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Matching and spanning trails in digraphs

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ABSTRACT

Let D be a digraph and let $\alpha(D)$, $\alpha'(D)$ and $\lambda(D)$ be independence number, the matching number and the arc-strong connectivity of D, respectively. Bang-Jensen and Thommassé in 2011 conjectured that every digraph D with $\lambda(D) \geq \alpha(D)$ is supereulerian. In [J. Graph Theory, 81(4), (2016) 393-402], it is shown that every digraph D with $\lambda(D) \geq \alpha'(D)$ is supereulerian. In this paper, we introduced the symmetric core of a digraph and use it to show that each of the following holds for a strong digraph D on $n \geq 3$ vertices with $\lambda(D) \geq \alpha'(D) - 1$.

(i) There exists a family $\mathcal{D}(n)$ of well-characterized digraphs such that for any digraph D with $\alpha'(D) \leq 2$, D has a spanning trial if and only if D is not a member in $\mathcal{D}(n)$.

(ii) If $\alpha'(D) \geq 3$, then D has a spanning trail.

(iii) If $\alpha'(D) \geq 3$ and $n \geq 2\alpha'(D) + 3$, then D is supereulerian.

(iv) If $\lambda(D) \geq \alpha'(D) \geq 4$ and $n \geq 2\alpha'(D) + 3$, then for any pair of vertices u and v of D, D contains a spanning (u, v)-trail.

1. Introduction

Throughout this paper, we use G to denote a graph and D a digraph. Graphs and digraphs considered are finite with undefined terms and notation will follow [9] for graphs and [3] for digraphs. As in [3], a digraph D is one that does not have loops and parallel arcs. Thus $\kappa(G)$, $\kappa'(G)$, $\alpha(G)$ and $\alpha'(G)$ denote the connectivity, the edge connectivity, the stability number (also called the independence number), and the matching number of a graph G; and $\kappa(D)$ and $\lambda(D)$ denotes the vertex-strong connectivity and the arc-strong connectivity of a digraph D, respectively. The **indegree** and **outdegree** of a vertex v in a digraph D are denoted by $d^-_D(v)$ and $d^+_D(v)$, respectively. We often use $G(D)$ to denote the underlying graph of D, the graph obtained from D by erasing all orientation on the arcs of D. The stability number and the matching number of a digraph D are defined as

$$\alpha(D) = \alpha(G(D)) \text{ and } \alpha'(D) = \alpha'(G(D)),$$

respectively. Throughout this paper, we use paths, cycles, and trails as defined in [9] when the discussion is on an undirected graph G, and to denote directed paths, directed cycles and directed trails when the discussion is on a digraph D. A directed trail (or path, respectively) from a vertex u to a vertex v in a digraph D is often refereed as to a (u, v)-trail (a (u, v)-path, respectively).

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The supereulerian problem was introduced by Boesch, Suffel, and Tindell in [8], seeking to characterize graphs that have spanning Eulerian subgraphs. Pulleyblank in [19] proved that determining whether a graph is supereulerian, even within planar graphs, is NP-complete. There have been lots of researches on this topic. For more literature on supereulerian graphs, see Catlin's informative survey [10], as well as the later updates in [11] and [17]. The supereulerian problem in digraphs is considered by Gutin [13,14]. A strong digraph D is **eulerian** if for any $v \in V(D)$, $d_D^+(v) = d_D^-(v)$. A digraph D is **supereulerian** if D contains a spanning eulerian subdigraph, or equivalently, a spanning closed trail. Thus supereulerian digraphs must be strong, and every hamiltonian digraph is also a supereulerian digraph.

The supereulerian digraph problem is to characterize the strong digraphs that contains a spanning closed trail. Other than the researches on hamiltonian digraphs, a number of studies on supereulerian digraphs have been conducted recently. In particular, Hong et al. in [15,16] and Bang-Jensen and Maddaloni [5] presented some best possible sufficient degree conditions for supereulerian digraphs. Several researches on various conditions of supereulerian digraphs can be found in [2,4,18], among others.

A well known theorem of Chvátal and Erdös [12] states that every 2-connected graph G with $\kappa(G) \geq \alpha(G)$ is hamiltonian. Thomassen [20] indicated that the Chvátal–Erdös Theorem does not extend to digraphs by presenting an infinite family of non hamiltonian (but supereulerian) digraphs D with $\kappa(D) = \alpha(D) = 2$. This motivates Bang-Jensen and Thommassé (2011, unpublished, see [6]) to make the following conjecture.

Conjecture 1.1 (*Bang-Jensen and Thommassé [5,6]*). *Let D be a digraph. If $\lambda(D) \geq \alpha(D)$, then D is supereulerian.*

A number of studies have been conducted towards Conjecture 1.1, In [5], Bang-Jensen and Maddaloni verified the validity of Conjecture 1.1 for several families of digraphs, including semicomplete multipartite digraphs and quasitransitive digraphs. The following have been proved.

Theorem 1.2. *Let D be a strong digraph.*
(i) (Alfegari and Lai, Theorem 1.5 of [1]) If $\lambda(D) \geq \alpha'(D)$, then D is supereulerian.
(ii) (Zhang et al. Theorem 1.5 of [21]) If $G(D)$ is a bipartite digraph and $\lambda(D) \geq \left\lfloor \frac{\alpha(D)}{2} \right\rfloor + 1$, then D is supereulerian.

A digraph D is **strongly trail-connected** if for any two vertices u and v of D, D possess both a spanning (u, v)-trail and a spanning (v, u)-trail. As the case when $u = v$ is possible, every strongly trail-connected digraph is also supereulerian. In Section 3, we shall introduce a digraph family $\mathcal{D}(n)$ each of whose members does not have a spanning trail with its underlying graph spanned by a $K_{2,n-2}$. The following is our main result.

Theorem 1.3. *Let D be a strong digraph on $n \geq 12$ vertices satisfying $\lambda(D) \geq \alpha'(D) - 1$. Each of the following holds.*
(i) If $\alpha'(D) \leq 2$, then D has a spanning trail if and only if D is not a member in $\mathcal{D}(n)$.
(ii) If $\alpha'(D) \geq 3$, then D has a spanning trail.
(iii) If $\alpha'(D) \geq 3$ and $n \geq 2\alpha'(D) + 3$, then D is supereulerian.
(iv) If $\lambda(D) \geq \alpha'(D) \geq 4$ and $n \geq 2\alpha'(D) + 3$, then D is strongly trail-connected.

Theorem 1.3 (*iii*) and (*iv*) extended Theorem 1.2 (*i*) when $\alpha'(D)$ and $|V(D)|$ are sufficiently large. In the next section, we present some preliminaries including several structural analysis lemmas. The proof of the main result will be given in the last section.

2. Preliminaries

Let D be a digraph on n vertices, and let $k = \alpha'(D)$. Thus $n \geq 2k$. If $G = G(D)$ for a digraph D, then as D may possess a 2-cycle, it is possible for G to have parallel edges. Throughout our discussions, we use the notation (u, v) to denote an arc oriented from u to v in a digraph D; and use $[u, v]$ to denote either (u, v) or (v, u). When $[u, v] \in A(D)$, we say that u and v are adjacent. If two arcs of D have a common vertex, we say that these two arcs are adjacent in D. If X is a vertex subset or an arc subset of D, we use $D[X]$ to denote the subdigraph of D induced by X. If e is an edge in a graph G or an arc in a digraph D incident with vertices u and v, define $V(e) = \{u, v\}$. As in [3], we define, for a vertex $v \in V(D)$, $N_D^+(v) = \{w \in V(D) : (v, w) \in A(D)\}$, $N_D^-(v) = \{u \in V(D) : (u, v) \in A(D)\}$ and $N_D(v) = N_D^+(v) \cup N_D^-(v)$. For a subset $X \subseteq V(D)$, define $N_D(X) = \cup_{x \in X} N_D(x)$.

For an arc subset $F \subseteq A(D)$, define $V(F) = \cup_{e \in F} V(e)$ to be the set of vertices incident with an edge of F in D. For subsets $X, Y \subseteq V(D)$, define

$$(X, Y)_D = \{(x, y) \in A(D) : x \in X, y \in Y\}, \text{ and } (X, Y)_{G(D)} = (X, Y)_D \cup (Y, X)_D.$$

If $X = \{x\}$ or $Y = \{y\}$, we often use $(x, Y)_D$ for $(X, Y)_D$ or $(X, y)_D$ for $(X, Y)_D$, respectively. Hence $(x, y)_D = (\{x\}, \{y\})_D$. For a vertex $v \in V(D)$, let $\partial_D^+(v) = (v, V(D) - v)_D$ and $\partial_D^-(v) = (V(D) - v, v)_D$. Thus $d_D^+(v) = |\partial_D^+(v)|$ and $d_D^-(v) = |\partial_D^-(v)|$. We further define

$$d_D(v) = d_D^+(v) + d_D^-(v) \text{ and } \delta(D) = \min\{d_D(v) : v \in V(D)\}.$$

Let M be a matching in a graph G. A path P is an M-**augmenting path** if the edges of P are alternately in M and in $E(G) - M$, and if both end vertices of P are not in $V(M)$. An M-augmenting path of a digraph D is an M-augmenting path of $G(D)$. The following theorem is fundamental.

Theorem 2.1 (*Berge,* [7]). *A matching M in G is a maximum matching if and only if G does not have M-augmenting paths.*

2.1. The symmetric core of a digraph

Let $D = (V(D), A(D))$ be a digraph. An arc $(u, v) \in A(D)$ is **symmetric** in D if $(u, v), (v, u) \in A(D)$, and asymmetric otherwise. Notice that a symmetric arc (u, v) together with the arc (v, u) form a pair of symmetric arcs of D. A digraph D is **symmetric** if every arc of D is symmetric. Let $S(D) = \{e \in A(D) : e \text{ is symmetric in } D\}$. If $A(D) = S(D)$, then D is symmetric. The **symmetric core** of D, denoted by $J(D)$, has vertex set $V(D)$ and arc set $S(D)$. When D is understood from the context, we often use J for $J(D)$.

Let $e = (v_1, v_2) \in A(D)$ be an arc of D. Define D/e to be the digraph obtained from $D - e$ by identifying v_1 and v_2 into a new vertex v_e, and deleting the possible resulting loop(s). If $W \subseteq A(D)$ is a symmetric arc subset, then define the **contraction** D/W to be the digraph obtained from D by contracting each arc $e \in W$, and deleting any resulting loops. Thus even D does not have parallel arcs, a contraction D/W is loopless but may have parallel arcs, with $A(D/W) \subseteq A(D) - W$. If H is a subdigraph of D, then we often use D/H for $D/A(H)$. If L is a connected symmetric component of H and v_L is the vertex in D/H onto which L is contracted, then L is the **contraction preimage** of v_L. We adopt the convention to define $D/\emptyset = D$, and define a vertex $v \in V(D/W)$ to be a **trivial vertex** if the preimage of v is a single vertex (also denoted by v) in D. Hence we often view trivial vertices in a contraction D/W as vertices in D. We use $\mathbb{Z}_k$ to denote the (additive) group of integers modulo k.

