 + (10n-114)(n-10) + 20 > 0, \\ g'(n-10) &= 5\lambda^4 + 4(11-n)\lambda^3 + 3(3-n)\lambda^2 + 2(9n-105)\lambda + (10n-106)\big|_{\lambda=n-10} \\ &= (n^2 - 19n + 69)(n-10)^2 + (18n-210)(n-10) + 10n - 106 > 0, \\ g''(n-10) &= 20\lambda^3 + 12(11-n)\lambda^2 + 6(3-n)\lambda + 2(9n-105)\big|_{\lambda=n-10} \\ &= (8n^2 - 154n + 698)(n-10) + 18n - 210 > 0, \\ g'''(n-10) &= 60\lambda^2 + 24(11-n)\lambda + 6(3-n)\big|_{\lambda=n-10} = 36(n-10)^2 + 18n - 222 > 0, \\ g^{(4)}(n-10) &= 120\lambda + 24(11-n)\big|_{\lambda=n-10} = 96(n-10) + 24 > 0, \\ g^{(5)}(n-10) &= 120 > 0. \end{aligned}$$

Thus by the Fourier–Budan theorem [30], there is no root of the polynomial $g(\lambda)$ in the interval $[n-10, +\infty)$. Then $\lambda(G_2/\pi') < n - 10$. As the partition π' is equitable, by Lemma 3.3, we have $\lambda(G_2) < n - 10$. Then for any $G \in \mathcal{P}_1(n) \setminus \mathcal{P}(n)$, $\lambda(G) < n - 10$. This proves Claim 2.

Claim 13: *For any $G \in \mathcal{P}_1(n) \setminus \mathcal{P}(n)$ with $n \geq 30$, $q(G) < 2n - 20$.*

Let $Q(G_2)/\pi'$ be the quotient matrix of G_2 of the partition $\pi' = (U, V, X, W, Y)$. Then by definition, we get this partition is equitable and

$$Q(G_2)/\pi' = \begin{pmatrix} 5 & 1 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 \\ 0 & 3 & n-7 & n-12 & 2 \\ 0 & 0 & 1 & 2n-23 & 2 \\ 0 & 0 & 1 & n-12 & n-11 \end{pmatrix}.$$

Thus the characteristic polynomial $det(qI_n - Q(G_2)/\pi')$ of matrix $Q(G_2)/\pi'$ is equal to

$$\begin{aligned} g(q) &= q^5 + (33-4n)q^4 + (5n^2 - 71n + 207)q^3 - (2n^3 - 22n^2 - 152n + 1681)q^2 \\ &\quad + (16n^3 - 437n^2 + 3789n - 10080)q - 26n^3 + 836n^2 - 8882n + 31104. \end{aligned}$$

As $n \geq 30$, by algebraic manipulations, we have the following inequalities holds:

$$\begin{aligned} g(2n-20) &= (2n^2 - 85n + 776)(2n-20) + 6n - 176 > 0, \\ g'(2n-20) &= 5q^4 + 4(33-4n)q^3 + 3(5n^2 - 71n + 207)q^2 \\ &\quad - 2(2n^3 - 22n^2 - 152n + 1681)q \\ &\quad + 16n^3 - 437n^2 + 3789n - 10080\big|_{q=2n-20} \\ &= (n-15)(n^2 - 18n + 83)(4n-40) + 3n^2 + 5n - 240 > 0, \\ g''(2n-20) &= 20q^3 + 12(33-4n)q^2 + 6(5n^2 - 71n + 207)q - 2(2n^3 - 22n^2 - 152n \\ &\quad + 1681)\big|_{q=2n-20} \\ &= (3n^2 - 68n + 373)(8n-80) + 4n + 38 > 0, \\ g'''(2n-20) &= 60q^2 + 24(33-4n)q + 6(5n^2 - 71n + 207)\big|_{q=2n-20} \\ &= 6(13n^2 - 287n + 1567) > 0, \\ g^{(4)}(2n-20) &= 120q + 24(33-4n)\big|_{q=2n-20} = 24(6n-67) > 0, \\ g^{(5)}(2n-20) &= 120 > 0. \end{aligned}$$

