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A NOTE ON GROUP COLORINGS AND GROUP STRUCTURE*

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Abstract. Abelian group colorings were first introduced by Jaeger et al. in [*J. Combin. Theory Ser. B*, 56 (1992), pp. 165–182] as the dual concept of group connectivity of graphs. For given groups Γ_1 and Γ_2 with $|\Gamma_1| = |\Gamma_2|$, the dual version of a problem raised by Jaeger et al. suggests to investigate whether every Γ_1-colorable graph G is also Γ_2-colorable. Recently, Hůsek, Mohelníková, and Šámal [*J. Graph Theory*, 93 (2019), pp. 317–327] used computer testing to find the first examples of $\mathbb{Z}_4$-connected but not $\mathbb{Z}_2^2$-connected graphs as well as $\mathbb{Z}_2^2$-connected but not $\mathbb{Z}_4$-connected graphs. As their examples are nonplanar, the group coloring problem remains unanswered. Group coloring was extended to non-abelian groups in Li and Lai [*Discrete Math.*, 313 (2013), pp. 101–104]. We introduce a group coloring local structure (defined as a snarl in the paper) and use it to construct infinitely many ordered triples (G, Γ_1, Γ_2) in which G is a graph and Γ_1 and Γ_2 are groups with $|\Gamma_1| = |\Gamma_2|$, such that G is Γ_1-colorable but not Γ_2-colorable.

Key words. graph coloring, group coloring, group chromatic number, group connectivity

AMS subject classifications. 05C15, 05C21, 05C40

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1. Introduction.

We consider finite graphs without loops but permitting multiple edges and follow [1] for undefined terms and notation in graph theory and [3] for those in group theory. We define a relation "$\sim$" on $E(G)$ such that $e_1 \sim e_2$ if $e_1 = e_2$, or if e_1 and e_2 form a cycle in G. It is routine to check that $\sim$ is an equivalence relation and edges in the same equivalence class are parallel edges with the same endpoints. We use $[uv]$ to denote the set of all edges between u and v in a graph or a digraph, without regard to orientation, and shorten $|[uv]|$ to $|uv|$. For a multigraph G, we denote the *simplification* of G (i.e., the simple graph formed by replacing each parallel class of edges $[uv]$ with a single edge uv) by $\hat{G}$. In a digraph D, we use (u, v) to denote an arc oriented from u to v. For an integer $k > 0$, define kG to be the graph obtained by replacing each edge in G with k parallel edges joining the same end vertices. Throughout this paper, Γ will denote a (multiplicative) group with identity 1, and 2^Γ will denote the power set of Γ.

Group coloring of a graph is introduced in [5] for an abelian group as the dual concept of group connectivity of graphs. In [8], the group coloring problem is relaxed to arbitrary groups. Let $F(G, \Gamma) = \{f : E(G) \to \Gamma\}$ denote the set of all mappings from $E(G)$ to Γ. Fix an orientation D of G. For an $f \in F(G, \Gamma)$, a mapping $c : V(G) \to \Gamma$ is a (Γ, f)-*coloring* if $c(u)c(v)^{-1} \neq f(e)$ for any arc $e = (u, v) \in A(D)$. If for any $f \in F(G, \Gamma)$, G always admits a (Γ, f)-coloring, then G is Γ-*colorable*. It is indicated in [5, 7] that whether G is Γ-colorable is independent of the choice of the orientation. The *group chromatic number*, denoted by $\chi_g(G)$, is the minimum m such that G is Γ-colorable for any group Γ with $|\Gamma| \geq m$. When $f : E(G) \to \{1\}$ is the

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constant map, a (Γ, f)-coloring is a proper $|\Gamma|$-coloring. Therefore, it is known that group coloring is a generalization of vertex coloring and furthermore that $\chi(G) \leq \chi_g(G)$.

Early research on group chromatic number and group choice number was almost entirely restricted to simple graphs and usually considered only abelian groups in the definition of $\chi_g(G)$. To make a distinction here, we will define $\chi_a(G)$ to be the minimum m such that G is A-colorable for any *abelian* group A with $|A| \geq m$. While it follows from the definitions that $\chi_a(G) \leq \chi_g(G)$ for any given graph G, it has been remarked in [6] that the difference between these two values is not well understood. Indeed, we are not aware of an example of a graph G for which $\chi_a(G) < \chi_g(G)$.

In [5], a problem is posed: for any two abelian groups Γ_1 and Γ_2 with $|\Gamma_1| = |\Gamma_2|$, is a Γ_1-connected graph also Γ_2-connected? In [4], Hŭsek, Mohelníková, and Šámal first answer this question with the following conclusion.