Lemma 2.2. *Let D be a digraph, $J = J(D)$ and J_0 be a symmetric subdigraph of J.*

(i) For any $v \in V(J_0)$, $d^+_{J_0}(v) = d^-_{J_0}(v)$.

(ii) If J_0 is connected, then J_0 is an eulerian subdigraph of D and so J_0 is strongly connected.

(iii) Suppose that J_0 is connected. Then for any vertices $u, v \in V(J_0)$, J_0 contains a spanning (u, v)-trail.

(iv) If D is strong and for some vertices $u, v \in V(D)$, D has a (u, v)-trail P such that $D - A(P)$ contains a connected symmetric subdigraph J' of J such that $V(P) \cup V(J') = V(D)$, $u, v \notin V(J')$ and there exist two vertices $v^+, v^- \in V(J')$ with $(v, v^+), (v^-, u) \in A(D)$, then D is supereulerian.

(v) If D/J_0 has a hamiltonian cycle, then D is supereulerian. In particular, if D is strong and J_0 is a spanning subdigraph of D with at most two connected components, then D is supereulerian.

(vi) If D is strong and $D[A(D) - A(J_0)]$ has a trail T' that intersects every component of J_0 with $V(D) - V(J_0) \subseteq V(T')$, then $T = D[A(T') \cup A(J_0)]$ is a spanning trail in D.

(vii) Suppose $\lambda(D) \geq 2$. If $G(D - V(J_0))$ is spanned by a 3-cycle, then D is supereulerian.

Proof. As (i) and (ii) are immediate consequences of the definitions, it suffices to justify the other conclusions. Let $u, v \in V(J_0)$. By (ii), we assume that J_0 is strong and $u \neq v$. Let P be a shortest (v, u)-path in J_0. As P is shortest, if an arc $e = (x, y) \in A(P)$, then $(y, x) \notin A(P)$. By (i), $T = J_0 - A(P)$ is a connected digraph such that $d^+_T(u) = d^-_T(u) + 1$, $d^+_T(v) = d^-_T(v) - 1$ and for any vertex $w \in V(T) - \{u, v\}$, $d^+_T(w) = d^-_T(w)$. Thus T is a spanning (u, v)-trail of J_0. This proves (iii).

By assumption, J' is a connected symmetric subdigraph, and so J' is the symmetric core of itself. By (iii) with $J_0 = J'$, J' contains a spanning (v^+, v^-)-trail T. As $A(T) \cap A(P) \subseteq A(J') \cap A(P) = \emptyset$, the arc set $A(T) \cup A(P) \cup \{(v, v^+), (v^-, u)\}$ induces a spanning closed trail of D, and so D is supereulerian. Hence (iv) is justified.

To prove (v), let $D' = D/J_0$ and denote $n = |V(D')|$. Suppose that D' has a hamiltonian cycle C with $V(C) = \{v_1, v_2, \ldots, v_n\}$ and $A(C) = \{e_i = (v_i, v_{i+1}) : i \in \mathbb{Z}_n\}$. Let $J_1, J_2, \ldots, J_n$ be the preimage of $v_1, v_2, \ldots, v_n$, respectively. By definition, each J_i is a connected component of J_0, and so a connected symmetric subdigraph of J. By the definition of contraction, $A(D') \subseteq A(D)$, and so for each $i \in \mathbb{Z}_n$, the arc $e_i \in A(D)$. Therefore, there exist vertices $v'_i \in V(J_i)$ and $v''_{i+1} \in V(J_{i+1})$ with $e_i = (v'_i, v''_{i+1}) \in A(D)$. Since each J_i is a connected symmetric subdigraph of J, it follows by (iii) that J_i has a spanning (v''_i, v'_i)-trail T_i. Let $A_1 = \{(v'_i, v''_{i+1}) : i \in \mathbb{Z}_n\}$. Then $H = D[A_1 \cup (\bigcup_{i \in \mathbb{Z}_n} A(T_i))]$ is a spanning closed trail of D, and so D is supereulerian. Now we assume that D is strong and J_0 is a spanning subdigraph of D with at most two connected components. Then D/J_0 is strong with $|V(D/J_0)| \leq 2$. It follows that D/J_0 is hamiltonian, and so D is supereulerian. Thus (v) follows.

Let T' be a trail of $D[A(D) - A(J_0)]$ that intersects every component of J_0 with $V(D) - V(J_0) \subseteq V(T')$, and let $J_1, J_2, \ldots, J_c$ be the connected components of J_0. Since for each i with $1 \leq i \leq c$, $V(T') \cap V(J_i) \neq \emptyset$ and so $T = D[A(T') \cup A(J_0)]$ is connected. As $V(D) - V(J_0) \subseteq V(T')$, $T = D[A(T') \cup A(J_0)]$ is spanning in D. Let $v \in V(T)$. If $v \in V(D) - V(T')$, we define $d^+_{T'}(v) = d^-_{T'}(v) = 0$. By (i), $d^+_T(v) = d^+_{T'}(v) + d^+_{J_0}(v) = d^-_{T'}(v) + d^-_{J_0}(v) = d^-_T(v)$, and so T is a spanning trail of D. This justifies (vi).

To prove (vii), we assume that $\lambda(D) \geq 2$ and $V(D - V(J_0)) = \{v_1, v_2, v_3\}$ such that $G(D - V(J_0))$ has a hamiltonian cycle. Suppose first that $D[\{v_1, v_2, v_3\}]$ is spanned by a 3-cycle. Then as D is strong, there must be arcs $(v', v^-), (v^+, v'') \in A(D)$ for some $v', v'' \in \{v_1, v_2, v_3\}$ and $v^-, v^+ \in V(J_0)$. It follows by Lemma 2.2 (iv) that D is supereulerian. Hence we assume that $D[\{v_1, v_2, v_3\}]$ does not contain a 3-cycle. Since D is a digraph, we may assume, by symmetry, that $(v_1, v_2), (v_2, v_3), (v_1, v_3) \in A(D)$ and $(v_3, v_1) \notin A(D)$. Since $d^-_D(v_1) \geq \lambda(D) \geq 2$, we must have $(v^+, v_1) \in A(D)$ for some $v^+ \in V(J_0)$. Likewise, as $d^+_D(v_3) \geq \lambda(D) \geq 2$, we must have $(v_3, v^-) \in A(D)$ for some $v^- \in V(J_0)$. It follows by Lemma 2.2 (iv) that D is supereulerian. This justifies (vii) and completes the proof of the lemma. ■

2.2. Structural properties

The rest of this section is devoted to the structural analysis for strong digraphs whose arc-strong connectivity is at least as big as the stability number minus one. We start with a definition.

Definition 2.3. Let M be a matching of a digraph D. For each $w \in V(D) - V(M)$, define

$$\begin{aligned} M_w^{2,2} &= \{e = [u_w(e), v_w(e)] \in M : \ |(w, \{u_w(e), v_w(e)\})_{G(D)}| = 4\}, \\ M_w^{2,1} &= \{e = [u_w(e), v_w(e)] \in M : \ |(w, \{u_w(e), v_w(e)\})_{G(D)}| = 3\}, \\ M_w^{2,0} &= \{e = [u_w(e), v_w(e)] \in M : \\ &\qquad \text{for some } v \in \{u_w(e), v_w(e)\}, |(w, v)_{G(D)}| = |(w, \{u_w(e), v_w(e)\})_{G(D)}| = 2\}, \\ M_w^{1,1} &= \{e = [u_w(e), v_w(e)] \in M : \ |(w, u_w(e))_{G(D)}| = |(w, v_w(e))_{G(D)}| = 1\}, \\ M_w^{1,0} &= \{e = [u_w(e), v_w(e)] \in M : \\ &\qquad \text{for some } v \in \{u_w(e), v_w(e)\}, |(w, v)_{G(D)}| = |(w, \{u_w(e), v_w(e)\})_{G(D)}| = 1\}, \\ M_w^{0,0} &= \{e = [u_w(e), v_w(e)] \in M : \ |(w, u_w(e))_{G(D)}| = |(w, v_w(e))_{G(D)}| = 0\}. \end{aligned} \tag{1}$$

The following observation follows from Definition 2.3 and Theorem 2.1.

Observation 2.4. *Let $n = |V(D)|$ and $M = \{[u_1, v_1], [u_2, v_2], \ldots, [u_k, v_k]\}$ be a maximum matching of a digraph D.*

(i) As M is a maximum matching, $V(D) - V(M)$ is a stable set. This implies that for any $w \in V(D) - V(M)$, $N_D(w) \subseteq V(M)$, and so by Definition 2.3, $d_D(w) = 4|M_w^{2,2}| + 3|M_w^{2,1}| + 2(|M_w^{2,0}| + |M_w^{1,1}|) + |M_w^{1,0}|$, and $|M_w^{2,2}| + |M_w^{2,1}| + |M_w^{2,0}| + |M_w^{1,1}| + |M_w^{1,0}| + |M_w^{0,0}| = k$.

(ii) Let $x, y \in V(D) - V(M)$ are distinct vertices, and $[u, v] \in M$. By Theorem 2.1, D does not have an M-augmenting path, and so if $x \in N_D(u)$, then $y \notin N_D(v)$.

(iii) As a consequence of (ii), if $x, y \in V(D) - V(M)$ are distinct vertices, then

$$(M_x^{2,2} \cup M_x^{2,1} \cup M_x^{1,1}) \cap (M_y^{2,2} \cup M_y^{2,1} \cup M_y^{2,0} \cup M_y^{1,1} \cup M_y^{1,0}) = \emptyset.$$

Throughout the rest of this section, we always assume that D is a digraph with $k = \alpha'(D) \geq 3$, $n = |V(D)| \geq 2k + 3$, $J = J(D)$ is the symmetric core of D, and let $X = V(D) - V(M)$. For each $x \in X$, define

$$k_1(x) = |M_x^{2,2}| + |M_x^{2,1}| + |M_x^{1,1}| \text{ and } k_2(x) = |M_x^{2,0}| + |M_x^{1,0}|. \tag{2}$$

Lemma 2.5. *Let D be a digraph with $k = \alpha'(D) \geq 3$ and $\delta(D) \geq 2k - 2$, and M be a maximum matching of D. If for some vertex $x_1 \in X$, both $d_D(x_1) \geq 2k - 1$ and $k_1(x_1) > 0$, then each of the following holds.*

(i) $k_1(x_1) = 1$, $k_2(x_1) \in \{k - 2, k - 1\}$, and for any vertex $x \in X - \{x_1\}$, $k_1(x) = 0$.

