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Group Colorings and DP-Colorings of Multigraphs Using Edge-Disjoint Decompositions

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Abstract

In (J Graph Theory 4:241–242, 1980), Burr proved that $\chi(G) \leq m_1 m_2 \ldots m_k$ if and only if G is the edge-disjoint union of k graphs $G_1, G_2, \ldots, G_k$ such that $\chi(G_i) \leq m_i$ for $1 \leq i \leq k$. This result established the practice of describing the chromatic number of a graph G which is the edge-disjoint union of k subgraphs $G_1, G_2, \ldots, G_k$ in terms of the chromatic numbers of these subgraphs, and more specific results and conjectures followed. We investigate possible extensions of this theorem of Burr to group coloring and DP-coloring of multigraphs, as well as extensions of another vertex coloring theorem involving arboricity. In particular, we determine the DP-chromatic number of all Halin graphs. In (J Graph Theory 50:123–129, 2005), it is conjectured that for any graph G, the list chromatic number is not higher than the group chromatic number of G. As related results, we show that the group list chromatic number of all multigraphs is at most the DP-chromatic number, and present an example G for which the group chromatic number of G is less than the DP-chromatic number of G.

Keywords Group-coloring · Group chromatic number · List group coloring · Edge-disjoint union of graphs · DP-coloring · Correspondence coloring

Mathematics Subject Classification 05C15

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1 Notation and Background

Graphs in this note are finite, loopless with multiple edge permitted. We follow Bondy and Murty [2] for undefined terms and notation. Thus $\chi(G)$, $\chi_L(G)$, $\delta(G)$ and $\kappa(G)$ denote the chromatic number, the list chromatic number, the minimum degree, and the connectivity of a graph G. For a positive integer n, let C_n and K_n denote the cycle of order n and the complete graph of order n, respectively. We use $H \subseteq G$ to mean that H is a subgraph of G. Let G be a graph and let X be a set of edges not in $E(G)$ and with ends in $V(G)$, then $G \cup X$ denotes the graph (possibly with multiple edges) with vertex set $V(G)$ and edges set $E(G) \cup X$. Given a graph G, define an equivalence relation " $\sim$ " on $E(G)$ such that $e_1 \sim e_2$ if and only if either $e_1 = e_2$ or e_1 and e_2 form a cycle of length two in G. Each equivalence is a **parallel class**. For $u, v \in V(G)$, let $m_G(u, v)$ denote the number of parallel edges between u and v. For any graph G we define the **simplification** of G, denoted by $\hat{G}$, to be the simple graph formed by removing all but one edge from each parallel class. For a graph H and a positive integer k, we define kH to be the graph obtained by replacing each edge of H by a class of k parallel edges joining the same pair of end vertices. For $V_1, V_2 \subseteq V(G)$ and $V_1 \cap V_2 = \emptyset$, let $G[V_1]$ denote the subgraph induced by V_1 and let $E_G[V_1, V_2]$ denote the set of edges between V_1 and V_2. We abbreviate $E_G[\{u\}, V_2]$ to $E_G[u, V_2]$, and omit the subscript when it is clear from context.

Throughout this paper, A denotes an (additive) abelian group with identify 0. Following [8], for any subset $A' \subseteq A$, define $F(G, A') = \{f : E(G) \to A'\}$. Let $D = D(G)$ be an orientation of G. We use (u, v) to denote an arc oriented from u to v. For a function $f \in F(G, A)$, an **(A, f)-coloring** of G under the orientation D is a function $c : V(G) \to A$ such that for any arc $e = (u, v) \in A(D)$, $c(u) - c(v) \neq f(e)$. A graph G is **A-colorable** under the orientation D if for any $f \in F(G, A)$, G has an (A, f)-coloring. It has been observed that (see Section 4 of [8], or Proposition 4.1 of [16]), for a given A, whether G is A-colorable is independent of the choice of the orientation D. The **group chromatic number** $\chi_g(G)$ of a graph G is the minimum integer m such that G is A-colorable for any group A with $|A| \geq m$ under the orientation D.

By definition, for any graph G, $\chi(G) \leq \chi_g(G)$. As noted in [14], $\chi_g(G) - \chi(G)$ can be arbitrarily large. It is observed that, unlike classical coloring, $\chi_g(G) > \chi_g(\hat{G})$ is possible. For example, it follows from the results below in Theorem 1.1 that $\chi_g(2K_2) = 3$ and $\chi_g(K_2) = 2$, though surely $\chi(2K_2) = \chi(K_2) = 2$.

Theorem 1.1 *Let G be a graph.*

(i) (*Jaeger, Linial, Payan, and Tarsi*, [8]) *If G is simple and planar, then* $\chi_g(G) \leq 6$.
(ii) (*Theorem* 1.2 *of* [13]) *If G is simple and does not have a* K_5*-minor, then* $\chi_g(G) \leq 5$.
(iii) (*Theorem* 2.8 *of* [15]) *If G simple and does not have a* $K_{3,3}$*-minor, then* $\chi_g(G) \leq 5$.

(iv) (*Theorem* 4.2 *of* [14]) *If G is simple and connected, then* $\chi_g(G) \leq \Delta(G) + 1$, *where equality holds if and only if G is a cycle or a complete graph.*

(v) (*Theorem* 1.3 *of* [18]) *If G is connected, then* $\chi_g(G) \leq \Delta(G) + 1$, *where equality holds if and only if G is a* kC_n *or a* kK_n, *for some positive integers k and n.*

The list coloring version of some of the results above can be found in [5] and [4].

Yet another type of coloring called **DP-coloring** or **correspondence coloring** has been recently introduced by Dvořák and Postle in [6]. We use a notation developed in [1].

Definition 1.2 Let G be a multigraph. A **cover** of G is a pair (L, H), where L is an assignment of pairwise disjoint sets to the vertices of G and H is a graph with vertex set $\bigcup_{v \in V(G)} L(v)$ satisfying the following:

1. $H[L(v)]$ is a complete graph for each $v \in V(G)$.
2. For each $u, v \in V(G)$, $E_H[L(u), L(v)]$ is the union of $m_G(u, v)$ (possibly empty) matchings.

Now, let (L, H) be a cover of a multigraph G. An (L, H)-**coloring** of G is an independent set in H of size $|V(G)|$, and a graph G with such a coloring is called (L, H)-**colorable**. A cover (L, H) of a graph G is said to be a k-**cover** if $|L(v)| = k$ for each $v \in V(G)$. If a graph G is (L, H)-colorable for any k-cover (L, H) then G is **DP-k-colorable**. Similarly, we say that G is **DP-degree-colorable** if it is (L, H)-colorable for any L where $|L(v)| = d(v)$ for each $v \in V(G)$. The **DP-chromatic number** is defined as $\chi_{DP}(G) = \min\{k \in \mathbb{N} : G \text{ is DP-}k\text{-colorable}\}$. In some other papers a slightly different definition for χ_{DP} is given, where we consider any cover such that $|L(v)| \geq k$ for any $v \in V(G)$. However, unlike in group colorings, it is worth noting that if G is DP-k-colorable, then G is (L, H)-colorable for any (L, H) such that $|L(v)| \geq k$ for each $v \in V(G)$. Therefore the two definitions are equivalent, and we state the simpler version here.