THEOREM 1.1 (Hŭsek, Mohelníková, and Šámal, Theorem 3 of [4]). *Let $\mathbb{Z}_n$ denote the cyclic group of order n, and let $\mathbb{Z}_n^2 = \mathbb{Z}_n \times \mathbb{Z}_n$. Each of the following holds:*
(i) *There exists a graph that is $\mathbb{Z}_4$-connected but not $\mathbb{Z}_2^2$-connected.*
(ii) *There exists a graph that is $\mathbb{Z}_2^2$-connected but not $\mathbb{Z}_4$-connected.*

Naturally, there is a corresponding problem in group coloring (see Problems 4.11 and 4.12 of [6]): for given groups Γ_1 and Γ_2 with $|\Gamma_1| = |\Gamma_2|$, is a Γ_1-colorable graph also Γ_2-colorable? Equivalently, we may ask whether any aspect of group structure (other than the cardinality of the group) plays a role in group colorings. The purpose of this research is an attempt to address this question. More precisely, we are to investigate answers to the following question.

QUESTION 1. *Is the Γ-colorability of a graph G entirely dependent on G and $|\Gamma|$?*

As the graphs showing the validity of Theorem 1.1 in [4] are nonplanar, we cannot apply duality and quote Theorem 1.1 to answer Question 1. In this paper, we show that there are infinitely many ordered triples (G, Γ_1, Γ_2) in which G is a graph, Γ_1 and Γ_2 are groups with $|\Gamma_1| = |\Gamma_2|$, and G is Γ_1-colorable but not Γ_2-colorable. While the justification of Theorem 1.1 uses computer checking, we develop a concept called snarls which characterizes certain phenomena observed in group colorings and use the snarl to construct our examples. The next section will be devoted to the development of snarls, followed by a section dedicated to constructing examples negating Question 1, and finally a section outlining possible refinements of Question 1.

2. The snarls. To give a negative answer to Question 1, it is useful to first introduce the concept of a *snarl*. This section will define this phenomenon as well as give sufficient and necessary conditions for a snarl to occur. Throughout this section, G denotes a graph, Γ denotes a multiplicative group with identity 1, and as in [3], for a subset $F \subseteq \Gamma$, we define $F^{-1} = \{x^{-1} : x \in F\}$. Before we consider the definition of a snarl, it will be useful to prove a small proposition that will remove some cases from consideration and significantly simplify the notation.

DEFINITION 2.1. *Let G be a multigraph under an orientation D such that any two parallel edges have the same direction, and let Γ be a group such that $|uv| \leq |\Gamma|$ for each $uv \in E(\hat{G})$. If $f \in F(G, \Gamma)$ such that $f|_{[uv]}$ is injective for each $uv \in E(\hat{G})$, then we say that f is* piecewise injective.

PROPOSITION 2.2. *Let G be a multigraph, and let Γ be a group such that $|uv| \leq |\Gamma|$ for each $uv \in E(\hat{G})$. If G is (Γ, f)-colorable for every piecewise injective $f \in F(G, \Gamma)$, then G is Γ-colorable.*

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Proof. Let G and Γ be as described above, and fix some orientation D of G such that any two parallel edges have the same direction. Assume that G is (Γ, f)-colorable for any $f \in F(G,\Gamma)$ such that f is piecewise injective. Let $f_0 \in F(G,\Gamma)$ such that $f_0|_{[uv]}$ is *not* piecewise injective. We now recursively define, for some $p \in \mathbb{N}$, $f_1, f_2, \ldots, f_p$ as follows: Choose $[uv]$ such that $uv \in E(\hat{G})$ and any $e_1, e_2 \in [uv]$ with $e_1 \neq e_2$ such that $f_i(e_1) = f_i(e_2)$. Let $f_{i+1}(e_1) = f_i(e_1)$ and $f_{i+1}(e_2) \in \Gamma - \{f_i(e) : e \in [uv]\}$. Since we consider only finite graphs, this recursion will eventually halt, yielding $f_p \in F(G,\Gamma)$ such that $f_p|_{[uv]}$ is piecewise injective. By assumption, there exists a (Γ, f_p)-coloring c of G under D. Since $f_0([uv]) \subseteq f_p([uv])$ for each $uv \in E(\hat{G})$, c must also be a (Γ, f_0)-coloring of G under D. Since f_0 was arbitrarily chosen, it follows directly from the definition that G is Γ-colorable. ∎

With Proposition 2.2 in mind, for the remainder of the paper we need only consider piecewise injective $f \in F(G,\Gamma)$. While doing so, we will also work under the assumption that G is under an orientation D such as is described in Definition 2.1.