(ii) D has a stable set $\{v_1, v_2, \ldots, v_k\}$ such that $M = \{[u_1, v_1], [u_2, v_2], \ldots, [u_k, v_k]\}$ with $M_{x_1}^{2,2} \cup M_{x_1}^{2,1} \cup M_{x_1}^{1,1} = \{[u_1, v_1]\}$ and $\{u_1, u_2, \ldots, u_{k-1}, v_1\} \subseteq N_D(x_1) \subseteq \{u_1, u_2, \ldots, u_k, v_1\}$, and such that J has a connected component J' with $(X - \{x_1\}) \cup \{u_2, u_3, \ldots, u_k\} \subseteq V(J')$.

(iii) $\{v_2, \ldots, v_k\} \subseteq V(J')$. Moreover, if $k \geq 4$, then v_1 lies in a nontrivial connected component of J.

(iv) If $\lambda(D) \geq 2$, then D is supereulerian.

(v) If, in addition, $d_D(x_1) \geq 2k$, then either $(x_1, v_1), (v_1, x_1) \in A(D)$, or there exist at least $k - 1$ vertices $u \in \{u_1, u_2, \ldots, u_k\}$ with $(x_1, u), (u, x_1) \in A(D)$.

Proof. Throughout the proof of this lemma, we let $k_1 = k_1(x_1)$ and $k_2 = k_2(x_1)$. Denote $M_{x_1}^{2,2} \cup M_{x_1}^{2,1} \cup M_{x_1}^{1,1} = \{[u_1, v_1], \ldots, [u_{k_1}, v_{k_1}]\}$ and $M_{x_1}^{2,0} \cup M_{x_1}^{1,0} = \{[u_{k_1+1}, v_{k_1+1}], \ldots, [u_{k_1+k_2}, v_{k_1+k_2}]\}$ with $\{u_{k_1+1}, \ldots, u_{k_1+k_2}\} \subseteq N_D(x_1)$.

Choose $x_2 \in X - \{x_1\}$ such that

$$k_1(x_2) = \max\{k_1(x) : x \in X - \{x_1\}\}, \text{ and let } k_2'' = \left|\bigcup_{j=1}^{2}(M_{x_j}^{2,0} \cup M_{x_j}^{1,0})\right|.$$

By Observation 2.4 (*i*) and (*iii*),

$$\begin{aligned} 2k - 1 &\leq d_D(x_1) = 4|M_{x_1}^{2,2}| + 3|M_{x_1}^{2,1}| + 2(|M_{x_1}^{2,0}| + |M_{x_1}^{1,1}|) + |M_{x_1}^{1,0}| \leq 4k_1 + 2k_2, \\ 2k - 2 &\leq d_D(x_2) = 4|M_{x_2}^{2,2}| + 3|M_{x_2}^{2,1}| + 2(|M_{x_2}^{2,0}| + |M_{x_2}^{1,1}|) + |M_{x_2}^{1,0}| \leq 4k_1(x_2) + 2k_2''. \end{aligned}$$

By adding the inequalities above side by side, and by Observation 2.4 (*iii*), we have

$$4k - 3 \leq 4(k_1 + k_1(x_2) + k_2'') \leq 4k - 4(|M_{x_1}^{0,0}| + |M_{x_2}^{0,0}|).$$

It follows that $|M_{x_1}^{0,0}| + |M_{x_2}^{0,0}| = 0$. By Observation 2.4 (*iii*),

$$\bigcup_{j=1}^{2}(M_{x_j}^{2,0} \cup M_{x_j}^{1,0}) \subseteq M - \left(\bigcup_{j=1}^{2}(M_{x_j}^{2,2} \cup M_{x_j}^{2,1} \cup M_{x_j}^{1,1})\right),$$

and so by Observation 2.4 (i) and by $k_1 > 0$, we have

$$N_D(x) \subseteq \bigcup_{j=1}^{2}\left(V(M_{x_j}^{2,0} \cup M_{x_j}^{1,0}) \cap N_D(x_j)\right), \text{ for any } x \in X - \{x_1, x_2\}, \tag{3}$$

$$k - 1 - k_1(x_2) \geq k - (k_1 + k_1(x_2)) \geq \left|\bigcup_{j=1}^{2}(M_{x_j}^{2,0} \cup M_{x_j}^{1,0})\right|. \tag{4}$$

If $k_1 = 1$ and $k_1(x_2) = 0$, then as $d_D(x_1) \geq 2k - 1$, it would follow that $k_2 \in \{k-2, k-1\}$. Hence to prove Lemma 2.5 (i), it suffices to show that $k_1 = 1$ and $k_1(x_2) = 0$. By contradiction, we assume that either $k_1 \geq 2$ or $k_1(x_2) > 0$. Then by (4), $2(k-2) \geq |\bigcup_{j=1}^{2} V(M_{x_j}^{2,0} \cup M_{x_j}^{1,0})|$. Since $n = |V(D)| \geq 2k+3$, there exists a vertex $x_3 \in X - \{x_1, x_2\}$. By $\delta(D) \geq 2k-2$, (3) and by Observation 2.4 (iii), $2(k-1) \leq |d_D(x_3)| \leq |\bigcup_{j=1}^{2} V(M_{x_j}^{2,0} \cup M_{x_j}^{1,0})| \leq 2(k-2)$, a contradiction. This proves that Lemma 2.5 (i).

By (i), $k_1 = 1$. Let $[u_1, v_1]$ denote the only arc in $M_{x_1}^{2,2} \cup M_{x_1}^{2,1} \cup M_{x_1}^{1,1}$. As $k_2 \in \{k-2, k-1\}$, we can label the vertices and denote $M = \{[u_1, v_1], [u_2, v_2], \ldots, [u_k, v_k]\}$ such that $\{u_1, u_2, \ldots, u_{k-1}\} \subseteq N_D(x_1)$, and such that if $(X, \{u_k, v_k\})_{G(D)} \neq \emptyset$, then $(X, \{u_k\})_{G(D)} \neq \emptyset$. Hence $\{u_1, u_2, \ldots, u_{k-1}, v_1\} \subseteq N_D(x_1) \subseteq \{u_1, u_2, \ldots, u_k, v_1\}$. Fix a vertex $x \in X - \{x_1\}$. By $k_1 = 1$ and by Observation 2.4 (i) and (ii), $(x, \{u_1, v_1, v_2, \ldots, v_k\})_D = \emptyset$, and so by $\delta(D) \geq 2k-2$, $N_D(x) = \{u_2, \ldots, u_k\}$. It follows by $\delta(D) \geq 2k-2$ that $\{(u_j, x), (x, u_j) \in A(D)\}$ for any $2 \leq j \leq k$, and so J has a connected component J' containing the vertices $(X - \{x_1\}) \cup \{u_2, u_3, \ldots, u_k\}$. As $N_D(x) = \{u_2, u_3, \ldots, u_k\}$, $k \geq 3$ and $u_1, v_1 \in N_D(x_1)$, We conclude by Theorem 2.1 that $\{v_1, v_2, \ldots, v_k\}$ is a stable set of D as any arc in D incident with two distinct vertices in $\{v_1, v_2, \ldots, v_k\}$ would give rise to an M-augmenting path in D. This proves Lemma 2.5 (ii).

For any v_i with $2 \leq i \leq k$, as $\{v_1, v_2, \ldots, v_k\}$ is a stable set, $N_D(v_i) \subseteq V(D) - \{v_1, \ldots, v_k\}$. By Observation 2.4 ($iii$) and by Lemma 2.5 ($ii$), we further conclude that $N_D(v_i) \subseteq \{u_2, u_3, \ldots, u_k\}$. This, together with $\delta(D) \geq 2k-2$, forces that $\{(u_j, v_i), (v_i, u_j)\} \subseteq A(D)$, for any j with $2 \leq j \leq k$. Hence $\{v_2, \ldots, v_k\} \subseteq V(J')$. By Observation 2.4, $(\{X - \{x_1\}\}, \{v_1\})_{G(D)} = \emptyset$, and so $N_D(v_1) \subseteq \{u_1, u_2, u_3, \ldots, u_k, x_1\}$. It follow that $|(\{u_1, u_2, u_3, \ldots, u_k, x_1\}, \{v_1\})_{G(D)}| \geq |d_D(v_1)| \geq 2k-2$, and so there exist at least $(2k-2)-(k+1) \geq k-3$ vertices $z \in \{u_1, u_2, u_3, \ldots, u_k, x_1\}$ satisfying $(z, v_1), (v_1, z) \in A(D)$. Hence if $k \geq 4$, then v_1 lies in a nontrivial connected component of J. This proves Lemma 2.5 (iii).

Let $J_0 = J[V(D) - \{u_1, v_1, x_1\}]$. By ($ii$) and ($iii$), J_0 is a connected symmetric subdigraph of J. As $[u_1, v_1], [v_1, x_1], [x_1, u_1] \in A(D)$, it follows by $\lambda(D) \geq 2$ and Lemma 2.2 (vii) that D is supereulerian. This proves (iv).

Finally, we assume that $d_D(x_1) \geq 2k$ but $|(\{x_1\}, \{v_1\})_{G(D)}| = 1$. Then $|(\{x_1\}, \{u_1, \ldots, u_k\})_{G(D)}| \geq 2k-1$, implying that there exist at least $k-1$ vertices $u \in \{u_1, u_2, \ldots, u_k\}$ with $(x_1, u), (u, x_1) \in A(D)$. Hence ($v$) holds. This completes the proof of Lemma 2.5. ■

A digraph D with vertex set $V = V(D)$ is a **complete digraph** if for any pair of distinct vertices $u, v \in V$, $(u, v), (v, u) \in A(D)$. A complete digraph on n vertices will be denoted by K_n^*. Define D_0 to be the vertex disjoint union of three complete digraphs of order 3.

Lemma 2.6. *Let D be a digraph with $k = \alpha'(D) \geq 3$, $\delta(D) \geq 2k-2$ and M be a maximum matching of D. Then each of the following holds.*

(i) If for some vertex $x_1 \in X$, $d_D(x_1) \geq 2k-1$ and $k_1(x_1) = 0$, then for any $x \in X$, $k_1(x) = 0$.

(ii) If for some vertex $x_1 \in X$, $k_1(x_1) > 0$, then either $D \cong D_0$, or $k_1(x_1) = 1$ and $k_1(x) = 0$ for any $x \in X - \{x_1\}$.