Definition 1.3 A k-cover (L, H) of a simple graph G is **full** if $E_H[L(u), L(v)]$ is a perfect matching for each $uv \in E(G)$. Similarly, a k-cover (L, H) of a multigraph G is full if $E_H[L(u), L(v)]$ is a union of $m_G(u, v)$ perfect matchings for each $uv \in E(G)$.

If we only use the definition for χ_{DP} given above, then we only need to consider k-covers. Furthermore, we only need to consider full covers of a graph G when determining $\chi_{DP}(G)$, since if a cover (L, H) is not full then there exists a full cover (L, H') where H is a proper subgraph of H' so that if G is (L, H')-colorable then it must also be (L, H)-colorable.

For a graph G, a group Γ, and some $f \in F(G, \Gamma)$, the Γ-colorability of a multigraph G corresponds to its (L, H)-colorability under a certain cover (L, H): We identify $L(v)$ with Γ for each $v \in V(G)$ and for each $e \in E(G)$ with endpoints $u, v \in V(G)$ we use an appropriate matching in H between $L(u)$ and $L(v)$ to mimic $f(e)$. Therefore when G is DP-k-colorable, it is Γ-colorable for any Γ with $|\Gamma| = k$, though the converse does not necessarily hold. It follows from the definitions then that for any graph G we have $\chi_g(G) \leq \chi_{DP}(G)$.

For a graph G, we define $\bar{\delta}(G) = \max\{\delta(H) : H \text{ is a subgraph of } G\}$. This is sometimes called the **degeneracy** of G. Below are collected some coloring results, some of which will be used throughout this paper. Part (ii) is not formally stated, but mentioned in passing by Dvořák and Postle in [6]. The proof closely resembles other similar degeneracy arguments, such as that for Theorem 1.4(i). Part (iii) is not used in this paper, but serves as some good background, along with [10] which specifies exactly which covers disallow a coloring in the context of (iii).

Theorem 1.4 *Let G be a graph and let Γ be a group.*

(i) ([18]) $\chi_g(G) \leq \bar{\delta}(G) + 1$
(ii) ([6]) $\chi_{DP}(G) \leq \bar{\delta}(G) + 1$.
(iii) (*Theorem* 9 *of* [1]) *If G is connected, then G is not DP-degree-colorable if and only if each block of G is one of the graphs kK_n and kC_n for some n and k.*
(iv) (*Theorem* 2.2 *of* [14]) *A graph G is Γ-colorable if and only if each block of G is Γ-colorable.*

Section 2 is devoted to investigations motivated by a theorem of Burr that describes the chromatic number of a graph G in terms of those of its edge-disjoint subgraphs. In Sect. 3, we determine the DP-chromatic number of all Halin graphs, and apply this result to study the difference between DP-coloring and group coloring. In particular we present a graph G for which $\chi_g(G) \neq \chi_{DP}(G)$.

2 Edge-Disjoint Decompositions

The research in this section is motivated by the coloring result proved by Burr in [3], stated as follows:

Theorem 2.1 [3] *Let G be a simple graph and $m_1, m_2, \ldots, m_k$ be positive integers. Then $\chi(G) \leq m_1 m_2 \ldots m_k$ if and only if G is the edge-disjoint union of k graphs $G_1, G_2, \ldots, G_k$ such that $\chi(G_i) \leq m_i$ for $1 \leq i \leq k$.*

It is natural to consider to what extent a group coloring version or even a DP-coloring version of Theorem 2.1 can be justified. For a graph G that is the edge-disjoint union of its subgraphs $G_1, G_2, \ldots, G_k$, we may ask more generally: How else can we describe $\chi_g(G)$ and $\chi_{DP}(G)$ using $\chi_g(G_1), \chi_g(G_2), \ldots, \chi_g(G_k)$ and $\chi_{DP}(G_1), \chi_{DP}(G_2), \ldots, \chi_{DP}(G_k)$, respectively? This section will have three subsections, presenting different attempts towards these goals.

2.1 Partial Group Coloring and DP-Coloring Versions of Burr's Theorem

We will first show that Theorem 2.1 can only be partially extended to group colorings. One direction closely resembles the proof given by Burr in [3] for

Theorem 2.1. Unlike Burr's theorem above, which concerns standard vertex coloring, it is meaningful with group coloring to consider multigraphs as well.

Definition 2.2 [7] For a collection $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ of groups, we define the **direct product** of these groups, denoted by $(\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_k)$, as a group whose elements are the Cartesian product of the elements of $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ and whose operation is given by $(a_1, a_2, \ldots, a_k) + (b_1, b_2, \ldots, b_k) = (a_1 +_1 b_1, a_2 +_2 b_2, \ldots, a_k +_k b_k)$, where $a_i, b_i \in \Gamma_i$ and $+_i$ is the group operation for Γ_i.

Theorem 2.3 *If a multigraph G is the edge-disjoint union of k multigraphs $G_1, G_2, \ldots, G_k$ such that each G_i is Γ_i-colorable, then G is $(\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_k)$-colorable.*

Proof Let G be the edge-disjoint union of k graphs $G_1, G_2, \ldots, G_k$ such that each G_i is Γ_i-colorable. Let $\Gamma = (\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_k)$ and $f \in F(G, \Gamma)$. To prove the theorem, it is sufficient to find a (Γ, f)-coloring of G. As Γ is a product of k groups, f can be viewed as a k-tuple $f = (f_1, f_2, \ldots, f_k)$, where $f_i \in F(G, \Gamma_i)$, for each $i \in \{1, 2, \ldots, k\}$. Fix an i with $1 \leq i \leq k$. Since each G_i is Γ_i-colorable, there exists a (Γ_i, f_i)-coloring $c_i : V(G_i) \to \Gamma_i$ of G_i. Define $c : V(G) \to \Gamma$ by $c(v) = (c_1(v), c_2(v), \ldots, c_k(v))$. Since G is the edge-disjoint union of $G_1, G_2, \ldots, G_k$, for any $e = uv \in E(G)$, there exists a unique $i_e \in \{1, 2, \ldots, k\}$ such that $uv \in E(G_{i_e})$. It follows that $c_{i_e}(u) - c_{i_e}(v) \neq f_{i_e}(uv)$, and so $c(u) - c(v) \neq f(uv)$ at the i_e's component. It follows that c is a proper (Γ, f)-coloring of G. $\square$

The converse of Theorem 2.3 does not hold, as we will see below. First we introduce a useful result:

Lemma 2.4 *For any nontrivial graph G, each of the following holds.*

(i) (*Theorem* 3.1 *of* [14]) $\chi_g(G) = 2$ *if and only if G is a forest.*
(ii) $\chi_{DP}(G) = 2$ *if and only if G is a forest.*