DEFINITION 2.3. *For a fixed piecewise injective $f \in F(G,\Gamma)$ and an edge $uv \in E(G)$, let $f(A_{uv}) = \{f(e) : e \in A_{uv}\}$, where $A_{uv} \subseteq [uv]$. A $(u, v; f)$-*snarl *is a 3-tuple (L_u, L_v, A_{uv}) with $|L_u| = |L_v| = |A_{uv}|$ and $L_u, L_v \subseteq \Gamma$, satisfying the property that for any $a \in L_u$, $f(A_{uv})^{-1}a = L_v$.*

To simplify our notation, when f is understood or assumed in the context, we often use a (u, v)-snarl for a $(u, v; f)$-snarl. We further define that a (u, v)-snarl with $A_{uv} = [uv]$ is a *full width (u, v)-snarl*.

PROPOSITION 2.4. *For a given piecewise injective $f \in F(G,\Gamma)$, if there exists a (u, v)-snarl (L_u, L_v, A_{uv}), then each of the following holds:*

(i) *For any $e_p \in A_{uv}$ and $a_q \in L_u$, there exists an element $b_z \in L_v$ such that $f(e_p)^{-1}a_q = b_z$.*

(ii) *For any $a_p \in L_u$ and $b_q \in L_v$, there exists an $e_z \in A_{uv}$ such that $f(e_z)^{-1}a_p = b_q$.*

(iii) *For any $e_p \in A_{uv}$ and $b_q \in L_v$, there exists an $a_z \in L_u$ such that $f(e_p)^{-1}a_z = b_q$.*

(iv) *For any $a_0 \in \Gamma - L_u$, we have $\{f(e)^{-1}a_0 : e \in A_{uv}\} \cap L_v = \emptyset$.*

Proof. By Definition 2.3, (i) and (ii) follow directly. We argue by contradiction to prove (iii) and assume that for any $a_z \in L_u$, $f(e_p)^{-1}a_z \neq b_q$. Then $f(e_p)^{-1}L_u \subsetneq L_v$. This implies that there exist distinct a_r and a_s in L_u and $b_t \in L_v$ such that $f(e_p)^{-1}a_r = f(e_p)^{-1}a_s = b_t$, which forces $a_r = a_s$, a contradiction. Hence (iii) must hold.

We also argue by contradiction to prove (iv) and assume that there exists an $a_0 \in \Gamma - L_u$ and an $e_p \in A_{uv}$ such that $f(e_p)^{-1}a_0 = b_q$ for some $b_q \in L_v$. Then, by (iii), there must be an $a_z \in L_u$ such that $f(e_p)^{-1}a_z = b_q$, forcing $a_0 = a_z \in L_u$, a contradiction. ∎

We will now work towards necessary and sufficient conditions for a snarl to occur. Through the rest of this subsection, we assume that G is a given graph and that $uv \in E(G)$, and unless otherwise stated, we always use (L_u, L_v, A_{uv}) to denote a (u, v)-snarl, assuming the existence of it.

DEFINITION 2.5. *For such a (u, v)-snarl, (L_u, L_v, A_{uv}), we may define $H_1 = \{a_i a_j^{-1} : a_i, a_j \in L_u\}$, $H_2 = \{a_i^{-1}a_j : a_i, a_j \in L_u\}$, and $H_3 = \{b_i b_j^{-1} : b_i, b_j \in L_v\}$.*

PROPOSITION 2.6. *For a given piecewise injective $f \in F(G,\Gamma)$ and a (u, v)-snarl (L_u, L_v, A_{uv}), each of the following holds:*

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(i) *If* $F_1 = \{f_i f_j^{-1} : f_i, f_j \in f(A_{uv})\}$, *then* $H_1 = F_1$.
(ii) *If* $F_2 = \{b_i^{-1} b_j : b_i, b_j \in L_v\}$, *then* $H_2 = F_2$.
(iii) *If* $F_3 = \{f_i^{-1} f_j : f_i, f_j \in f(A_{uv})\}$, *then* $H_3 = F_3$.
(iv) *Given any* $a_i, a_j, a_k \in L_u$, *there exists* $a_l \in L_u$ *such that* $a_i a_j^{-1} = a_l a_k^{-1}$.
(v) *Given any* $b_i, b_j, b_k \in L_v$, *there exists* $b_l \in L_v$ *such that* $b_i b_j^{-1} = b_l b_k^{-1}$.

Proof. For any $a_i, a_j \in L_u$, by Proposition 2.4(ii), there exist $f_y, f_z \in f(A_{uv})$ and $b_q \in L_v$ such that $a_i a_j^{-1} = f_y b_q^{-1}(f_z b_q^{-1})^{-1} = f_y f_z^{-1} \in F_1$, implying $H_1 \subseteq F_1$. Conversely, for any $f_i, f_j \in f(A_{uv})$, by Proposition 2.4(iii) there exist $a_i, a_j \in L_u$ and $b_q \in L_v$ such that $f_i f_j^{-1} = a_i b_q (a_j b_q^{-1})^{-1} = a_i a_j^{-1} \in H_1$, implying $F_1 \subseteq H_1$. This proves (i). The proofs for (ii) and (iii) are similar and so are omitted.