Thus by the Fourier–Budan theorem [30], there is no root of the polynomial $g(q)$ in the interval $[2n-20, +\infty)$. Then $q(Q(G_2)/\pi') < 2n - 20$. As the partition π' is equitable, by Lemma 3.3, we have $q(G_2) < 2n - 20$. Then for any $G \in \mathcal{P}_1(n) \setminus \mathcal{P}(n)$, $q(G) < 2n - 20$. This proves Claim 3. ■

Proof of Theorem 1.8(1): By Lemma 3.1 and $\delta(G) \geq 3$, we have

$$n - 10 \leq \lambda(G) \leq \frac{3 - 1 + \sqrt{(3+1)^2 + 4(2|E(G)| - 3n)}}{2},$$

which implies that

$$|E(G)| \geq \frac{n^2 - 19n + 117}{2}.$$

As $n \geq 15$, we have

$$|E(G)| \geq \frac{n^2 - 19n + 117}{2} \geq \binom{n-10}{2} + 17.$$

Then by Theorem 1.3, G is supereulerian if and only if $G \notin \mathcal{K}_{2,3}(n) \cup \mathcal{P}_1(n) \cup \mathcal{K}'_{2,3}(n)$. Thus in the rest of the proof, we assume that $G \notin \mathcal{P}(n)$ to prove that G is supereulerian.

As for any graph $G \in \mathcal{P}(n)$, $K_{n-9} \subsetneq G$, it follows that $\lambda(G) > n - 10$. And by Lemma 3.5, for any $G \in \mathcal{K}_{2,3}(n) \cup \mathcal{K}'_{2,3}(n) \cup \mathcal{P}_1(n) \setminus \mathcal{P}(n)$, $\lambda(G) < n - 10$. Therefore, if $G \notin \mathcal{P}(n)$, then G is supereulerian. This proves (1). ■

Proof of Theorem 1.8(2): By Lemma 3.2, we have

$$2n - 20 \leq q(G) \leq \frac{2|E(G)|}{n-1} + n - 2,$$

which implies that

$$|E(G)| \geq \frac{n^2 - 19n + 18}{2}.$$

As $n \geq 63$, it follows that

$$|E(G)| \geq \frac{n^2 - 19n + 18}{2} \geq \binom{n-10}{2} + 17.$$

Then by Theorem 1.3, G is supereulerian if and only if $G \notin \mathcal{K}_{2,3}(n) \cup \mathcal{P}_1(n) \cup \mathcal{K}'_{2,3}(n)$.

As for any graph $G \in \mathcal{P}(n)$, $K_{n-9} \subsetneq G$, it follows that $q(G) > 2n - 20$. And by Lemma 3.5, for any $G \in \mathcal{K}_{2,3}(n) \cup \mathcal{K}'_{2,3}(n) \cup \mathcal{P}_1(n) \setminus \mathcal{P}(n)$, $q(G) < 2n - 20$. Therefore, if $G \notin \mathcal{P}(n)$, then G is supereulerian. This proves (2). ■

Proof of Theorem 1.9(1): By Lemma 3.1 and $\delta(G) \geq k$, we have

$$\begin{aligned}\frac{k - 1 + \sqrt{4n^2 - 36(k+1)n + 81(k+1)^2 + 48}}{2} &< \lambda(G) \\ &\leq \frac{k - 1 + \sqrt{(k+1)^2 + 4(2|E(G)| - nk)}}{2},\end{aligned}$$

which implies that

$$|E(G)| > \binom{n-4k-4}{2} + 2k(k+1) + 6.$$

Thus by Theorem 1.5, G is supereulerian if and only if $G \notin \mathcal{K}'_{2,3}(n,k) \cup \mathcal{K}''_{2,3}(n,k)$. And for any $G \in \mathcal{K}'_{2,3}(n,k) \cup \mathcal{K}''_{2,3}(n,k)$, $|E(G)| = \binom{n-4k-4}{2} + 2k(k+1) + 6$. Thus if $|E(G)| > \binom{n-4k-4}{2} + 2k(k+1) + 6$, then G is supereulerian. This proves (1). ■