Proof We only need to justify (ii). If $\chi_{DP}(G) \leq 2$ then $\chi_g(G) \leq 2$ and thus by Lemma 2.4(i) that G must be a forest. If G is a forest then $\bar{\delta}(G) = 1$ and thus by Theorem 1.4(ii) we must have $\chi_{DP}(G) \leq 2$. Since G is assumed to be nontrivial, $\chi_{DP}(G) = 2$. $\square$

Proposition 2.5 *The graph $K_{4,4}$ is $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-colorable, but cannot be decomposed into two edge-disjoint $\mathbb{Z}_2$-colorable subgraphs.*

Proof By Theorem 1.1, $\chi_g(K_{4,4}) \leq \Delta(K_{4,4}) = 4$, so that $K_{4,4}$ must be $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-colorable. Now assume by way of contradiction that $K_{4,4}$ is the edge-disjoint union of two graphs G_1 and G_2, both of which are $\mathbb{Z}_2$-colorable. By Lemma 2.4, G_1 and G_2 must both be forests, and thus they each have at most $|V(G)| - 1 = 7$ edges. Then, since we assumed that $K_{4,4}$ is the edge-disjoint union of G_1 and G_2, we must have $|E(K_{4,4})| \leq 14$, a contradiction. $\square$

Utilizing Theorem 1.4(iv), it is possible to construct infinitely many graphs each of which is $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-colorable, but cannot be decomposed into two edge-disjoint

$\mathbb{Z}_2$-colorable subgraphs. Furthermore, it is important to recall now a property of group colorings that will show us Theorem 2.3 is not quite as strong as it appears. Knowing that a graph G is Γ-colorable does not necessarily imply that $\chi_g(G) \leq |\Gamma|$. To see an example of this, albeit in the dual group connectivity form, see [8].

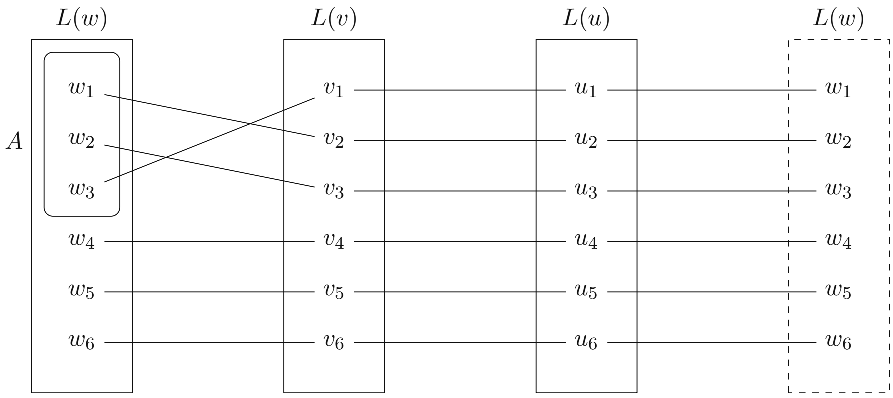
We will now attempt another generalization of Theorem 2.1, this time to DP-coloring. Just as Theorem 2.3 is stated in terms of Γ-colorability rather than χ_g, we will again prove only a partial result, and this time the result is even weaker. We include it here in the interest of examining the natural generalization of Burr's methods to DP-coloring and to introduce the idea of contracting portions of a cover which we will revisit for more interesting results in a future paper. First, we will introduce a specific type of cover.

Definition 2.6 Let G be a multigraph, let $m_1, m_2 \in \mathbb{N}$, and let (L, H) be a full m_1m_2-cover of G. We say that (L, H) is **uniformly** m_1**-collapsible** if there exist partitions $\{L_1(v), L_2(v), \ldots, L_{m_1}(v)\}$ for each $v \in V(G)$ such that:

1. $|L_1(v)| = |L_2(v)| = \cdots = |L_{m_1}(v)| = m_2$ for each $v \in V(G)$.
2. If u and v are adjacent vertices in G, and $i \in \{1, 2, \ldots, m_1\}$ then $N_H(L_i(u)) \cap L(v) = L_j(v)$ for some $j \in \{1, 2, \ldots, m_1\}$.

For acyclic graphs, it is not difficult to show that any m_1m_2-cover is uniformly m_1-collapsible. First, we may designate some arbitrary vertex v_0 in a given component as a root, then partition $L(v_0)$ appropriately according to the first property of Definition 2.6. Now, we partition the neighbors of v_0 so that the second property of Definition 2.6 is satisfied, and continue in this way to neighbors of neighbors until each $L(v)$ is partitioned.

However, for a cycle it is possible to construct an m_1m_2-cover that is not uniformly m_1-collapsible. The figure below illustrates a 6-cover for the graph K_3 that is not uniformly 3-collapsible. To be uniformly 3-collapsible, we would need to partition $L(w)$ into 3 parts of size 2, but the second property of Definition 2.6 implies that, in this particular cover, no part may mix elements from inside and outside of the set A. Therefore, a divisibility argument indicates that $L(w)$ cannot be appropriately partitioned.



Theorem 2.7 *Let G be a multigraph, and let (L, H) be a uniformly m_1-collapsible m_1m_2-cover of G. If G is the edge-disjoint union of G_1 and G_2 such that $\chi_{DP}(G_i) \le m_i$ for $1 \le i \le 2$, then G is (L, H)-colorable.*

Proof Let G, (L, H), G_1 and G_2 be as described. To prove the theorem, we will construct an (L, H)-coloring for G. For each $v \in V(G)$, we have an appropriate partition $\{L_1(v), L_2(v), \ldots, L_{m_1}(v)\}$ of $L(v)$ that satisfies the two properties of Definition 2.6. We now modify (L, H) to form a new m_1-cover (L', H') as follows: For each $uv \in E(G_2)$, remove $E_H[L(u), L(v)]$ from H. For each $v \in V(G)$ we identify the vertices in each part $L_i(v)$ of the partition, removing any loops and new parallel edges formed, to create the vertex $v_i \in V(H')$. Thus $L'(v) = \{v_1, v_2, \ldots, v_{m_1}\}$ for each $v \in V(G)$. The second property of Definition 2.6 implies that H' is indeed a cover of G_1. If we now consider only the edges of G_1, we may note that $\chi_{DP}(G_1) \le m_1$ implies that there exists an independent set $I' \subseteq V(H')$ with $|I'| = |V(G_1)|$. We use this set I' to construct another graph H'' from H as follows: Remove all $u \in V(H)$ except those contracted to form the vertices in I', and for each $uv \in E(G_1)$ remove $E_H[L(u), L(v)]$. We may also define, for each $v \in V(G)$, $L''(v) = L(v) \cap V(H'')$. Then (L'', H'') is an m_2-cover of G_2, and $\chi_{DP}(G_2) \le m_2$ implies that there is an independent set $I'' \subseteq V(H'')$ with $|I''| = |V(G_2)| = |V(G)|$. We may now note that I'' is independent in H as well: by the construction of H' and H'', for any two $u, v \in I''$ we have $uv \notin E(H)$, since uv cannot cover an edge of G_1 nor of G_2. Thus, we have by definition an (L, H)-coloring of G.□

It would be possible to write Theorem 2.7 with slightly weaker assumptions; specifically we only need to require that the second property in Definition 2.6 holds along edges of G_1, but for the sake of clarity we elect to leave the theorem as written. Likewise, it is possible to perform an induction to prove a version with more subgraphs in the decomposition, but it would be cumbersome to read due to the collapsibility requirements at each step.