By Proposition 2.4(i), there exists an element $b_x \in L_v$ such that $f_z = a_k b_x^{-1}$, and by Proposition 2.4(iii), there exists $a_l \in L_u$ such that $f_y = a_l b_x^{-1}$. Therefore, $a_i a_j^{-1} = f_y f_z^{-1} = (a_l b_x^{-1})(a_k b_x^{-1})^{-1} = a_l a_k^{-1}$. This proves (iv).

Finally, by Proposition 2.4(iii) there exists $a_z \in L_u$ such that $f_x^{-1} a_z = b_k$ and by Proposition 2.4(i) there exists $b_l \in L_v$ such that $f_w^{-1} a_z = b_l$. Thus $b_i b_j^{-1} = f_w^{-1} f_x = (a_z b_l^{-1})^{-1}(a_z b_k^{-1}) = b_l a_z^{-1} a_z b_k^{-1} = b_l b_k^{-1}$. This proves (v). ∎

There is some symmetry at play here, so there are more properties similar to those written in Proposition 2.6, including an "f" property corresponding to (iv) and (v). In the interest of brevity, Proposition 2.6 is limited to include only what is needed to prove Theorem 2.8.

LEMMA 2.7. *The subsets* H_1, H_2, H_3 *defined in Definition* 2.5 *are conjugate subgroups of* Γ.

Proof. Given $h_1, h_2 \in H_1$, by Definition 2.5, there exist some $a_i, a_j, a_p, a_q \in L_u$ such that we have $h_1 = a_i a_j^{-1}$ and $h_2 = a_p a_q^{-1}$. By Proposition 2.6(iv), we may also write $h_1 = a_l a_q^{-1}$ for some element a_l so that $h_1 h_2^{-1} = a_l a_q^{-1}(a_p a_q^{-1})^{-1} = a_l a_p^{-1} \in H_1$. As $H_1 \neq \emptyset$, H_1 is a subgroup of Γ. To prove the lemma, we first justify the following two claims.

Claim 1. For any element $a \in L_u$, $aH_2 = L_u = H_1 a$.

By Definition 2.5, we have $H_2 = \{a_i^{-1} a_j : a_i, a_j \in L_u\}$. Thus for any $a_j \in L_u$, as $a \in L_u$, we have $a_j = aa^{-1} a_j \in aH_2$, and so $L_u \subseteq aH_2$. Conversely, take an arbitrary element $aa_i^{-1} a_j \in aH_2$. By Proposition 2.6(iv), there exists an element $a_l \in L_u$, with which we may write $aa_i^{-1} a_j = a_l a_j^{-1} a_j = a_l \in L_u$. This implies that $aH_2 \subseteq L_u$, and so $aH_2 = L_u$. Again by Definition 2.5, we have $H_1 = \{a_i a_j^{-1} : a_i, a_j \in L_u\}$. Thus for any $a_i \in L_u$, as $a \in L_u$, we have $a_i = a_i a^{-1} a \in H_1 a$ as well, and so $L_u \subseteq H_1 a$. Conversely, pick an arbitrary element $a_i a_j^{-1} a \in H_1 a$. By Proposition 2.6(iv), there exists an element $a_l \in L_u$ so that $a_i a_j^{-1} a = a_l a^{-1} a = a_l \in L_u$. Thus $H_1 a \subseteq L_u$, and so $H_1 a = L_u$. This completes the proof of the claim.

Claim 2. For any element $b \in L_v$, $bH_2 = L_v = H_3 b$.

By Proposition 2.6(ii), we have $H_2 = \{b_i^{-1} b_j : b_i, b_j \in L_v\}$. Thus for any $b_j \in L_v$, as $b \in L_v$, we have $b_j = bb^{-1} b_j \in bH_2$, and so $L_v \subseteq bH_2$. Conversely, take an arbitrary element $bb_i^{-1} b_j \in bH_2$. By Proposition 2.6(v), there exists an element $b_l \in L_v$ such that $bb_i^{-1} b_j = b_l b_j^{-1} b_j = b_l \in L_v$. This proves $bH_2 = L_v$. By Definition 2.5, we have $H_3 = \{b_i b_j^{-1} : b_i, b_j \in L_v\}$. Thus, for any $b_i \in L_v$, as $b \in L_v$, we have $b_i = b_i b^{-1} b \in H_3 b$, and so $L_v \subseteq H_3 b$. Conversely, for any $b_i b_j^{-1} b \in H_3 b$, by Proposition 2.6(v), there exists an element $b_l \in L_v$ such that $b_i b_j^{-1} b = b_l b^{-1} b = b_l \in L_v$, and so $H_3 b \subseteq L_v$. This validates the claim.