Proof. Arguing by contradiction to prove (i), we may assume that $x_2 \in X - \{x_1\}$ and $k_1(x_2) > 0$. Let $[u_2, v_2] \in M_{x_2}^{2,2} \cup M_{x_2}^{2,1} \cup M_{x_2}^{1,1}$. Then by Observation 2.4 (i), $N_D(x_1) \subseteq V(M - \{[u_2, v_2]\})$. As $d_D(x_1) \geq 2k-1$, and as $|M - \{[u_2, v_2]\}| = k-1$, there exists an arc $[u_1, v_1] \in M - \{[u_2, v_2]\}$ such that $|(x_1, \{u_1, v_1\})_D| \geq 3$. Hence we must have $k_1(x_1) > 0$, contrary to the assumption that $k_1(x_1) = 0$. This proves Lemma 2.6 (i).

Now assume that for some vertex $x_1 \in X$, $k_1(x_1) > 0$. Then there exists an arc $[u_1, v_1] \in M$ such that $u_1, v_1 \in N_D(x_1)$. By Observation 2.4 (ii), for any $x \in X - \{x_1\}$, $u_1, v_1 \notin N_D(x)$. Suppose that we have another vertex $x_2 \in X - \{x_1\}$ with $k_1(x_2) > 0$, or we have $k_1(x_1) \geq 2$. Then there must be an arc $[u_2, v_2] \in M - \{[u_1, v_1]\}$ such that $u_2, v_2 \in N_D(x_2)$ (if $k_1(x_2) > 0$), or $u_2, v_2 \in N_D(x_1)$ (if $k_1(x_1) \geq 2$). If there exists a vertex $x \in X$ with $k_1(x) = 0$, then by $d_D(x) \geq 2k-2$, either $(x, \{u_1, v_1\})_{G(D)} \neq \emptyset$ or $(x, \{u_2, v_2\})_{G(D)} \neq \emptyset$. In either case, a contradiction to Observation 2.4 (ii) is obtained. Thus, either $k_1(x) > 0$ for any $x \in X$, or $k_1(x_1) = 1$ and $k_1(x) = 0$ for any $x \in X - \{x_1\}$.

To complete the proof of (ii), in the following we assume that $k_1(x) > 0$ for any $x \in X$. If $D \cong D_0$, then done. Hence we by contradiction assume that $D \not\cong D_0$. Define $S = \cup_{x \in X}(M_x^{2,0} \cup M_x^{1,0})$, $m' = \min\{k_1(x) : x \in X\}$ and $m'' = \sum_{x \in X, k_1(x) > 0}(k_1(x) - 1)$. Since $k_1(x) > 0$ for any $x \in X$, $m' > 0$. By Observation 2.4 (iii), $(\bigcup_{x \in X}(M_x^{2,2} \cup M_x^{2,1} \bigcup M_x^{1,1})) \cup S$ is a disjoint union and is a subset of M. This, together with $|X| = n - 2k$, implies that

$$k = |M| \geq \sum_{x \in X} k_1(x) + |S| = m'' + (n - 2k) + |S|. \tag{5}$$

Claim 2.7. *We have $m'' = 0$, $n = 2k+3$, $|X| = 3$.*

By (5), $k \geq m'(n-2k)+|S|$. Let $x' \in X$ satisfying $k_1(x') = m'$. Then $4m'+2|S| \geq d_D(x') \geq 2k-2$, and so $|S| \geq k-1-2m'$. Hence we have

$$k \geq m'(n-2k)+|S| \geq m'(n-2k)+k-1-2m' = m'(n-2k-2)+k-1. \tag{6}$$

With $n \geq 2k+3$, (6) leads to the conclusion that $1 \geq m'(n-2k-2) \geq m' \geq 1$, forcing $m' = 1$ and $n = 2k+3$. Thus $|X| = n-2k = 3$. By (5) and by $|S| \geq k-1-2m' = k-3$, we have $k \geq m''+3+(k-3) = m''+k$. This implies $m'' = 0$ and proves Claim 2.7.

By Claim 2.7, we may assume that $X = \{x_1, x_2, x_3\}$. As $m'' = 0$, for any $x \in X$, $k_1(x) = 1$. Fix an $x_i \in X$ for $1 \leq i \leq 3$. As $k_1(x_i) = 1$, we may assume that $u_i, v_i \in N_D(x_i)$, and $(\{x_i\}, \{v_j\})_{G(D)} = \emptyset$ for any j with $j \neq i$. By Observation 2.4 (ii), we observe that $(\{x_i\}, \{u_h, v_h\})_{G(D)} = \emptyset$ for any $1 \leq i \leq 3$ and $h \neq i$. This implies that $4+2(k-3) \geq |(\{x_i\}, \{u_i, v_i\})_{G(D)}| + \sum_{j=4}^{k} |(x_i, u_j)_{G(D)}| = d_D(x_i) \geq 2k-2$, and so we must have $d_D(x_i) = 2k-2$, $|(\{x_i\}, \{u_i, v_i\})_{G(D)}| = 4$, and for j with $4 \leq j \leq k$, $|(x_i, u_j)_{G(D)}| = 2$.

We further claim that $\{v_1, \ldots, v_k\}$ is a stable set in D. By contradiction, we assume that there exists an arc $[v_i, v_j] \in A(D)$ for some $1 \leq i < j \leq k$. If $j \leq 3$, then $\{[x_i, u_i], [u_i, v_i], [v_i, v_j], [v_j, u_j], [u_j, x_j]\}$ induces an M-augmenting path in D. If $i \leq 3 < j$, then choosing an index $i' \neq i$ and $1 \leq i' \leq 3$, then $\{[x_i, u_i], [u_i, v_i], [v_i, v_j], [v_j, u_j], [u_j, x_{i'}]\}$ induces an M-augmenting path in D. If $i \geq 4$, then $\{[x_1, u_i], [u_i, v_i], [v_i, v_j], [v_j, u_j], [u_j, x_2]\}$ induces an M-augmenting path in D. In any case, Theorem 2.1 is violated. Hence $\{v_1, \ldots, v_k\}$ must be a stable set.

If $k \geq 4$, then $N_D(v_4) \subseteq \{u_1, u_2, \ldots, u_k\}$. Since $d_D(v_4) \geq 2k-2$, there must be an i with $1 \leq i \leq 3$ such that $[u_i, v_4] \in A(D)$. Pick $i' \neq i$ and $1 \leq i' \leq 3$. Then $\{[x_i, v_i], [u_i, v_i], [u_i, v_4], [v_4, u_4], [u_4, x_{i'}]\}$ induces an M-augmenting path in D, violating Theorem 2.1. Hence we must have $k = 3$. Recall that for each $i \in \{1, 2, 3\}$, $|(\{x_i\}, \{u_i, v_i\})_{G(D)}| = 4$. Since $D \ncong D_0$ and $d_D(u_i) \geq 2k-2 = 4$, we may assume that, either $[u_i, v_j] \in A(D)$ or $[u_i, u_j] \in A(D)$, for $1 \leq i, j \leq 3$ with $i \neq j$. Once again, $\{[x_i, v_i], [v_i, u_i], [u_i, v_j], [v_j, u_j], [u_j, x_j]\}$ or $\{[x_i, v_i], [v_i, u_i], [u_i, u_j], [u_j, v_j], [v_j, x_j]\}$ induces an M-augmenting path in D. These contradictions indicate that if $k_1(x) > 0$ for any $x \in X$, then we must have $D \ncong D_0$. This proves Lemma 2.6(ii). ■

Corollary 2.8. *Let $k \geq 4$ be an integer, D be a digraph with $\lambda(D) \geq \alpha'(D) = k$, $\delta(D) \geq 2k-2$ and $n = |V(D)| \geq 2k+3$. Then $J = J(D)$ is connected.*

Lemma 2.9. *Let D be a digraph with $k = \alpha'(D) \geq 3$ and M be a maximum matching of D. Suppose that for some vertex $x_1 \in X$, $d_D(x_1) \geq 2k-1$ with $k_1(x_1) = 0$. If $\delta(D) \geq 2k-2$, then there exists a labeling of the vertices of $V(M)$ such that $M = \{[u_1, v_1], [u_2, v_2], \ldots, [u_k, v_k]\}$ and each of the following holds.*

(i) $N_D(x_1) = \{u_1, u_2, u_3, \ldots, u_k\}$, $(X, \{v_1, v_2, \ldots, v_k\})_{G(D)} = \emptyset$, and there exist at least $k-1$ vertices $u \in \{u_1, u_2, \ldots, u_k\}$ with $(x_1, u), (u, x_1) \in A(D)$. Moreover, if $d_D(x_1) \geq 2k$, then for any $u \in \{u_1, u_2, \ldots, u_k\}$, we have $(x_1, u), (u, x_1) \in A(D)$.

(ii) For any $x \in X - \{x_1\}$, $N_D(x) \subseteq \{u_1, u_2, \ldots, u_k\}$; and there exist at least $k-2$ vertices $u \in \{u_1, u_2, \ldots, u_k\}$ satisfying $(x, u), (u, x) \in A(D)$.

(iii) The vertex subset $\{v_1, v_2, \ldots, v_k\}$ is a stable set in D. Furthermore, for each v_j with $1 \leq j \leq k$, $N_D(v_j) \subseteq \{u_1, u_2, \ldots, u_k\}$ and there exist at least $k-2$ vertices $u \in \{u_1, u_2, \ldots, u_k\}$ satisfying $(v_j, u), (u, v_j) \in A(D)$.

(iv) J has at most two components; and if $\lambda(D) \geq 1$, then D is supereulerian.

Proof. By Lemma 2.6 (i), for any $x \in X$, $k_1(x) = 0$. By Observation 2.4 (i), $N_D(x_1) \subseteq V(M)$. Hence by $d_D(x_1) \geq 2k-1$ and $k_1(x_1) = 0$, we can label $M = \{[u_1, v_1], [u_2, v_2], \ldots, [u_k, v_k]\}$ so that $N_D(x_1) = \{u_1, u_2, u_3, \ldots, u_k\}$. Again by $d_D(x_1) \geq 2k-1$, there must be at least $k-1$ vertices $u \in \{u_1, u_2, \ldots, u_k\}$ satisfying $(x_1, u), (u, x_1) \in A(D)$. Similarly, if $d_D(x_1) \geq 2k$, then for any $u \in \{u_1, u_2, \ldots, u_k\}$, we have $(x_1, u), (u, x_1) \in A(D)$. It follows by $N_D(x_1) = \{u_1, u_2, u_3, \ldots, u_k\}$ and by Observation 2.4 that $(X, \{v_1, v_2, \ldots, v_k\})_{G(D)} = \emptyset$. This verifies Lemma 2.9 (i).

By (i), $N_D(x_1) = \{u_1, u_2, u_3, \ldots, u_k\}$. For any $x \in X - \{x_1\}$, by Observation 2.4 (i) and (ii), $N_D(x) \subseteq \{u_1, u_2, \ldots, u_k\}$. By $\delta(D) \geq 2k-2$, $d_D(x) \geq 2k-2$, and so there must be at least $k-2$ vertices $u \in \{u_1, u_2, \ldots, u_k\}$ with $(x, u), (u, x) \in A(D)$. This proves Lemma 2.9 (ii).