Just as in Theorem 2.3, the converse of Theorem 2.7 does not hold. Indeed the same graph $K_{4,4}$ serves as a counter-example. By Theorem 1.4 (ii), $\chi_{DP}(K_{4,4}) \le \Delta(K_{4,4}) = 4$, implying that any uniformly collapsible 4-cover (L, H)

of $K_{4,4}$ is colorable. We may be assured that a uniformly collapsible cover exists, as it is evident that the covers equivalent to standard vertex coloring will be uniformly collapsible for any graph. However, as previously demonstrated, $K_{4,4}$ cannot be decomposed into two edge-disjoint forests. Therefore by Corollary 2.4 $K_{4,4}$ cannot be decomposed into two graphs G_1 and G_2 with $\chi_{DP}(G_1) \leq 2$ and $\chi_{DP}(G_2) \leq 2$.

2.2 Forests and Arboricity

In this subsection we consider extending a well-known vertex coloring result involving arboricity defined below. All results apply to both group coloring and DP-coloring, with only trivial modifications of the proofs. Therefore the group coloring result will be, for the most part, mentioned as a corollary.

Definition 2.8 The **arboricity** of a multigraph G, here denoted by $a(G)$, is the minimum number of edge-disjoint forests required to cover G.

The next result appears in part as an exercise in Section 21.4 of [2]. The proof is omitted, since Theorem 2.9 follows from Corollary 2.12.

Theorem 2.9 *For any multigraph G, if $a(G) \leq k$ then $\chi(G) \leq 2k$.*

To work towards a generalization, we revisit the proof using $\bar{\delta}(G)$.

Lemma 2.10 *If a multigraph G is the edge-disjoint union of k multigraphs $G_1, G_2, \ldots, G_k$ such that $\bar{\delta}(G_i) \leq p_i$ for $1 \leq i \leq k$, then $\bar{\delta}(G) \leq \lfloor 2(1 - \frac{1}{|V(G)|}) \sum_{i=1}^{k} p_i \rfloor$.*

Proof Assume that G is the edge-disjoint union of k graphs $G_1, G_2, \ldots, G_k$ such that $\bar{\delta}(G_i) \leq p_i$ for $1 \leq i \leq k$. Then $|E(G_i)| \leq (|V(G)| - 1)p_i$ and $|E(G)| \leq (|V(G)| - 1) \sum_{i=1}^{k} p_i$. Likewise, for each subgraph H of G we can get $|E(H)| \leq (|V(H)| - 1) \sum_{i=1}^{k} p_i$ in the same way, since any such H is also the edge-disjoint union of some $\{H_i : 1 \leq i \leq k\}$ which are subgraphs of $\{G_i : 1 \leq i \leq k\}$, respectively. Now for any subgraph H of G we have the following inequality:

$$\begin{aligned}\delta(H) &\leq \left\lfloor \frac{2|E(H)|}{|V(H)|} \right\rfloor \leq \left\lfloor \frac{2(|V(H)| - 1)}{|V(H)|} \sum_{i=1}^{k} p_i \right\rfloor \\ &= \left\lfloor 2\left(1 - \frac{1}{|V(H)|}\right) \sum_{i=1}^{k} p_i \right\rfloor \leq \left\lfloor 2\left(1 - \frac{1}{|V(G)|}\right) \sum_{i=1}^{k} p_i \right\rfloor.\end{aligned}$$

This completes the proof. □

Since graphs with $\chi_{DP}(G) = 2$ are so well-characterized by Corollary 2.4, we can use Lemma 2.10 to further refine Theorem 2.7 in the special case where $m_1 = m_2 = \cdots = m_k = 2$.

Theorem 2.11 *If a multigraph G is the edge-disjoint union of k graphs $G_1, G_2, \ldots, G_k$ such that each $\chi_{DP}(G_i) \leq 2$, then $\chi_{DP}(G) \leq \lfloor 2k(1 - \frac{1}{|V(G)|}) \rfloor + 1$.*

Proof Assume that G is the edge-disjoint union of k graphs $G_1, G_2, \ldots, G_k$ such that each G_i is $\mathbb{Z}_2$-colorable. Then by Theorem 2.4, each G_i is a forest and thus $\bar{\delta}(G_i) \leq 1$. Now by Lemma 2.10 we have $\bar{\delta}(G) \leq \lfloor 2k(1 - \frac{1}{|V(G)|}) \rfloor$, and thus $\chi_{DP}(G) \leq \lfloor 2k(1 - \frac{1}{|V(G)|}) \rfloor + 1$. □

The above theorem will seem more familiar when restated in other terms; Corollary 2.12 follows directly from Theorem 2.11 and Definition 2.8:

Corollary 2.12 *For a multigraph G, if $a(G) \leq k$ then we have $\chi_g(G) \leq \chi_{DP}(G) \leq \lfloor 2k(1 - \frac{1}{|V(G)|}) \rfloor + 1 \leq 2k$.*

Ordinarily, when dealing with standard vertex colorings, there is no reason not to simply round down the bound on chromatic number to $2k$, since $2k$ will always be smaller than $|V(G)|$ for a minimum covering. However when dealing with multigraphs it is possible for $2k$ to be greater than $|V(G)|$, so we have included the intermediate result here to ensure that the bound remains tight for a larger family of multigraphs. We may apply this bound in the following application on multigraphs that are slightly larger than complete graphs.

Proposition 2.13 *Let $p > 0$ be odd and let G be a multigraph of the form $(2p+1)K_{2t+1} \cup H$, where H is an acyclic graph with at most t edges. Then $\chi_g(G) = \chi_{DP}(G) = 2t(2p+1) + 1$.*

Proof Let $G = (2p+1)K_{2t+1} \cup H$, where H is an acyclic graph with at most t edges. First, we demonstrate that $a(G) \leq (p+1)(t+1)$. The subgraph $2pK_{2t+1}$ can be covered by $p(2t+1)$ forests. Next, we cover the remaining edges which induce $K_{2t+1} \cup H$ as follows: Pick some vertex v that is not on H and remove it from K_{2t+1} to form K_{2t}. This copy of K_{2t} can be covered by t trees which also span K_{2t}. Each of these trees can be extended so that it spans v as well by adding edges between t vertices of K_{2t} and v, but we will defer this choice until deciding which edges are appropriate. Consider the graph H on the vertex set $V(G) - v$. Since H has at most t edges, it must have at least t components, some of which may be isolated vertices. Select t of these components, and select one vertex from each selected component yielding $\{u_i : 1 \leq i \leq t\}$. We form a forest F by starting with H and drawing the edges $u_i v$ for $1 \leq i \leq t$. Finally we return to the choice we deferred earlier and extend our spanning t trees to cover v as well by drawing edges from each $u \in V(G) - (\{v\} \cup \{u_i : 1 \leq i \leq t\})$ to v. Now $K_{2t+1} \cup H$ is covered by $t+1$ forests, and thus G is covered by $p(2t+1) + t + 1$ forests so $a(G) \leq p(2t+1) + t + 1$.