By Claims 1 and 2, for elements $a \in L_u$ and $b \in L_v$, we have $H_1 = aH_2 a^{-1}$

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and $H_3 = bH_2b^{-1}$. Thus H_2 and H_3 are conjugates of H_1, and so H_1, H_2, and H_3 are conjugates of one another. It is well known (see [3, Chapter II, section 4]) that the conjugates of a subgroup is itself a subgroup. Thus H_1, H_2, H_3 are conjugate subgroups of Γ. ∎

Now we may show some sufficient and necessary conditions for a snarl to occur.

THEOREM 2.8. *Let $f \in F(G, \Gamma)$ be a given piecewise injective mapping. For some $uv \in E(G)$, let $L_u, L_v, F \subseteq \Gamma$ such that for some $A_{uv} \subseteq [uv]$, we have $F = f(A_{uv})$. Then the following are equivalent:*

(i) *(L_u, L_v, A_{uv}) is a (u, v)-snarl.*

(ii) *There exist conjugate subgroups H_1, H_2, and H_3 of Γ and elements $a, b, f \in \Gamma$ satisfying $ab^{-1} = f$ such that $L_u = H_1a = aH_2$, $L_v = H_3b = bH_2$, and $F = H_1f = fH_3$.*

Proof. By Lemma 2.7, (i) implies (ii). Assume that (ii) holds. We are to show that Definition 2.3 must hold. For any $a_j \in L_u$, there exists an element $h_1 \in H_1$ such that $a_j = h_1a$, and so $\{f(e)^{-1}a_j : e \in A_{uv}\} = F^{-1}a_j = (H_1f)^{-1}h_1a$. As $h_1 \in H_1$ and as H_1 is a subgroup, $H_1^{-1}h_1 = H_1^{-1}$, and so we have $(H_1f)^{-1}h_1a = f^{-1}H_1^{-1}h_1a = f^{-1}H_1^{-1}a = (H_1f)^{-1}a = (fH_3)^{-1}a = H_3^{-1}f^{-1}a = H_3b = L_v$. It follows, by Definition 2.3, that (L_u, L_v, A_{uv}) is a (u, v)-snarl. ∎

Corollary 2.9 follows from Theorem 2.8 immediately.

COROLLARY 2.9. *Let G be a graph, let A be an abelian group, and let $f \in F(G, A)$ be a fixed piecewise injective mapping. For some $uv \in E(G)$, let $L_u, L_v, F \subseteq A$ be subsets of A such that for some $A_{uv} \subseteq [uv]$, we have $F = f(A_{uv})$. Then the following are equivalent:*

(i) *(L_u, L_v, A_{uv}) is a (u, v)-snarl.*

(ii) *There exists a subgroup H of A and elements $a, b, f \in A$ satisfying $ab^{-1} = f$ such that $L_u = aH$, $L_v = bH$, and $F = fH$.*

Proof. By Theorem 2.8, Corollary 2.9(i) holds if and only if there exist conjugate subgroups H_1, H_2, and H_3 of A and elements $a, b, f \in A$ satisfying $ab^{-1} = f$ such that $L_u = H_1a = aH_2$, $L_v = H_3b = bH_2$, and $F = H_1f = fH_3$. Since A is abelian, and since H_1, H_2, and H_3 are conjugate subgroups of A, we have $H := H_1 = H_2 = H_3$, and so in this case, Theorem 2.8(ii) holds if and only if Corollary 2.9(ii) holds. ∎

3. Counterexamples. We now use the characterization of a snarl developed in the previous section to give a counterexample for Question 1 in the two propositions that follow. We denote by A_4 the alternating group of degree 4. Recall that $|A_4| = 12$. As in [1], $\delta(G)$ denotes the minimum degree of a graph G. Define $\bar{\delta}(G) = \max\{\delta(H) :$ for any subgraph H of $G\}$. This value is known as the *degeneracy* of G, and $\bar{\delta}(G) + 1$ is often called the *coloring number* of G, sometimes denoted by $col(G)$. It is shown in Lemma 4.2 of [7] (see also Corollary 2.3 of [8]) that for any multigraph G,

$$\chi_g(G) \leq \bar{\delta}(G) + 1. \tag{1}$$

The inequality holds for standard vertex coloring, as well as many other coloring notions, including list coloring and DP-coloring. The proof is essentially the same for any of these and can be routinely verified.