Proof of Theorem 1.9(2): By Lemma 3.2, we have

$$2n - 8k - 10 + \frac{20k^2 + 32k + 24}{n-1} < q(G) \leq \frac{2|E(G)|}{n-1} + n - 2,$$

which implies that

$$e(G) > \binom{n-4k-4}{2} + 2k(k+1) + 6.$$

Thus by Theorem 1.5, G is supereulerian if and only if $G \notin \mathcal{K}'_{2,3}(n,k) \cup \mathcal{K}''_{2,3}(n,k)$. And for any $G \in \mathcal{K}'_{2,3}(n,k) \cup \mathcal{K}''_{2,3}(n,k)$, $|E(G)| = \binom{n-4k-4}{2} + 2k(k+1) + 6$. Thus if $|E(G)| > \binom{n-4k-4}{2} + 2k(k+1) + 6$, then G is supereulerian. This proves (2). ■

Proof of Theorem 1.10(1): By Lemma 3.1 and $\delta(G) \geq k$, we have

$$\frac{k-1+\sqrt{4n^2-76(k+1)n+361(k+1)^2+120}}{2} < \lambda(G) \leq \frac{k-1+\sqrt{(k+1)^2+4(2|E(G)|-nk)}}{2},$$

which implies that

$$|E(G)| > \binom{n-9k-9}{2} + \frac{9}{2}k(k+1) + 15.$$

Thus by Theorem 1.7, G is supereulerian if and only if $G \notin \mathcal{P}(n,k)$. As for any $G \in \mathcal{P}(n,k)$, $|E(G)| = \binom{n-9k-9}{2} + \frac{9}{2}k(k+1) + 15$, then we have G is supereulerian when $|E(G)| > \binom{n-9k-9}{2} + \frac{9}{2}k(k+1) + 15$. This proves (1). ■

Proof of Theorem 1.10(2): By Lemma 3.2, we have

$$2n - 20 - 18k + \frac{90k^2+162k+102}{n-1} < q(G) \leq \frac{2|E(G)|}{n-1} + n - 2,$$

which implies that

$$|E(G)| > \binom{n-9k-9}{2} + \frac{9}{2}k(k+1) + 15.$$

Thus by Theorem 1.7, G is supereulerian if and only if $G \notin \mathcal{P}(n,k)$. As for any $G \in \mathcal{P}(n,k)$, $|E(G)| = \binom{n-9k-9}{2} + \frac{9}{2}k(k+1) + 15$, then we have G is supereulerian when $|E(G)| > \binom{n-9k-9}{2} + \frac{9}{2}k(k+1) + 15$. This proves (2). ■

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

This project is supported in part by the China Scholarship Council [grant number 201906150105], Guangzhou Science and Technology Bureau Project [grant number 201904010339], Guangdong Basic and Applied Basic Research Foundation [grant number 2021A1515012046], and by the National Science Foundation of China [grant numbers 11771039, 11771443].

ORCID

Hong-Jian Lai http://orcid.org/0000-0001-7698-2125

References

[1] Bondy JA, Murty USR. Graph theory. New York (NY): Springer; 2008.
[2] Catlin PA. A reduction method to find spanning eulerian subgraphs. J Graph Theory. 1988;12:29–45.