Now, by Corollary 2.12 we have $\chi_g(G) \leq \chi_{DP}(G) \leq \lfloor 2(p(2t+1) + t + 1)(1 - \frac{1}{2t+1}) \rfloor + 1 = 2t(2p+1) + 1$. By Theorem 1.1 (v), $\chi_g((2p+1)K_{2t+1}) = \Delta((2p+1)K_{2t+1}) + 1 = 2t(2p+1) + 1$, and thus $\chi_g(G) = \chi_{DP}(G) = 2t(2p+1) + 1$ exactly. □

A similar (and much simpler) calculation shows that just as we may add an acyclic graph with t edges to $(2p+1)K_{2t+1}$ without increasing the group chromatic number or DP-chromatic number, we may say the same about adding an acyclic graph with t edges to rK_{2t} for any integers r and t. In general, the value of $\bar{\delta}(G)$ is not always obvious so for some classes of graphs, like in Proposition 2.13, it may be easier to bound the value of $\bar{\delta}(G)$ with $a(G)$.

2.3 Analysis

In this subsection we attempt to generalize Theorems 2.3 and 2.7 to look more like Theorem 2.1, while relaxing the conclusion. To do so using methods similar to Sect. 2.2, a lower bound on $\chi_g(G)$ in terms of $\bar{\delta}(G)$ is necessary so we may say $\chi_g(G) \leq N$ implies $\bar{\delta}(G) \leq M$ for some M depending only on N. It seems that theorems of this form are more difficult than the converse form, since they cannot rely on techniques developed for vertex coloring. We may note that a graph with $\chi(G) = 2$ may have an arbitrarily high $\bar{\delta}(G)$. This is not true of group chromatic number, and in general it has been shown that sufficiently high $\bar{\delta}(G)$ will imply that $\chi_g(G)$ is as large as required, which is the type of bound that is useful here.

Theorem 2.14 [11] *If a graph G contains a subgraph H with $\delta(H) \geq 2$ then $\chi_g(G) > \delta(H)/2\ln(\delta(H))$.*

Corollary 2.15 *For any integer $N \in \mathbb{N}$, there exists an integer $M \in \mathbb{N}$, depending only on N, such that $\chi_g(G) \leq N$ implies $\bar{\delta}(G) < M$.*

Proof We prove an equivalent statement: For each $N \in \mathbb{N}$ there exists $M \in \mathbb{N}$, depending only on N, such that $\bar{\delta}(G) \geq M$ implies $\chi_g(G) > N$. By Theorem 2.14, $\chi_g(G) > \delta(H)/2\ln(\delta(H))$ if $\delta(H) \geq 2$ for some subgraph H of G. Therefore, since $x/2\ln(x)$ is increasing on (e, ∞), we may say that for $\bar{\delta}(G) > e$ we have the following:

$$\chi_g(G) > \max_{H \subseteq G} \frac{\delta(H)}{2\ln(\delta(H))} = \frac{\bar{\delta}(G)}{2\ln(\bar{\delta}(G))}.$$

Furthermore since $\lim_{x\to\infty} x/2\ln(x) = \infty$ we may say that sufficiently high $\bar{\delta}(G)$ will ensure that $\chi_g(G)$ is as large as required. □

Theorem 2.16 *For any $m_1, m_2, \ldots, m_k \in \mathbb{N}$ there exists $M \in \mathbb{N}$ such that if G is the edge-disjoint union of k graphs $G_1, G_2, \ldots, G_k$ with $\chi_g(G_i) \leq m_i$ for $1 \leq i \leq k$ then $\chi_g(G) \leq M$.*

Proof This follows from Corollary 2.15 and Lemma 2.10. □

Furthermore, using only the fact that $\chi_g(G) \leq \chi_{DP}(G)$ for any graph G, we can adapt the results from Theorem 2.15 for DP-coloring to produce a bound analogous to Theorem 2.16:

Theorem 2.17 *For any $m_1, m_2, \ldots, m_k \in \mathbb{N}$ there exists $M \in \mathbb{N}$ such that if G is the edge-disjoint union of k graphs $G_1, G_2, \ldots, G_k$ with $\chi_{DP}(G_i) \leq m_i$ for $1 \leq i \leq k$ then $\chi_{DP}(G) \leq M$.*

Now that the existence of such bounds has been established, it is natural to seek the best bounds. We might ask the following questions:

Question 2.18 *What is the smallest integer $a \in \mathbb{N}$ such that if G is the edge-disjoint union of k* graphs $G_1, G_2, \ldots, G_k$ *with* $\chi_g(G_i) \leq m_i$ *for* $1 \leq i \leq k$ *then* $\chi_g(G) \leq am_1m_2...m_k$?

Question 2.19 *What is the smallest integer $a \in \mathbb{N}$ such that if G is the edge-disjoint union of k graphs $G_1, G_2, \ldots, G_k$ with $\chi_{DP}(G_i) \leq m_i$ for $1 \leq i \leq k$ then $\chi_{DP}(G) \leq am_1m_2...m_k$?*

It is clear that in both questions we have $a \geq 1$ since K_4 can be written as the disjoint union of two paths T_1 and T_2, and $\chi_g(K_4) = \chi_{DP}(K_4) = 4 = \chi_g(T_1)\chi_g(T_2) = \chi_{DP}(T_1)\chi_{DP}(T_2)$. We are not aware of an example that requires $a > 1$, so it may well be that $a = 1$ as with Theorem 2.1.

3 A Comparison of DP-Coloring and Group Coloring of Graphs

So far in this paper, we have seen methods that work just as well for DP-coloring as they do for group coloring. It is important to note that, as with many such generalizations, some properties are lost. In this section, we explore methods and constructions that have been applied to group coloring, but break when applied to DP-coloring. The first, and possibly most important construction that is lost is the dual. Group coloring, from its conception in [8], has been closely tied to group connectivity. This leads us to extend a former group chromatic number result on Halin graphs to its DP-coloring version, for which no duality argument is known. In [11], it is conjectured that for any graph G, $\chi_L(G) \leq \chi_g(G)$. In [12], Král and Nejedlý first introduced the list group chromatic number $\chi_{gl}(G)$ of a graph G (to be formally defined in Sect. 3.2). Further studies of $\chi_{gl}(G)$ can be found in [4, 5, 12, 17], among others. As the invariants $\chi_L(G)$, $\chi_g(G)$, $\chi_{gl}(G)$ and $\chi_{DP}(G)$ are considered to be closely related, we in this section will also investigate their relationships, and present an example of a graph G such that $\chi_g(G) < \chi_{DP}(G)$.