PROPOSITION 3.1. *The graph $6K_3$ is A_4-colorable.*

Proof. Let $f \in F(6K_3, A_4)$, and let v_1, v_2, and v_3 denote the three vertices of $6K_3$. To show that $6K_3$ is A_4-colorable, it suffices to construct an (A_4, f)-coloring

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of $6K_3$. First, we arbitrarily choose $c(v_1) \in A_4$. With Proposition 2.2 in mind, we may assume that f is piecewise injective. This leaves us with exactly six choices for $c(v_2)$ (which we denote by L_2) and exactly six choices for $c(v_3)$ (which we denote by L_3). We may also assume without loss of generality that each arc in $[v_2v_3]$ is oriented towards v_3.

Assume by way of contradiction that for each $\gamma \in L_2$ we have $(f[v_2v_3])^{-1}\gamma = L_3$. Then we have, by Definition 2.3, a full-width v_2v_3-snarl. By Theorem 2.8, A_4 must then have a subgroup H with $|H| = 6$, although it is well known that A_4 has no subgroup of order 6. We assume then that there exists $\gamma \in L_2$ such that $L_3 \not\subset (f[v_2v_3])^{-1}\gamma$, and so we may set $c(v_2) = \gamma$ and $c(v_3) \in (f[v_2v_3])^{-1}\gamma - L_3$ to complete the coloring. ◻

PROPOSITION 3.2. *The graph $6K_3$ is not $\mathbb{Z}_{12}$-colorable.*

Proof. Define $\Gamma_e = 2\mathbb{Z}_{12}$ to be the cyclic subgroup of $\mathbb{Z}_{12}$ with six elements, and let $\Gamma_o = \mathbb{Z}_{12} - \Gamma_e$. Let v_1, v_2, and v_3 denote the three vertices of $6K_3$. We may assume without loss of generality that arcs are oriented from v_1 to v_2, v_2 to v_3, and v_3 to v_1. We define $f \in F(G, \mathbb{Z}_{12})$ with $f[v_1v_2] = f[v_2v_3] = f[v_3v_1] = \Gamma_e$. To complete the proof, it suffices to show that $6K_3$ is not $(\mathbb{Z}_{12}, f)$-colorable. Indeed, if we choose $c(v_1) \in \Gamma_o$, then by the definition of f, we must have $c(v_2) \in \Gamma_e$ (yielding a conflict at $[v_3v_1]$), and so we cannot choose $c(v_1) \in \Gamma_o$. A similar argument shows that $c(v_1) \in \Gamma_e$ cannot be chosen either, and thus there can be no $(\mathbb{Z}_{12}, f)$-coloring of $6K_3$. ◻

Later, we shall show that for any abelian group Γ with $|\Gamma| = 12$, $6K_3$ is not Γ-colorable. The important structural distinction here lies in the fact that A_4 has no subgroup of order 6, whereas $\mathbb{Z}_{12}$ does. Since $|A_4| = |\mathbb{Z}_{12}|$, it is clear that group structure does play some role in group coloring, and we can answer Question 1 in the negative.

Before constructing more counterexamples it will be useful to define some terminology. First, consider the following well-known theorem of Lagrange.

THEOREM 3.3. *If H is a subgroup of a finite group Γ, then $|H|$ divides $|\Gamma|$.*

It is important to note that the converse of Theorem 3.3 does not always hold; there exist many groups Γ such that h divides $|\Gamma|$ but Γ has no subgroup of order h. Furthermore, this failure of the converse was vital to constructing the counterexample for Question 1 in Propositions 3.1 and 3.2. The converse of Theorem 3.3 has in fact been studied widely, but there is no known characterization of all groups for which it holds.

DEFINITION 3.4. *A group Γ is called a* CLT group *or a* Lagrangian group *if for each h such that h divides $|\Gamma|$ there is a subgroup H of Γ with $|H| = h$.*

It is well known (see [3, Chapter II, Corollary 2.4]) that any abelian group is a CLT group. Therefore, further counterexamples for Question 1 using the same technique as above must have at least one of Γ_1 or Γ_2 be non-abelian. Any such counterexample must specifically include a group that is not CLT. The following lemma appears as an exercise in [9] and gives an infinite set of such non-CLT groups.

LEMMA 3.5. *For $n \geq 5$, the symmetric group S_n has no subgroup H with $2 < [S_n : H] < n$.*

We shall use Lemma 3.5 to construct an infinite set of counterexamples for Question 1.

THEOREM 3.6. *For each $n \geq 5$ and $2 < t < n$, $\frac{n!}{t}K_{t+1}$ is S_n-colorable.*

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Proof. Let $G = \frac{n!}{t}K_{t+1}$, and let $f \in F(G, S_n)$ be piecewise injective.