To prove (iii), we argue by contradiction and assume that for some $1 \leq i < j \leq k$, an arc $[v_i, v_j]$ is in $A(D)$. Since $n \geq 2k+3$, there exists a vertex $x_2 \in X - \{x_1\}$. By Lemma 2.9 (ii), $N_D(x_2) \subseteq \{u_1, u_2, \ldots, u_k\}$. As $d_D(x_2) \geq 2k-2$, we may assume that $u_i \in N_D(x_2)$, and so $\{[x_2, u_i], [u_i, v_i], [v_i, v_j], [v_j, u_j], [u_j, x_1]\}$ induced an M-augmenting path in D, contrary to Theorem 2.1. Hence $\{v_1, v_2, \ldots, v_k\}$ must be a stable set in D. Likewise, by Lemma 2.9 (i) and (ii), and arc in $(X, \{v_1, v_2, \ldots, v_k\})_{G(D)}$ will give rise to an M-augmenting path, contrary to Theorem 2.1. Thus $(X, \{v_1, v_2, \ldots, v_k\})_{G(D)} = \emptyset$. Consequently, for each v_j with $1 \leq j \leq k$, $N_D(v_j) \subseteq \{u_1, u_2, \ldots, u_k\}$. By $d_D(v_j) \geq 2k-2$, there exist at least $k-2$ vertices $u \in \{u_1.u_2, \ldots, u_k\}$ satisfying $(v_j, u), (u, v_j) \in A(D)$.

To show (iv), we first assume by (i) and by symmetry that for any i with $1 \leq i \leq k-1$, (x_1, u_i) is a symmetric arc in D and $[x_1, u_k] \in A(D)$. Thus J has a connected component of J' with $\{x_1, u_1, \ldots, u_{k-1}\} \subseteq V(J')$. Let J'' denote the connected component of J with $u_k \in V(J'')$. As $k \geq 3$, it follows by (ii) that, for every $x \in X - \{x_1\}$, either $x \in V(J')$ or $x \in V(J'')$. Similarly, by (iii), for every $v \in \{v_1, v_2, \ldots, v_k\}$, either $v \in V(J')$ or $v \in V(J'')$. Hence J has at most two connected components J' and J''. It now by Lemma 2.2 (v) that if D is strong, then D must be supereulerian. This completes the proof of the lemma. ■

Lemma 2.10. *Let D be a digraph with $k = \alpha'(D) \geq 3$, $\delta(D) \geq 2k-2$ and let M be a maximum matching of D and $J = J(D)$ be the symmetric core of D. If for any $x \in X$, $k_1(x) = 0$, and if there exists an arc $e \in M$ with $(X, V(e))_{G(D)} = \emptyset$, then there exists a labeling of the vertices of $V(M)$ with $M = \{[u_1, v_1], [u_2, v_2], \ldots, [u_k, v_k]\}$ and $e = [u_k, v_k]$ such that each of the following holds.*

(i) $(X, \{v_1, v_2, \ldots, v_k\})_{G(D)} = \emptyset$, $\{v_1, v_2, \ldots, v_{k-1}\}$ is a stable set in D and J has a connected component J' with $X \cup \{u_1, u_2, \ldots, u_{k-1}\} \subseteq V(J')$.

(ii) If $\{v_1, v_2, \ldots, v_k\}$ is a stable set in D, then for any $j \in \{1, 2, \ldots, k\}$, there exist $k-2$ vertices $u \in \{u_1, u_2, \ldots, u_k\}$ with $(v_j, u), (u, v_j) \in A(D)$, and J has at most two connected components.

(iii) Suppose that $\{v_1, v_2, \ldots, v_k\}$ is not a stable set in D and $[v_{k-1}, v_k] \in A(D)$. Then $(u_k, \{v_1, \ldots, v_{k-2}\})_{G(D)} = \emptyset$. Moreover, if $k \geq 4$, then $\{v_1, \ldots, v_{k-2}\} \subseteq V(J')$; and if $\lambda(D) \geq 2$, then D is supereulerian.

Proof. By Observation 2.4 (i), for any $x \in X$, $N_D(x) \subseteq V(M)$. As for some $e \in M$, we have $(X, V(e))_{G(D)} = \emptyset$, and by $k_1(x) = 0$ and $d_D(x) \geq 2k-2$, we can label $M = \{[u_1, v_1], [u_2, v_2], \ldots, [u_k, v_k]\}$ with $e = [u_k, v_k]$ such that for any $x \in X$, $N_D(x) = \{u_1, u_2, \ldots, u_{k-1}\}$, and for any i with $1 \leq i \leq k-1$, $(x, u_i), (u_i, x) \in A(D)$. As $k \geq 3$ and $|X| = n - 2k \geq 3$, it follows that J has a connected component J' with $X \cup \{u_1, u_2, \ldots, u_{k-1}\} \subseteq V(J')$. As $k_1(x) = 0$ for any $x \in X$, we conclude that $(X, \{v_1, v_2, \ldots, v_k\})_{G(D)} = \emptyset$.

We argue by contradiction to show that $\{v_1, v_2, \ldots, v_{k-1}\}$ is a stable set in D. Suppose that for some $1 \leq i < j \leq k-1$, $[v_i, v_j] \in A(D)$. As $n-2k \geq 3$, $D[\{[x_1, u_i], [u_i, v_i], [v_i, v_j], [v_j, u_j], [u_j, x_2]\}]$ is an M-augmenting path, contrary to Theorem 2.1. This proves (i).

In the proof of (ii) and (iii), we let J^2, J^3 and J^4 be connected components of J such that $u_k \in V(J^2)$, $v_k \in V(J^3)$ and $v_{k-1} \in V(J^4)$.

Assume that $\{v_1, v_2, \ldots, v_k\}$ is a stable set in D. Fix an arbitrary vertex v_j with $1 \leq j \leq k$. By (i), we have $N_D(v_j) \subseteq \{u_1, u_2, \ldots, u_{k-1}, u_k\}$, and so by $\delta(D) \geq 2k-2$, there must be at least $k-2$ vertices $u \in \{u_1, u_2, \ldots, u_k\}$ with $(v_j, u), (u, v_j) \in A(D)$. It follows by $k \geq 3$ and by (i) that either $v_j \in V(J')$ (if $u \neq u_k$) or $v_j \in V(J^2)$ (if $u = u_k$). Hence every vertex in D is either in J' or in J^2, and so J has at most two connected components. This proves (ii).

To prove (iii), we assume by symmetry that $[v_{k-1}, v_k] \in A(D)$. Fix a vertex v_j with $1 \leq j \leq k-2$. If $[u_k, v_j] \in A(D)$, then by (i) and by $n \geq 2k+3$, $D[\{[x_1, u_j], [u_j, v_j], [v_j, u_k], [u_k, v_k], [v_k, v_{k-1}], [v_{k-1}, u_{k-1}], [u_{k-1}, x_2]\}]$ is an M-augmenting path, contrary to Theorem 2.1. Hence $(u_k, v_j)_{G(D)} = \emptyset$. This proves that $(u_k, \{v_1, \ldots, v_{k-2}\})_{G(D)} = \emptyset$, and so $N_D(v_j) \subseteq \{u_1, \ldots, u_{k-1}, v_k\}$. By $d_D(v_j) \geq 2k-2$, there exist at least $k-2$ vertices $u' \in \{u_1, \ldots, u_{k-1}, v_k\}$ such that $(u', v_j), (v_j, u') \in A(D)$. If $k \geq 4$ then $u' \in \{u_1, \ldots, u_{k-1}\} \subseteq V(J')$, and so $v_j \in V(J')$. Thus $\{v_1, \ldots, v_{k-2}\} \subseteq V(J')$.

In the following, we assume that $\lambda(D) \geq 2$ to prove the following claim, which completes the proof of the lemma.

Claim 2.11. *Under the assumption of Lemma 2.10 (iii), if $\lambda(D) \geq 2$, then each of the following holds.*

(a) If $k \geq 5$, then J has at most two components, and so by Lemma 2.2(v), D is supereulerian.

(b) If $[u_k, v_{k-1}] \in A(D)$, then $(\{v_k\}, \{v_1, \ldots, v_{k-2}\})_{G(D)} = \emptyset$.

(c) If $k = 4$, then J has at most two components, and so by Lemma 2.2(v), D is supereulerian.

(d) If $k = 3$, then J has a symmetric subdigraph J_0 such that $G(D - V(J_0))$ is spanned by a 3-cycle, and so by Lemma 2.2 (vii), D is supereulerian.

Assume that $k \geq 5$. If $J^2 = J^3 = J^4$, then J has at most two components. Hence we assume that either $J^2 \neq J^3$, whence $|(\{u_k\}, \{v_k\})_{G(D)}| \leq 1$; or $J^2 \neq J^4$, whence $|(\{u_k\}, \{v_{k-1}\})_{G(D)}| \leq 1$. Since $(u_k, \{v_1, \ldots, v_{k-2}\})_{G(D)} = \emptyset$ and $(X, \{u_k, v_k\})_{G(D)} = \emptyset$, we have $N_D(u_k) \subseteq \{u_1, \ldots, u_{k-1}, v_{k-1}, v_k\}$. This, together with $d_D(u_k) \geq 2k-2$, implies that $|(u_k, \{u_1, \ldots, u_{k-1}\})_{G(D)}| \geq 2k-5$, and so there exists at least $k-4$ vertices $u'' \in \{u_1, \ldots, u_{k-1}\}$ such that $(u_k, u''), (u'', u_k) \in A(D)$. As $k \geq 5$, $u_k \in V(J')$. Similarly, by (i), $N_D(v_{k-1}) \subseteq \{u_1, \ldots, u_{k-1}, u_k, v_k\}$ and so $|(v_{k-1}, \{u_1, \ldots, u_{k-1}, u_k\})_{G(D)}| \geq 2k-4$. Again by $k \geq 5$, there exists at least $k-4$ vertices $u^3 \in \{u_1, \ldots, u_{k-1}, u_k\}$ such that $(v_{k-1}, u^3), (u^3, v_{k-1}) \in A(D)$, and so $v_{k-1} \in V(J')$. This indicates that $V(D) - V(J') \subseteq \{v_k\}$, and so Claim 2.11 (a) follows.

By contradiction, we assume that $[u_k, v_{k-1}], [v_j, v_k] \in A(D)$ for some $j \in \{1, 2, \ldots, k-2\}$. Then $\{[x_1, u_j], [u_j, v_j], [v_j, v_k], [v_k, u_k], [u_k, v_{k-1}], [v_{k-1}, u_{k-1}], [u_{k-1}, x_2]\}$ induces an M-augmenting path in D, contrary to Theorem 2.1. Hence (b) holds.