[3] Boesch FT, Suffel C, Tindell R. The spanning subgraphs of eulerian graphs. J Graph Theory. 1977;1:79–84.
[4] Pulleyblank WR. A note on graphs spanned by eulerian graphs. J Graph Theory. 1979;3:309–310.
[5] Catlin PA. Supereulerian graphs: a survey. J Graph Theory. 1992;16:177–196.
[6] Chen ZH, Lai H-J. Reduction techniques for super-eulerian graphs and related topics-a survey. In: Combinatorics and graph theory '95. Vol. 1 (Hefei). River Edge: World Scientific Publishing; 1995. pp. 53–69.
[7] Lai H-J, Shao Y, Yan H. An update on supereulerian graphs. WSEAS Trans Math. 2013;12:926–940.
[8] Jaeger F. A note on subeulerian graphs. J Graph Theory. 1979;3:91–93.
[9] Cai XT. Connected eulerian spanning subgraphs. Chinese Quart J Math. 1990;5:78–84.
[10] Ore O. Arc coverings of graphs. Ann Mat Pura Appl. 1961;55(4):315–321.
[11] Catlin PA, Chen ZH. Nonsupereulerian graphs with large size, graph theory, combinatorics, algorithm and applications. In: Proceedings of the 2nd International Conference; 1991. p. 83–95.
[12] Erdős P. Remarks on a paper of Pósa. Magyar Tud Akad Mat Kut Int Közl. 1962;7:227–229.
[13] Benediktovich VI. Spectral condition for Hamiltonicity of a graph. Linear Algebra Appl. 2016;494:70–79.
[14] Li BL, Ning B. Spectral analogues of Erdős' and Moon-Moser's theorems on Hamilton cycles. Linear Multilinear Algebra. 2016;64:2252–2269.
[15] Liu M, Ch. Das K, Lai H-J. The (signless) Laplacian spectral radii of c-cyclic graphs with n vertices, girth g and k pendant vertices. Linear Multilinear Algebra. 2017;65(5):869–881.
[16] Liu M, Lai H-J, Das KCh. Spectral results on Hamiltonian problem. Discrete Math. 2019;342:1718–1730.
[17] Liu M, Wu Y, Lai H-J. Unified spectral Hamiltonian results of balanced bipartite graphs and complementary graphs. Graphs Comb. 2020;36:1363–1390.
[18] Wei J, You ZF, Lai H-J. Spectral analogues of Erdős' theorem on Hamilton-connected graphs. Appl Math Comput. 2019;340:242–250.
[19] Catlin PA. Supereulerian graph, collopsible graphs and 4-cycles. Congressus Numerantium. 1987;56:223–246.
[20] Catlin PA, Han ZY, Lai H-J. Graphs without spanning closed trails. Discrete Math. 1996;160:81–91.
[21] Liu Q, Hong Y, Gu X, et al. Note on edge-disjoint spanning trees and eigenvalues. Linear Algebra Appl. 2014;458:128–133.
[22] Chen ZH, Lai H-J. Supereulerian graph and the Petersen graph, II. Ars Combinatoria. 1998;48:271–282.
[23] Chen ZH. Supereulerian graphs and the Petersen graph. J Comb Math Comb Comput. 1991;9:70–89.
[24] Hong Y, Shu J, Fang K. A sharp upper bound of the spectral radius of graphs. J Combin Theory B. 2001;81:177–183.
[25] Nikiforov V. Some inequalities for the largest eigenvalue of a graph. Combin Probab Comput. 2002;11(2):179–189.
[26] Feng LH, Yu GH. On three conjectures involving the signless Laplacian spectral of graphs. Publ Inst Math. 2009;85(99):35–38.
[27] Godsil CD, Royle GF. Algebraic graph theory. New York: Springer-Verlag; 2001.
[28] Kelmans AK. On graphs with randomly deleted edges. Acta Math Acad Sci Hungar. 1981;37:77–88.
[29] Csikvári P. On a conjecture of V. Nikiforov. Discrete Math. 2009;309(13):4522–4526.
[30] Prasolov VV. Polynomials. Moscow: MTSNMO; 2001.