3.1 DP-Colorings of Halin Graphs

In this subsection, we will, in the lack of the duality argument, determine the DP-chromatic number of all Halin graphs. We start with the definition of Halin graphs and a former result.

Definition 3.1 A graph G is a Halin Graph if it is the edge-disjoint union of a tree T with at least 4 vertices, none of which has degree 2, and a cycle C, constructed by first embedding T on a plane, and then traversing the boundary of T in the clockwise direction connecting each leaf encountered with the next.

Theorem 3.2 [19] *Let G be a Halin graph. Then $\chi_g(G) = 4$ if G is isomorphic to an odd wheel and $\chi_g(G) = 3$ otherwise.*

The proof of Theorem 3.2 in [19] relies heavily on group connectivity of the dual graph. A corresponding theorem for DP-coloring would have to use much different methods. We will prove such a corresponding theorem for DP-coloring below, but we will need to introduce some notation and a lemma first. In the discussions below, for a positive integer k, we define $[k] = \{1, 2, 3, \ldots, k\}$, and let S_k denote the symmetric group in k letters, also viewed as the set of all permutations on $[k]$.

In [9], a generalization of DP-coloring is introduced which we will find useful. Let S be a subset of S_k, the symmetric group on k letters for some integer $k > 0$, such that S is closed under taking inverses. Let $\sigma : E(G) \to S$ such that $\sigma_{uv} = (\sigma_{vu})^{-1}$ for each pair u, v of adjacent vertices in G. We say that G is σ-**colorable** if there exists a function $c : V(G) \to [k]$ such that $\sigma_{uv}(c(u)) \neq c(v)$ for any $uv \in E(G)$. We say that G is S- k **-colorable** if G is σ-colorable for any $\sigma : E(G) \to S$ with the inverse value requirement as described above. We will demonstrate that, in some sense, this concept generalizes DP-coloring.

Proposition 3.3 *A graph G is S_k-k-colorable if and only if $\chi_{DP}(G) \leq k$.*

Proof First we assume that G is S_k-k-colorable and let (L, H) be a k-cover of G. We may assume without loss of generality that (L, H) is full, otherwise we might consider a more "difficult" H with the desired property. For each $v \in V(G)$, we may arbitrarily identify $L(v)$ with $[k]$. Now for each $uv \in E(G)$ we may convert $H[L(u), L(v)]$ into a permutation σ_{uv} where $\sigma_{uv}(i) = j$ when $ij \in E_H[L(u), L(v)]$. If c is a σ-coloring of G, then it is also an (L, H)-coloring of G. Since we assumed that G is S_k-k-colorable, and (L, H) was arbitrarily chosen, we have $\chi_{DP}(G) \leq k$.

Conversely, we assume that $\chi_{DP}(G) \leq k$. Let $\sigma : E(G) \to S_k$ be such that $\sigma_{uv} = (\sigma_{vu})^{-1}$ for each pair u, v of adjacent vertices in G. We covert σ to a cover (L, H) of G as follows: For each $v \in V(G)$ let $L(v) = [k]$. For each $uv \in E(G)$ let $E_H[L(u), L(v)] = \{xy : x \in L(u),\ y \in L(v), \text{ and } \sigma_{uv}(x) = y\}$. Finally, let $H[L(v)] = K_k$ for each $v \in V(G)$. No other edges are present in H, besides those described above. Now as c is an (L, H)-coloring of G, then it is also a σ-coloring of G. Since $\chi_{DP}(G) \leq k$, G is (L, H)-colorable for any k-cover (L, H) of G. Therefore G is σ-colorable for any σ as described, so by definition G is S_k-k-colorable. $\square$

Note that in the proof of Proposition 3.3 above, when converting a covering (L, H) of G into a mapping $\sigma : E(G) \to S_k$, we *arbitrarily* identify each $L(v)$ with $[k]$. Given a k-covering (L, H) of G, there are in fact $(k!)^{|V(G)|}$ different ways to identify all $L(v)$ with $[k]$. We will see below that if we are more careful with an assignment of $L(v)$ to $[k]$ then we can ensure that σ has some desirable properties.

Definition 3.4 For a k-cover (L, H) of G, we say that $\sigma : E(G) \to S_k$ **represents** (L, H) when, as in Proposition 3.3, G is σ-colorable if and only if G is (L, H)-colorable.

Lemma 3.5 *Given any full k-cover (L, H) of a graph G and any acyclic subgraph T of G, there exists some $\sigma : E(G) \to S_k$ such that σ represents (L, H) and $\sigma_{uv} = 1_{S_k}$ for each $uv \in E(T)$.*

Proof Let (L, H) be a k-cover of a graph G, and let T be an acyclic subgraph of G. Choose some $v \in V(T)$ and arbitrarily identify $L(v)$ with $[k]$. Now for each neighbor u of v in T, we identify $L(u)$ with $[k]$ so that for $i \in [k]$, we assign i to the element of $L(u)$ that is adjacent in H to the element of $L(v)$ which was assigned i. Now $\sigma_{uv}(i) = i$ for $i \in [k]$ (i.e. $\sigma_{uv} = 1_{S_k}$). We may repeat this process until we have identified $L(w)$ with $[k]$ for each $w \in V(T)$. Since T is acyclic, there can be no conflict in assigning $[k]$ to any $L(w)$ for $w \in V(T)$. For $w \notin V(T)$, we may arbitrarily assign $[k]$ to $L(w)$. Now σ represents (L, H) and $\sigma_{uv} = 1_{S_k}$ for each $uv \in E(T)$. $\square$

Lemma 3.5 is well-suited to Halin graphs, since Halin graphs can be decomposed into a spanning tree and a cycle along the leaves of the spanning tree. We apply it below to prove a theorem analogous to Theorem 3.2.

Theorem 3.6 *Let G be a Halin graph. Then $\chi_{DP}(G) = 4$ if G is isomorphic to an odd wheel and $\chi_{DP}(G) = 3$ otherwise.*

Proof Let W_k denote the wheel graph on $k+1$ vertices. Let G be a Halin graph. We consider the two cases depending on whether G is a wheel or not.

Case 1. G is a wheel. Thus $G = W_k$ for some integer $k \geq 3$.