Fix two vertices $u, v \in V(G)$, and define $J = G - [uv]$ to be the graph formed by removing all parallel edges in $[uv]$. Then $\bar{\delta}(J) = (t-1)\frac{n!}{t} < n! = |S_n|$. By (1), $|S_n| \geq \bar{\delta}(J) + 1 \geq \chi_g(J)$, and so J is S_n-colorable. In particular, J has an $(S_n, f|J)$-coloring c. We will show that at least one such $(S_n, f|J)$-coloring of J is also an (S_n, f)-coloring of G. Since $d_J(u) = d_J(v) = (t-1)\frac{n!}{t} < |S_n|$, we may modify c by changing $c(u)$ and $c(v)$, and it will still be an $(S_n, f|J)$-coloring of J. To be precise, we have at least $|S_n| - d_J(u) = \frac{n!}{t}$ choices for modifying $c(u)$ and likewise at least $\frac{n!}{t}$ choices for modifying $c(v)$. Let $L_u, L_v \subset S_n$ denote the sets of choices for $c(u)$ and $c(v)$, respectively. We may assume without loss of generality that each arc in $[uv]$ is oriented towards v.

First we consider the case where $|L_u| > \frac{n!}{t}$ or $|L_v| > \frac{n!}{t}$. Without loss of generality, we assume $|L_u| > \frac{n!}{t}$. Then we may arbitrarily choose $c(v) \in L_v$ and be assured that at least one choice remains for $c(u) \in u$. Now we consider the case where $|L_u| = |L_v| = \frac{n!}{t}$. Assume by way of contradiction that for each $\gamma \in L_u$ we have $(f[uv])^{-1}\gamma = L_v$. Then we have, by Definition 2.3, a full-width uv-snarl. By Theorem 2.8, S_n must then have a subgroup H with $|H| = |uv| = \frac{n!}{t}$. However, this would imply that $[S_n : H] = t$, contradicting Lemma 3.5. We assume then that there exists $\gamma \in L_u$ such that $L_v \not\subset (f[uv])^{-1}\gamma$, and so we may set $c(u) = \gamma$ and $c(v) \in (f[uv])^{-1}\gamma - L_v$. Now c is an (S_n, f)-coloring of G, and thus G is S_n-colorable. ∎

THEOREM 3.7. *Let Γ be a group with $|\Gamma| = st$, and let H be a subgroup of Γ such that $|H| = t$. Then tK_{s+1} is not Γ-colorable.*

Proof. Let Γ be a group with $|\Gamma| = st$ and H be a subgroup of Γ such that $|H| = t$ and $G = tK_{s+1}$. We shall construct a mapping $f \in F(G, \Gamma)$ such that G is not (Γ, f)-colorable. For each $u, v \in V(K_{s+1})$, let $f[uv] = H$. Assume by contradiction that there exists a (Γ, f)-coloring $c : V(G) \to \Gamma$.

Claim 1. For any pair of vertices $u, v \in V(tK_{s+1})$, $c(u)$ and $c(v)$ must be in different right cosets of H in Γ.

By way of contradiction, we assume there exist such vertices $u, v \in V(tK_{s+1})$ such that $c(u), c(v) \in Hg$ for some $g \in \Gamma$. We may assume without loss of generality that each arc in $[uv]$ is oriented towards v. It is well known that two cosets of a subgroup are either identical or disjoint since congruence modulo H is an equivalence relation (see, for example, Chapter I, Theorem 4.2, and Corollary 4.3 of [3]). Therefore, since $c(v) \in Hg$, we have $Hc(v) \cap Hg \neq \emptyset$, and so $Hc(v) = Hg$. As we also have $c(u) \in Hg = Hc(v)$, it follows that $c(u)[c(v)]^{-1} \in H$. As $f[uv] = H$, c cannot be a (Γ, f)-coloring, contrary to the assumption that c is a (Γ, f)-coloring. This proves Claim 1.

By Claim 1, for any $u, v \in V(tK_{s+1})$, $c(u)$ and $c(v)$ must be in different cosets of H in Γ. However, as there are $s+1$ vertices and only s cosets, we conclude that such a coloring c does not exist. ∎

COROLLARY 3.8. *Let A be an abelian group with $|A| = st$. Then tK_{s+1} is not A-colorable.*

Proof. Since A is abelian and t divides A, there must be a subgroup H with $|H| = t$ (see [3, Chapter II, Corollary 2.4]). Corollary 3.8 now follows directly from Theorem 3.7. ∎

By Corollary 3.8 with $s = 2$ and $t = 6$, we observe that the conclusion of Proposition 3.2 can be extended to the form that for every abelian group A with $|A| = 12$, $6K_3$ is not A-colorable, as mentioned earlier. Moreover, it is not difficult to see that

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Theorems 3.6 and 3.7 together constitute an infinite and somewhat diverse set of examples for graphs where group structure matters in group coloring. As stated before, the CLT property has not been characterized but has been otherwise studied. A look at [2] tells us that CLT groups are solvable, a more well known property than CLT. Therefore, one could create more examples using nonsolvable groups.