Assume that $k = 4$. Then $v_1, v_2 \in V(J')$ and $(u_k, \{v_1, v_2\})_{G(D)} = \emptyset$. Hence $N_D(u_4) \subseteq \{u_1, u_2, u_3, v_3, v_4\}$. Since $d_D(u_4) \geq 6$, for some $w \in \{u_1, u_2, u_3, v_3, v_4\}$, both $(w, u_4), (u_4, w) \in A(D)$. Hence either $J^2 = J'$ (if $w \in \{u_1, u_2, u_3\}$), or $J^2 = J^3$ (if $w = v_4$), or $J^2 = J^4$ (if $w = v_3$), and so J has at most three connected components J', J^3 and J^4. Similarly, $N_D(v_3) \subseteq \{u_1, u_2, u_3, u_4, v_4\}$. As $d_D(v_3) \geq 6$, for some $w' \in \{u_1, u_2, u_3, u_4, v_4\}$, both $(w', v_3), (v_3, w') \in A(D)$. Hence either $J^2 = J^4 = J'$, or $J^2 = J^4 = J^3$, or $J^2 = J^4$ with $V(J^4) \cap (V(J') \cup V(J^3)) = \emptyset$. It follows that either J has at most two connected components J' and J^3, or $J^2 = J^4$ and J has at most three connected components J', J^3 and J^4. When $J^2 = J^4$, we have $[u_4, v_3] \in A(D)$, and so by (b), $N_D(v_4) \subseteq \{u_1, u_2, u_3, u_4, v_3\}$. By $d_D(v_4) \geq 6$, we must have $J^3 = J'$ or $J^3 = J^4$ and so J has at most two connected components J' and J^4. This proves (c).

We now assume that $k = 3$. Assume first that $(u_3, v_2)_{G(D)} = \emptyset$. Then for each $z \in \{v_1, v_2, u_3\}$, as $N_D(z) \subseteq \{u_1, u_2, v_3\}$, $z \in V(J')$ or $z \in V(J^3)$. Hence J has at most two connected components J' and J^3. and so by Lemma 2.2 (v), D is supereulerian. Therefore, we assume that $[u_3, v_2] \in A(D)$. By (b), $|(\{v_1\}, \{v_3\})_{G(D)}| = 0$. By (i), $|(\{v_1\}, \{v_2\})_{G(D)}| = 0$. Hence $N_D(v_1) \subseteq \{u_1, u_2\}$. By $d_D(v_1) \geq 4$, $(v_1, u_1), (u_1, v_1) \in A(D)$, and so $v_1 \in V(J')$. Let $J_0 = J'[V(D) - \{v_1, u_1, u_2\}]$. As $[u_3, v_2], [v_2, v_3], [u_3, v_3] \in A(D)$, it follows from $\lambda(D) \geq 2$ and Lemma 2.2 (vii) that D is supereulerian. This completes the justification of Claim 2.11. ■

Lemma 2.12. *Let D be a digraph with $k = \alpha'(D) \geq 3$ and $\delta(D) \geq 2k-2$, and M be a maximum matching of D. If for any $x \in X$, $k_1(x) = 0$ and for any arc $e \in M$, $(X, V(e))_{G(D)} \neq \emptyset$, then there exists a labeling of the vertices of $V(M)$ such that $M = \{[u_1, v_1], [u_2, v_2], \dots, [u_k, v_k]\}$, $N_D(X) = \{u_1, u_2, \dots, u_k\}$, and each of the following holds.*

(i) $(X, \{v_1, v_2, \dots, v_k\})_{G(D)} = \emptyset$, and for any $x \in X$, there exist at least $k-2$ vertices $u \in \{u_1, u_2, \dots, u_k\}$ with $(x, u), (u, x) \in A(D)$.

(ii) $\{v_1, v_2, \dots, v_k\}$ is a stable set in D, and for any v_j with $1 \leq j \leq k$, there exist at least $k-2$ vertices $u \in \{u_1, u_2, \dots, u_k\}$ with $(u, v_j), (v_j, u) \in A(D)$.

(iii) If $\lambda(D) \geq 2$, then D is supereulerian.

Proof. For any vertex $x \in X$, by Observation 2.4 (i), $N_D(x) \subseteq V(M)$; by assumption, $k_1(x) = 0$ and

$$\text{for any arc } e \in M,\ (X, V(e))_{G(D)} \neq \emptyset. \tag{7}$$

This, together with Observation 2.4 (ii), implies that every arc in M has exactly one vertex in $N_D(X)$. Thus we can denote $V(M) \cap N_D(X) = \{u_1, u_2, \dots, u_k\}$ and $M = \{[u_1, v_1], [u_2, v_2], \dots, [u_k, v_k]\}$. This labeling of vertices in $V(M)$ implies that $N_D(X) = \{u_1, u_2, \dots, u_k\}$, and so $(X, \{v_1, v_2, \dots, v_k\})_{G(D)} = \emptyset$. Fix an $x \in X$. Since $d_D(x) \geq 2k-2$, for at least $k-2$ vertices $u \in \{u_1, u_2, \dots, u_k\}$, both (u, x) and (x, u) are in $A(D)$. Thus (i) holds.

By contradiction, assume that $\{v_1, v_2, \dots, v_k\}$ is not a stable set in D. By symmetry, we may assume that $[v_1, v_2] \in A(D)$. For i with $1 \leq i \leq k$, let $X_i = X \cap N_D(u_i)$. By (7), $X_i \neq \emptyset$, and so there exists a vertex $x_1 \in X_1$. If there exists a vertex $x_2 \in X_2 - \{x_1\}$, then $D[\{[x_1, u_1], [u_1, v_1], [v_1, v_2], [v_2, u_2], [u_2, x_2]\}]$ is an M-augmenting path, contrary to Theorem 2.1. Hence $X_2 = \{x_1\}$. By the same argument, we conclude that $X_1 = X_2 = \{x_1\}$. Since $n \geq 2k+3$, we have $|X| \geq 3$, and so $X - \{x_1\} \neq \emptyset$. For any vertex $x \in X - \{x_1\}$, as $N_D(X) \subseteq \{u_1, u_2, \dots, u_k\}$ and $X_1 = X_2 = \{x_1\}$, we conclude that $N_D(x) \subseteq \{u_3, u_4, \dots, u_k\}$, which implies that $2k-2 = 2\lambda(D) \leq d_D(x) \leq 2(k-2)$, a contradiction. Thus $\{v_1, v_2, \dots, v_k\}$ must be a stable set in D.

Fix a vertex v_j with $1 \leq j \leq k$. By (i), $(X, \{v_1, v_2, \dots, v_k\})_{G(D)} = \emptyset$. As $\{v_1, v_2, \dots, v_k\}$ is a stable set, we must have $N_D(v_j) \subseteq \{u_1, u_2, \dots, u_k\}$. Since $\delta(D) \geq 2k-2$, there exist at least $k-2$ vertices $u \in \{u_1, u_2, \dots, u_k\}$ with $(u, v_j), (v_j, u) \in A(D)$. This proves ($ii$).

We now assume that $\lambda(D) \geq 2$. By contradiction, we assume that D is not supereulerian. Pick a vertex $x_1 \in X$ and let J_1 be the connected component of J with $x_1 \in V(J_1)$. By (i), we may assume that $u_1, \dots, u_{k-2} \in V(J_1)$. Let J_2 and J_3 be connected components of J with $u_{k-1} \in V(J_2)$ and $u_k \in V(J_3)$. By (i) and (ii), and by $k \geq 3$, for every vertex $v \in X \cup \{v_1, v_2, \dots, v_k\}$, there exists an $i \in \{1, 2, 3\}$ such that either $v \in V(J_i)$. It follows that J has at most three connected components J_1, J_2 and J_3. By Lemma 2.2 (v), if J has at most two connected components, then D is supereulerian. Hence J must have exactly three components J_1, J_2 and J_3.

Case 1. $k \geq 4$.

If there exists a vertex $v \in X \cup \{v_1, v_2, \dots, v_k\}$ such that for distinct $i, j \in \{1, 2, 3\}$, $v \in V(J_i) \cup V(J_j)$, then as $k-2 \geq 2$, we have either $J_1 = J_2$, or $J_1 = J_3$, or $J_2 = J_3$, contrary to the assumption that J has exactly three components. Therefore, for any $k \geq 4$, we have

$$V(J_1) = V(D) - \{u_{k-1}, u_k\},\ V(J_2) = \{u_{k-1}\} \text{ and } V(J_3) = \{u_k\}. \tag{8}$$

Thus for any $x \in X$, and $u \in \{u_1, \dots, u_{k-2}\}$ and any $v \in \{v_1, v_2, \dots, v_k\}$, the arcs $(x, u), (u, v)$ are symmetric in D. As $\delta(D) \geq 2k-2$, we conclude that for any $v \in X \cup \{v_1, v_2, \dots, v_k\}$, $d_D(v) = 2k-2$ and $|(v, u_{k-1})_{G(D)}| = |(v, u_k)_{G(D)}| = 1$. If $[u_{k-1}, u_k] \in A(D)$, then by $\lambda(D) > 0$ and by Lemma 2.2 (iv), D is supereulerian. Thus $(u_{k-1}, u_k)_{G(D)} = \emptyset$. If $D - A(J_1)$ has a cycle C containing both u_{k-1} and u_k, then $D[A(J_1) \cup D(C)]$ is a spanning closed trail of D, and so D is supereulerian. Hence we assume $D - A(J_1)$ does not have a cycle or disjoint cycles containing both u_{k-1} and u_k.

Since $\lambda(D) \geq 2$, there exist vertices $v^-, v^+, w^-, w^+ \in V(J_1)$ such that

$$(v^-, u_{k-1}), (w^-, u_k), (u_{k-1}, v^+), (u_k, w^+) \in A(D). \tag{9}$$

Since J_1, J_2 and J_3 are distinct components of J, thus, we assume that $w^- \neq w^+$ and $v^- \neq v^+$.

If $v^-, w^+ \in X \cup \{v_1, \dots, v_k\}$, then $(w^+, u_1), (u_1, w^+), (u_1, v^-), (v^-, u_1) \in A(J_1)$. Let $J_1' = J_1 - \{(w^+, u_1), (u_1, w^+), (u_1, v^-), (v^-, u_1)\}$. As $|X| \geq 3$ and $k \geq 4$, J_1' is a connected symmetric subdigraph of D, and by (9), $D - A(J_1')$ has a trail $w^- u_k w^+ u_1 v^- u_{k-1} v^+$. By Lemma 2.2 ($iv$) with $J' = J_1'$, D is supereulerian.