Let (L, H) be a 3-cover of G. Without loss of generality, we may assume that (L, H) is full. We label the vertices of G so v_0 is the "hub" of the wheel and $v_1, v_2, \ldots, v_k$ are placed in order around the "rim". By Lemma 3.5, there exists $\sigma : E(G) \to S_3$ such that σ represents (L, H) and $\sigma_{v_0 v_i} = 1_{S_3}$ for each $i \in k$. To prove this case, we prove a stronger property: If G is not σ-colorable then $\sigma_{uv} = 1_{S_3}$ for each $uv \in E(G)$, and thus G must be an odd wheel. First, we construct two partial colorings. Let $c_0(v_0) = 0$ and $c_0(v_1) = 2$. Let $c_1(v_0) = 1$ and $c_1(v_1) = 2$. Note that our choice for $c_0(v_1)$ disallows exactly one choice of $c_0(v_2)$, so $c_0(v_2) \neq \sigma_{v_1 v_2}(c_0(v_1))$. Similarly, $c_0(v_2) \neq \sigma_{v_0 v_2}(c_0(v_0))$. Therefore there must be at least one choice remaining for $c_0(v_2)$. We may continue to extend c_0 and c_1 to $v_3 \ldots v_{k-1}$ in this way without fear of conflict. $\square$

Claim *If for some i with $1 \leq i \leq k-1$ our choices of $c_0(v_0)$ and $c_0(v_i)$ both forbid the same color at v_{i+1} (i.e. $\sigma_{v_i v_{i+1}}(c_0(v_i)) = \sigma_{v_0 v_{i+1}}(c_0(v_0))$) then c_0 can be extended to a σ-coloring of G. The same applies to c_1.*

Proof of Claim First assume that our choices of $c_0(v_0)$ and $c_0(v_{k-1})$ both forbid the same color at v_k. Then we have two choices remaining for $c_0(v_k)$ that are consistent with $c_0(v_0), c_0(v_1), \ldots, c_0(v_{k-1})$, if we disregard $\sigma_{v_k v_1}$. Only one of these two choices may cause conflict with respect to $\sigma_{v_k v_1}$, so the other must be a σ-coloring of G. Now we note that if for some i with $1 \leq i \leq k-2$ our choices of $c_0(v_0)$ and $c_0(v_i)$ both forbid the same color at v_{i+1}, then the two options for $c_0(v_{i+1})$ that are consistent with previous $c_0(v_0), c_0(v_1), \ldots, c_0(v_i)$ must yield two options for $c_0(v_{i+2})$. By induction we see that this reduces to the case where $i = k-1$, and thus c_0 can again be extended to a σ-coloring of G. This completes the proof of the claim. $\square$

We therefore assume that the condition of the claim does not hold, so that possible extensions of c_0 and c_1 to $v_2, v_3, \ldots, v_k$ are uniquely determined by the colors they first assign to v_0 and v_1. Now, $c_0(v_0) = 0$ implies $c_0(v_2) \neq 0$ and $c_1(v_0) = 1$ implies $c_1(v_2) \neq 1$. Therefore, since the condition of the claim does not hold and since $c_0(v_1) = c_1(v_1) = 2$, we must have $\sigma_{v_1v_2}(2) \neq 0$ and $\sigma_{v_1v_2}(2) \neq 1$. This leaves $\sigma_{v_1v_2}(2) = 2$. Similar arguments with different colors at v_0 and v_1 show that $\sigma_{v_1v_2}(0) = 0$ and $\sigma_{v_1v_2}(1) = 1$ so that $\sigma_{v_1v_2} = 1_{S_3}$. We may also repeat the same argument to show that $\sigma_{v_iv_{i+1}} = 1_{S_3}$ for $2 \leq i \leq k-1$. It remains to show that $\sigma_{v_kv_1} = 1_{S_3}$.

Now we consider two more partial colorings c_3 and c_4 where $c_3(v_0) = 2$, $c_3(v_1) = 0$, $c_4(v_0) = 2$ and $c_4(v_1) = 1$. Since $\sigma_{v_iv_{i+1}} = 1_{S_3}$ for $1 \leq i \leq k-1$, we must have:

$$\begin{cases} c_3(v_i) = 0 & \text{when } 1 \leq i \leq k \text{ and } i \text{ is odd,} \\ c_3(v_i) = 1 & \text{when } 1 \leq i \leq k \text{ and } i \text{ is even,} \\ c_4(v_i) = 1 & \text{when } 1 \leq i \leq k \text{ and } i \text{ is odd,} \\ c_4(v_i) = 0 & \text{when } 1 \leq i \leq k \text{ and } i \text{ is even.} \end{cases}$$

Since we assume that c_3 and c_4 are not σ-colorings, we can see that when k is even we must have $\sigma_{v_kv_1}(1) = 0$ and $\sigma_{v_kv_1}(0) = 1$, so $\sigma_{v_kv_1} = (01)$. However, we then have $c_0(v_1) = 2$ and $c_0(v_k) = 1$ so c_0 is a σ-coloring, a contradiction. If instead k is odd then we must have $\sigma_{v_kv_1}(0) = 0$ and $\sigma_{v_kv_1}(1) = 1$ so $\sigma_{v_kv_1} = 1_{S_3}$. To summarize, W_k is not σ-colorable if and only if $\sigma_{uv} = 1_{S_3}$ for each $uv \in E(G)$ and k is odd. Such a σ-coloring of G would be equivalent to a vertex coloring of an odd wheel with three colors, obviously not possible. Therefore, $\chi_{DP}(W_k) = 3$ when k is even, and $\chi_{DP}(W_k) > 3$ when k is odd. Since $\chi_{DP} \leq \bar{\delta}(W_k) + 1 = 4$, we must have $\chi_{DP}(W_k) = 4$ when k is odd.

Case 2: G is not a wheel.

By definition, G is an edge-disjoint union of a spanning tree T and a cycle C through the leaves of T. Let (L, H) be a full 3-cover of G, and let $\sigma : E(G) \to S_3$ such that σ represents (L, H) and $\sigma_e = 1_{S_3}$ for each $e \in E(T)$. Again, we assume by way of contradiction that G is not σ-colorable.

Let $J = T - V(C)$, and let w_2 be a leaf of J. We label $V(C)$ with $v_1, v_2, \ldots, v_k$ in order such that $v_2 \ldots v_t$ are adjacent to w_2 in G and $v_{t+1}, \ldots, v_k, v_1$ are not adjacent to w_2 in G. Let w_1 be the vertex of J adjacent to v_1. By the same reasoning as in Case 1, a fixed partial σ-coloring of $G - \{v_2, \ldots, v_k\}$ must uniquely determine any possible extension to $G - \{v_3, \ldots, v_k\}$. Thus, if c is a partial σ-coloring of $G - \{v_2, \ldots, v_k\}$, then $\sigma_{v_1v_2}(c(v_1)) \neq \sigma_{w_2v_2}(c(w_2)) = c(w_2)$. Now we consider a partial σ-coloring $c_0 : V(J) \cup \{v_1\} \to \{1, 2, 3\}$ with $c_0(v_1) = 0$, $c_0(w_1) = 1$, and $c_0(w_2) = 0$. Before proceeding, we will demonstrate that such a proper partial coloring exists.