All of the previous examples in this section have only one of Γ_1 and Γ_2 as a non-abelian group, but this is not a necessary condition for a counterexample to exist. We may note that there is a (non-abelian) dicyclic group Dic_3 (also written as Q_{12}) with order 12 and a subgroup of order 6. Using virtually the same proof as in Theorem 3.7, we see that $6K_3$ is not Dic_3-colorable. However, as previously stated, $6K_3$ is A_4-colorable even though $|\mathrm{Dic}_3| = |A_4| = 12$.

4. Further questions.

After answering Question 1 in the negative, there are some natural first steps toward determining whether $\chi_a(G)$ and $\chi_g(G)$ can differ. In this last section, we state these further questions and discuss them briefly.

Question 2. *Is there a graph G such that G is A-colorable but not Γ-colorable where A is an abelian group, Γ is a non-abelian group, and $|A| = |\Gamma|$?*

Question 3. *Is there a graph G such that G is A_1-colorable but not A_2-colorable where A_1 and A_2 are two abelian groups with $|A_1| = |A_2|$?*

Questions 2 and 3 can, like Question 1, be considered steps toward determining whether $\chi_a(G)$ and $\chi_g(G)$ can differ. The methods used in this note to answer Question 1 cannot be easily applied to these further questions since abelian groups are CLT; i.e., they always have subgroups of order k for any k that divides the order of the group (see [3, Chapter II, Corollary 2.4]). Indeed, Corollary 3.8 demonstrates that a new approach is needed in answering these further questions. Finally, the goal of all previously mentioned questions is to work towards answering the following.

Question 4. *Does there exist a graph G such that $\chi_a(G) < \chi_g(G)$?*

5. Appendix: Proof of Lemma 3.5.

Notation: For a group G, let G' denote the commutator subgroup of G, the subgroup generated by the subset $\{aba^{-1}b^{-1} : a, b \in G\}$ in G, and let the center of a G be $C(G) = \{a \in G : gag^{-1} = a \text{ for each } g \in G\}$.

Lemma 5.1. *Let $N \triangleleft G$ such that $N \cap G' = \{1\}$. Then $N \subseteq C(G)$.*

Proof. Let $n \in N$ and $g \in G$. Then $gng^{-1}n^{-1} = n'n^{-1} \in N$ (where $n' \in N$). By definition, $gng^{-1}n^{-1} \in G'$ as well. Since $N \cap G' = \{1\}$, we must have $gng^{-1}n^{-1} = 1$ so that $gng^{-1} = n$ and thus $n \in C(G)$. □

Lemma 5.2. *A_n is the only nontrivial normal subgroup of S_n for $n \geq 5$.*

Proof. Assume by way of contradiction that N is a nontrivial normal subgroup of S_n and $N \neq A_n$. Then $N \cap A_n \triangleleft A_n$. Since A_n is simple for $n \neq 4$ (see Chapter I and Theorem 6.10 of [3]), we must have either $N \cap A_n = \{1\}$ or $A_n \triangleleft N \triangleleft S_n$. Since $[S_n : A_n] = 2$, we cannot have $A_n \triangleleft N \triangleleft S_n$. Therefore, we may assume that $N \cap A_n = \{1\}$. In this case, we first note that $A_n = S_n'$. Now by Lemma 5.1, we have $N \subseteq C(S_n) = \{1\}$, a contradiction. □

Lemma 5.3. *For each $n \geq 5$, the symmetric group S_n has no subgroup H with $2 < [S_n : H] < n$.*

Proof. Let H be a nontrivial subgroup of S_n such that $[S_n : H] < n$. To complete the proof, it suffices to show that $[S_n : H] \leq 2$. First, we note that S_n acts on

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the set S_n/H of all left cosets of H in S_n by left translation. This group action induces a homomorphism $f : S_n \to A(S_n/H)$, where $A(S_n/H)$ is the group of all permutations on the set S_n/H, and furthermore $\ker(f) \subseteq H$ (see Chapter II, Theorem 4.5, and Proposition 4.8 of [3]). Now, $\ker(f)$ is a normal subgroup of S_n. Since $|S_n| = n! > [S_n : H]! = |A(S_n/H)|$, f cannot be injective, and thus $\ker(f) \neq \{\mathbb{1}\}$. Certainly, $\ker(f) \neq S_n$ since H is nontrivial and $\ker(f) \subseteq H$. Therefore, by Lemma 5.2, we must have $\ker(f) = A_n$. Since $[S_n : \ker(f)] = [S_n : A_n] = 2$ and $\ker(f) \subseteq H$, we must have $[S_n : H] \leq 2$. ∎

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