Suppose that $|\{u_1, \dots, u_{k-2}\} \cap \{v^-, w^+\}| = 1$ and $|(X \cup \{v_1, \dots, v_k\}) \cap \{v^-, w^+\}| = 1$ By symmetry, we assume that $v^- = u_1$ and $w^+ \in X \cup \{v_1, \dots, v_k\}$. As $(w^+, u_1) \in A(J_1)$ is symmetric arcs of D. Let $J_2' = J_1 - \{(w^+, u_1), (u_1, w^+)\}$. As $|X| \geq 3$ and $k \geq 4$, J_2' is a connected symmetric subdigraph of D, and by (9), $D - A(J_2')$ has a trail $w^- u_k w^+ u_1 u_{k-1} v^+$. It follows from Lemma 2.2 (iv) with $J' = J_2'$ that D is supereulerian. Hence we may assume that $v^-, w^+ \in \{u_1, \dots, u_{k-2}\}$. By (8), $(w^+, x_1), (x_1, v^-) \in A(J_1)$ are symmetric arcs of D. As $|X| \geq 3$ and $k \geq 4$, $J_1 - x_1$ is a connected symmetric subdigraph of D, and by (9), $D - A(J_1 - x_1)$ has a trail $w^- u_k w^+ x_1 v^- u_{k-1} v^+$. By Lemma 2.2 ($iv$) with $J' = J_1 - x_1$, D is supereulerian.

Case 2. $k = 3$.

By definition, for each $i \in \{1, 2, 3\}$, $u_i \in V(J_i)$. By relabeling the vertices u_1, u_2 and u_3, we assume that $u_i \in V(J_i)$. By (*ii*) and by $\delta(D) \geq 4$, every v_i is adjacent to a u_j by a pair of symmetric arcs. Therefore, we may relabel v_1, v_2, v_3 and assume that $(u_i, v_i) \in A(J_i)$ is a symmetric arc of D.

Let $D' = D/J$, and denote $V(D') = \{z_1, z_2, z_3\}$, where $z_i \in V(D')$ be the vertex onto which J_i is contracted. If D' has a hamiltonian cycle, then by Lemma 2.2 (v), D is supereulerian. Hence we may assume that D is not Hamiltonian. By (*i*), (*ii*), $\lambda(D) \geq 2$, and the fact that for $i \in \{1, 2, 3\}$, $d_D(v_i) = 4$, we observe that

$$\text{if } \{i', i'', i'''\} = \{1, 2, 3\}, \text{ then } |(v_{i'}, \{u_{i''}, u_{i'''}\})_D| = 1 \text{ and } |(\{u_{i''}, u_{i'''}\}, v_{i'})_D| = 1. \tag{10}$$

By (10) and by symmetry, we assume that $(v_1, u_2), (u_3, v_1) \in A(D)$. Thus $(z_1, z_2), (z_3, z_1) \in A(D')$. As D' is not hamiltonian, we assume that $(z_2, z_3) \notin A(D')$. By (10) and since $(z_2, z_3) \notin A(D')$, we conclude that $(u_3, v_2), (v_3, u_2) \in A(D)$. These force, by (10), that $(v_2, u_1), (u_1, v_3) \in A(D)$. As $(u_1, v_3), (v_3, u_2), (v_2, u_1) \in A(D)$, it follows that D' must be hamiltonian, a contradiction. This proves that in Case 2, D is also supereulerian. This completes the proof of the lemma. ■

Lemma 2.13. *Let $k \geq 3$ be an integer, D be a digraph with $k = \alpha'(D) \geq 3$, $\delta(D) \geq 2k - 2$, and M be a maximum matching of D. Suppose that for some $x_1 \in X$, $k_1(x_1) > 0$. Then each of the following holds.*

(i) Either $D \cong D_0$, or J has a connected component J' such that the subdigraph $D_1 = D - V(J')$ satisfies $|V(D_1)| \leq 3$ and that $G(D_1)$ is spanned by a 3-cycle or a K_2.

(ii) If, in addition, $\lambda(D) \geq 2$, then D is supereulerian.

Proof. As $k_1(x_1) > 0$, there exists an arc $e = [u_1, v_1] \in M$ with $u_1, v_1 \in N_D(x_1)$. By Lemma 2.6 (*ii*), $D \cong D_0$, or $k_1(x_1) = 1$ and $k_1(x) = 0$ for any $x \in X - \{x_1\}$. Thus to prove (*i*), it suffices to assume that $k_1(x_1) = 1$ and $k_1(x) = 0$ for any $x \in X - \{x_1\}$ to show that the desired J' and D_1 exist.

Fix a vertex $x \in X - \{x_1\}$. By Observation 2.4 (*ii*), $N_D(x) \subseteq V(M) - \{u_1, v_1\}$; and by $k_1(x) = 0$, for any $e \in M$, $|N_D(x) \cap V(e)| \leq 1$. Hence we can label $M = \{[u_1, v_1], [u_2, v_2], \dots, [u_k, v_k]\}$ such that $N_D(x) \subseteq \{u_2, \dots, u_k\}$. By $\delta(D) \geq 2k - 2$, we conclude that for any u_i with $2 \leq i \leq k$, $(x, u_i), (u_i, x) \in A(D)$. It follows that J has a connected component J' such that $(X - \{x_1\}) \cup \{u_2, \dots, u_k\} \subseteq V(J')$.

We claim that $\{v_1, v_2, \dots, v_k\}$ is a stale set. Assume by contradiction that for some $1 \leq i < j \leq k$, $[v_i, v_j] \in A(D)$. If $i = 1$, then $D[\{[x_1, u_1], [u_1, v_1], [v_1, v_j], [v_j, u_j], [u_j, x_2]\}]$ is an M-augmenting path; If $i > 1$, then $D[\{[x_2, u_i], [u_i, v_i], [v_i, v_j], [v_j, u_j], [u_j, x_3]\}]$ is an M-augmenting path. In either case, a contradiction to Theorem 2.1 is obtained. Hence $\{v_1, v_2, \dots, v_k\}$ is a stable set.

Fix a vertex v_j with $2 \leq j \leq k$. If $[u_1, v_j] \in A(D)$, then $\{[x_1, v_1], [v_1, u_1], [u_1, v_j], [v_j, u_j], [u_j, x_2]\}$ induces an M-augmenting path in D, contrary to Theorem 2.1. Hence $(u_1, \{v_2, \dots, v_k\})_{G(D)} = \emptyset$ and so $N_D(v_j) \subseteq \{u_2, .., u_k\}$. As $d_D(v_j) \geq 2k-2$, we conclude that for any $u \in \{u_2, \dots, u_k\}$ with $(u, v_j), (v_j, u) \in A(D)$, and so $(X - \{x_1\}) \cup \{u_2, \dots, u_k\} \cup \{v_2, \dots, v_k\} \subseteq V(J')$. As $[x_1, u_1], [x_1, v_1], [u_1, v_1] \in A(D)$, Lemma 2.13 (*i*) is justified.

By Lemma 2.13 (*i*) and since $\lambda(D) \geq 2$, we observe that $D \ncong D_0$ and so $J(D)$ has a connected component J' such that the subdigraph $D_1 = D - V(J')$ satisfies $|V(D_1)| \leq 3$ and that $G(D_1)$ is spanned by a 3-cycle or a K_2. If $G(D_1)$ is spanned by a 3-cycle, then by Lemma 2.2 (*vii*), D is supereulerian. If $G(D_1)$ is spanned by a K_2, then by Lemma 2.2 (*iv*), D is supereulerian. Hence Lemma 2.13 (*ii*) holds. ■

3. Spanning trails in digraphs

Let D be a digraph and let X denote a set of arcs not in $A(D)$ satisfying $\cup_{e \in X} V(e) \subseteq V(D)$. Define $D+X$ to be the digraph with vertex set $V(D)$ and arc set $A(D) \cup X$. If $X \subseteq A(D)$ (or $X \subseteq V(D)$, respectively), then define $D - X = D[A(D) - X]$ (or $D - X = D[V(D) - X]$, respectively). We often use $D + e$ for $D + \{e\}$, $D - e$ for $D - \{e\}$ and $D - v$ for $D - \{v\}$.

3.1. Spanning trails in digraphs with small matching numbers

In this subsection, we will identify a family $\mathcal{D}(n)$ of digraphs, and use it to prove Theorem 1.3 (*i*). We start with some examples.

Example 3.1. Let $n, t_1, t_1', t_1'', t_2, t_2', t_2'', t_3$ be nonnegative integers with $n = 2 + t_1 + t_1' + t_1'' + t_2 + t_2' + t_2'' + t_3$. Define mutually disjoint vertex sets X, Y and Z as follows,

$$X = \{x_1, x_2, \dots, x_{t_1}, x_1', x_2', \dots, x_{t_1'}', x_1'', x_2'', \dots, x_{t_1''}''\},$$
$$Y = \{y_1, y_2, \dots, y_{t_2}, y_1', y_2', \dots, y_{t_2'}', y_1'', y_2'', \dots, y_{t_2''}''\},$$
$$Z = \{z_1, z_2, \dots, z_{t_3}\},$$

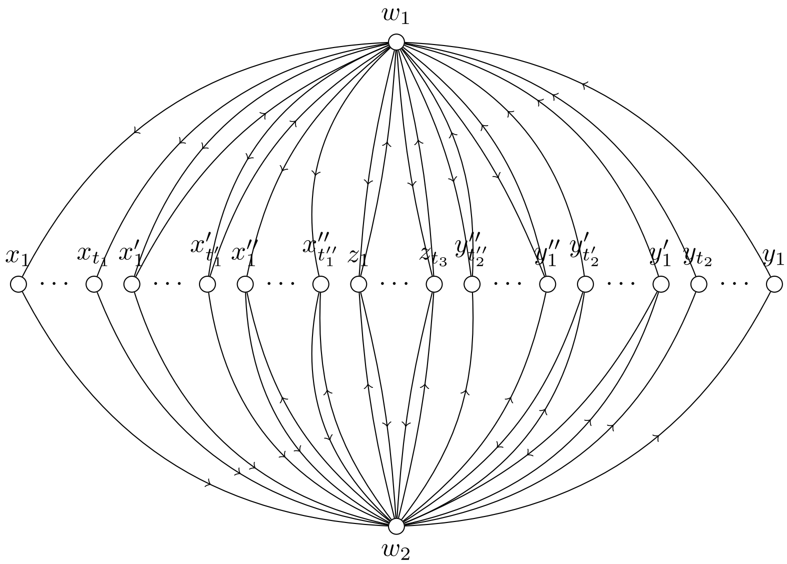

Fig. 1. Digraph $D(t_1, t'_1, t''_1, t_2, t'_2, t''_2, t_3)$.

and w_1, w_2 be two vertices not in $X \cup Y \cup Z$; and define mutually disjoint arc sets A_X, A_Y and A_Z as follows,

$$A_X = \left(\bigcup_{i=1}^{t_1}\{(w_1, x_i), (x_i, w_2)\}\right) \cup \left(\bigcup_{i=1}^{t'_1}\{(w_1, x'_i), (x'_i, w_1), (x'_i, w_2)\}\right) \tag{11}$$

$$\cup \left(\bigcup_{i=1}^{t''_1}\{(w_1, x''_i), (w_2, x''_i), (x''_i, w_2)\}\right),$$

$$A_Y = \left(\bigcup_{i=1}^{t_2}\{(w_2, y_i), (y_i, w_1)\}\right) \cup \left(\bigcup_{i=1}^{t'_2}\{(w_2, y'_i), (y'_i, w_2), (y'_i, w_1)\}\right)$$

$$\cup \left(\bigcup_{i=1}^{t''_2}\{(w_2, y''_i), (w_1, y''_i), (y''_i, w_1)\}\right),$$

$$A_Z = \bigcup_{i=1}^{t_3}\{(w_1, z_i), (z_i, w_1), (w_2, z_i), (z_i, w_2)\}.$$

Define a digraph $D = D(t_1, t'_1, t''_1, t_2, t'_2, t''_2, t_3)$ with $V(D) = \{w_1, w_2\} \cup X \cup Y \cup Z$ and arc set $A(D) = A_X \cup A_Y \cup A_Z$. (See Fig. 1.)