We define $J' = G[V(J) \cup \{v_1\}]$, a subgraph of the tree T, so that we have $\sigma_e = 1_{S_3}$ for each $e \in E(J')$, and thus c_0 is proper under σ if and only if c_0 is proper when considered as a standard vertex coloring problem. We further define P to be the unique (v_1, w_2)-path in J', and since v_1 is a leaf of J', we may write a $V(P)$ in order as $v_1, w_1, \ldots, w_2$. If the distance between w_1 and w_2 on P is odd, then we may color P by alternating 0 and 1. Note that we may then change the color of w_2 if necessary, though this is not required for c_0. If instead the distance between w_1 and w_2 is even, define w_p to be the unique vertex on P adjacent to w_2. As before, we color $v_1, w_1, \ldots, w_{p-1}$ by alternating 0 and 1, but then let $c_0(w_p) = 2$. Now we may set $c_0(w_2) = 0$ as required. In either case, after coloring P it is routine to greedily extend c_0 to J', since $\bar{\delta}(J') + 1 \leq 3$.

Now that this partial coloring is established, we may note that $\sigma_{v_1 v_2}(0) \neq 0$, or else our choice of $c_0(v_1)$ and $c_0(w_2)$ forbid the same choice for $c_0(v_2)$. By similar reasoning as with c_0, we may construct a partial coloring $c_1 : V(J) \cup \{v_1\} \to \{1, 2, 3\}$ with $c_1(v_1) = 0$, $c_1(w_1) = 2$, and $c_1(w_2) = 1$, so $\sigma_{v_1 v_2}(0) \neq 1$, and another partial coloring c_2 with $c_2(v_1) = 0$, $c_2(w_1) = 1$, and $c_2(w_2) = 2$, so $\sigma_{v_1 v_2}(0) \neq 2$. Since any $\sigma \in S_3$ must map 0, we have a contradiction. $\square$

So in the case of Halin graphs, while the dual methods used for group coloring cannot be applied to DP-coloring, the result still holds.

3.2 Comparing $\chi_g(G)$, $\chi_{gl}(G)$ and $\chi_{DP}(G)$

We now examine another result in group coloring which cannot be applied to DP-coloring, and give an example to show that a DP-coloring analogue does not hold. Recall that a **block** of a multigraph G is a maximal 2-connected subgraph. Theorem 3.7(ii) is a consequence of Theorem 3.7(i).

Theorem 3.7 *Let G be a multigraph.*

(i) [18] *G is Γ-colorable if and only if each block of G is Γ-colorable.*
(ii) $\chi_g(G) = \max\{\chi_g(B) : B \text{ is a block of } G\}$

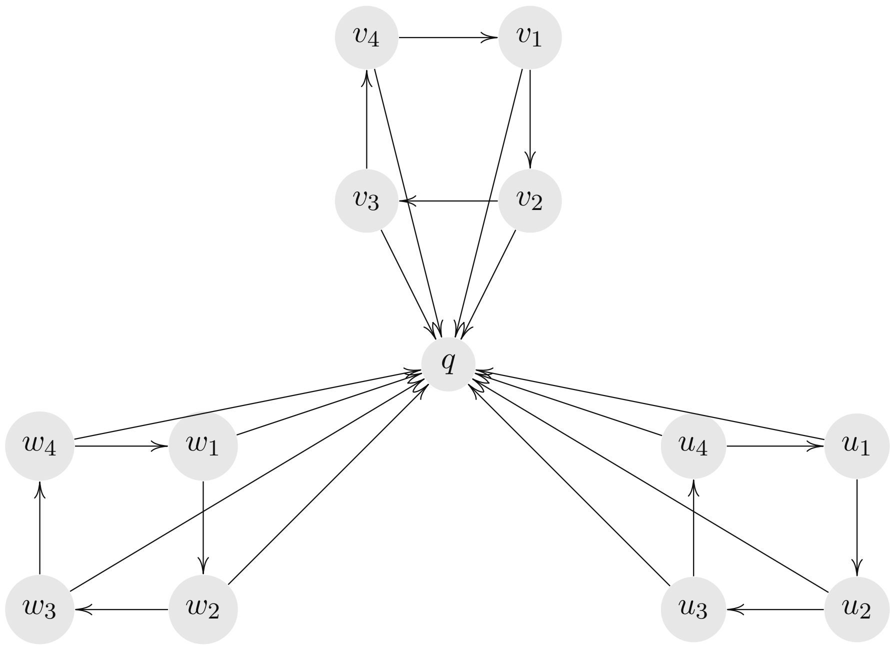
Theorem 3.7(ii) does not hold if we replace χ_g with χ_{DP}. To show this, we will re-use a counter-example found in [5]. First, we give some definitions for group list coloring and show that list coloring problems can be converted into DP-coloring problems. It is worth noting that we are not aware of a graph G where the inequality $\chi_{gl}(G) \leq \chi_{DP}(G)$ proved below is strict.

Definition 3.8 Let G be a multigraph and let Γ be a finite group. For a function $f \in F(G, \Gamma)$ and a list assignment $L : V(G) \to 2^{\Gamma}$, a (Γ, f, L)**-coloring** of G under the orientation D is a function $c : V(G) \to A$ such that for any arc $e = (u, v) \in A(D)$, $c(u) - c(v) \neq f(e)$ and such that $c(w) \in L(w)$ for each $w \in V(G)$. The **group list chromatic number** $\chi_{gl}(G)$ of a graph G is the minimum integer m such that G is (Γ, f, L)-colorable for any $f \in F(G, \Gamma)$ and list assignment $L : V(G) \to 2^{\Gamma}$ where $|\Gamma| \geq m$ and $|L(w)| \geq m$ for each $w \in V(G)$.

Proposition 3.9 *For any multigraph G we have $\chi_{gl}(G) \leq \chi_{DP}(G)$.*

Proof Let G be a multigraph with $\chi_{DP}(G) \leq k$. Let Γ be a finite group, let $f \in F(G, \Gamma)$ and let $L : V(G) \to 2^{\Gamma}$ with $|\Gamma| \geq k$ and $|L(w)| \geq k$. Now to show that $\chi_{gl}(G) \leq k$ (and thus prove the proposition) it suffices to show that G is (Γ, f, L)-colorable for such arbitrarily chosen (Γ, f, L). We construct a k-cover (K, H) as follows. For each $w \in V(G)$ label $K(w)$ with $L(w)$. For each $e \in E(G)$, define a matching J_e on $K(u) \cup K(v)$ by $E(J_e) : \{xy : x \in L(u), y \in L(v), x - y = f(uv)\}$. Now let $E_H[L(u), L(v)] = \bigcup\{J_e : e \in [uv]\}$. Then since $\chi_{DP}(G) \leq k$, G is (K, H)-colorable, so by the construction of (K, H) we see that G is (Γ, f, L)-colorable as well.$\square$

Now, consider the following graph G:



By Theorem 3.2 and Corollary 3.7 we have $\chi_g(G) = 3$, but it is proven in [5] that $\chi_{gl}(G) = 4$. Therefore, by Proposition 3.9 and Theorem 1.4 we must also have $\chi_{DP}(G) = 4$. Furthermore, we can see that Corollary 3.7 does not hold for DP-colorings, since $\chi_{DP}(B) = 3$ for each block B of G by Theorem 3.6.

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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