Observation 3.2. *Let $D = D(t_1, t'_1, t''_1, t_2, t'_2, t''_2, t_3)$ with that $n \geq 4$ and $\lambda(D) > 0$. Then each of the following holds.*
(i) D is supereulerian if and only if both $t_1 \leq t_2 + t'_2 + t''_2 + t_3$ and $t_2 \leq t_1 + t'_1 + t''_1 + t_3$.
(ii) D has a spanning trail if and only if one of the following holds.

$$\text{both } t_1 \leq t_2 + t'_2 + t''_2 + t_3 + 1 \text{ and } t_2 \leq t_1 + t'_1 + t''_1 + t_3; \tag{12}$$

$$\text{both } t_1 \leq t_2 + t'_2 + t''_2 + t_3 \text{ and } t_2 \leq t_1 + t'_1 + t''_1 + t_3 + 1. \tag{13}$$

Proof. We are to justify the conclusions of Observation 3.2. By inspection, the conclusions (i) and (ii) holds if $n = 4$. Thus we assume that $n \geq 5$. Let $J = J(D)$ be the symmetric core of D.

We assume that both $t_1 \leq t_2 + t'_2 + t''_2 + t_3$ and $t_2 \leq t_1 + t'_1 + t''_1 + t_3$ to show by induction on $t_1 + t_2$ that D is supereulerian. If $t_1 + t_2 = 0$, then J has at most two connected components, and so by Lemma 2.2 (v), D is supereulerian. Assume that $t_1 + t_2 > 0$ and that for smaller values of $t_1 + t_2$, D is supereulerian. By symmetry, we may assume that $t_1 \geq t_2$, and so $t_1 > 0$. If $t_2 > 0$, then let $D_1 = D - \{x_1, y_1\}$. Then as $D_1 = D(t_1 - 1, t'_1, t''_1, t_2 - 1, t'_2, t''_2, t_3)$, by induction, D_1 has a spanning eulerian subdigraph H_1, and so $D[A(H_1) \cup \{(w_1, x_1), (x_1, w_2), (w_2, y_1), (y_1, w_1)\}]$ is a spanning eulerian subdigraph of D. Hence we assume that $t_2 = 0$. Since $t_1 \le t_2 + t_2' + t_2'' + t_3 = t_2' + t_2'' + t_3$, there exists a $v \in \{y_1', y_2', \ldots, y_{t_2'}', y_1'', y_2'', \ldots, y_{t_2''}'', z_1, z_2, \ldots, z_{t_3}\}$ such that $(w_2, v), (v, w_1) \in A(D)$. Let $D_2 = D - \{x_1, v\}$. By induction, D_2 has a spanning eulerian subdigraph H_2, and so $D[A(H_2) \cup \{(w_1, x_1), (x_1, w_2), (w_2, v), (v, w_1)\}]$ is a spanning eulerian subdigraph of D.

Conversely, we assume that D has a spanning eulerian subdigraph H. We again argue by induction on $t_1 + t_2$ to show that both $t_1 \le t_2 + t_2' + t_2'' + t_3$ and $t_2 \le t_1 + t_1' + t_1'' + t_3$. As these inequalities hold when $t_1 = t_2 = 0$, we assume by symmetry, that $t_1 \ge t_2$ and $t_1 > 0$. If $t_2 > 0$, then $(w_1, x_1), (x_1, w_2), (w_2, y_1), (y_1, w_1) \in A(H)$, and so $H - \{x_1, y_1\}$ is a spanning eulerian subdigraph of $D - \{x_1, y_1\}$, and so by induction. $t_1 - 1 \le (t_2 - 1) + t_2' + t_2'' + t_3$ and $t_2 - 1 \le (t_1 - 1) + t_1' + t_1'' + t_3$. Hence we assume that $t_2 = 0$. As H is a spanning eulerian subdigraph, there must be a $v \in \{y_1', y_2', \ldots, y_{t_2'}', y_1'', y_2'', \ldots, y_{t_2''}'', z_1, z_2, \ldots, z_{t_3}\}$ such that $(w_2, v), (v, w_1) \in A(H)$. Let H' denote the nontrivial component of $H - \{(w_1, x_1), (x_1, w_2), (w_2, v), (v, w_1)\}$ and D' the nontrivial component of $D - \{(w_1, x_1), (x_1, w_2), (w_2, v), (v, w_1)\}$. Then H' is a spanning eulerian subdigraph of D', and so by induction, we have $t_2 = 0$ and $t_1 - 1 \le t_2' + t_2'' + t_3 - 1$. Hence (i) holds by induction.

To prove (ii), it suffices to investigate spanning trails in a nonsupereulerian D. By (i), any strong digraph $D(0, t_1', t_1'', 0, t_2', t_2'', t_3)$ is supereulerian, and so we assume that $\max\{t_1, t_2\} > 0$. We make the following claim.

Claim 3.3. *Let $D = D(t_1, t_1', t_1'', t_2, t_2', t_2'', t_3)$ with $\lambda(D) > 0$ be a nonsupereulerian digraph. If D has a spanning trail, then D has a spanning (u, v)-trail T satisfying*

$$\text{both } u \in \{x_1, x_2, \ldots, x_{t_1}\} \text{ and } v = w_2, \text{ or both } u \in \{y_1, y_2, \ldots, y_{t_2}\} \text{ and } v = w_1. \tag{14}$$

Proof. Since D is not supereulerian, by Observation 3.2 (i), $\max\{t_1, t_2\} > 0$, we may assume that $t_1 > 0$. Let T' be a spanning (u', v')-trail of D. We will construct a spanning trail satisfying (14) from the following cases.

We note that as T' is a (u', v')-trail, we have

$$d^+_{T'}(u') - d^-_{T'}(u') = 1 \text{ and } d^-_{T'}(v') - d^+_{T'}(v') = 1. \tag{15}$$

Case 1 $\{u', v'\} = \{w_1, w_2\}$.

If $u' = v'$, then D is supereulerian, contrary to the assumption of Claim 3.3. If T' is a (w_1, w_2)-trail and $d^+_{T'}(w_1) \ge 2$, then $T' - (w_1, x_1)$ is a spanning (x_1, w_2)-trail of D satisfying (14). If T' is a (w_1, w_2)-trail and $d^+_{T'}(w_1) = 1$, then there exists a vertex $y \in X \cup Y \cup Z$ such that $(y, w_2) \in A(T')$ and $(y, w_1) \in A(D) - A(T')$, so $T' - (y, w_2) + (y, w_1)$ is an eulerian subdigraph of D, contrary to the assumption of Claim 3.3. The proof for the case when both T' is a (w_2, w_1)-trail and $t_2 > 0$ is similar and so it is omitted. Hence we assume that T' is a (w_2, w_1)-trail and $t_2 = 0$. As $t_1 > 0$, $(w_1, x_1), (x_1, w_2) \in A(T')$. Since $n \ge 4$ and T' is spanning in D, there must be a vertex $y \in V(D)$ such that $(w_2, y), (y, w_1) \in A(T')$. It follows that $y \in Y \cup Z$ and $T' - y$ is an eulerian subdigraph of D. Since $t_2 = 0$, we have $y \in \{y_1', y_2', \ldots, y_{t_2'}', y_1'', y_2'', \ldots, y_{t_2''}''\} \cup Z$, and so y is incident with a pair of symmetric arcs $(y, w), (w, y)$ for some $w \in \{w_1, w_2\}$. It follows that $(T' - y) + \{(y, w), (w, y)\}$ is a spanning closed trail of D, contrary to the assumption of Claim 3.3.

Case 2 Both $u' \in \{w_1, w_2\}$ and $v' \in X \cup Y \cup Z$, or both $u' \in X \cup Y \cup Z$ and $v' \in \{w_1, w_2\}$.

Suppose first that $u' \in \{w_1, w_2\}$ and $v' \in X \cup Y \cup Z$. If $d^-_{T'}(v') = 1$, then by (15), for some $i \in \{1, 2\}$, $(v', w_i) \in A(D) - A(T')$. It follows that $T' + (v', w_i)$ is a spanning (u', w_i)-trail. By Case 1, we are done. Hence we assume that $d^-_{T'}(v') = 2$. Then by (15) and by (11), for some $i' \in \{1, 2\}$, $(w_1, v'), (w_2, v'), (v', w_{i'}) \in A(T')$. It follows that $T' - (w_{3-i'}, v')$ is a spanning $(u', w_{3-i'})$-trail. By Case 1, we are done. The proof for the case when both $u' \in X \cup Y \cup Z$ and $v' \in \{w_1, w_2\}$ is similar and so it is omitted.

Case 3 $u', v' \in X \cup Y \cup Z$.

By (15), either $d^+_{T'}(u') = 1$ and for some $j_1 \in \{1, 2\}$, $(w_{j_1}, u') \in A(D) - A(T')$, or $d^+_{T'}(u') = 2$ and for some $j_2 \in \{1, 2\}$, $(u', w_1), (u', w_2), (w_{j_2}, u') \in A(T')$. Likewise, either $d^-_{T'}(v') = 1$ and for some $j_3 \in \{1, 2\}$, $(v', w_{j_3}) \in A(D) - A(T')$, or $d^-_{T'}(v') = 2$ and for some $j_4 \in \{1, 2\}$, $(w_1, v'), (w_2, v'), (v', w_{j_4}) \in A(T')$. It follows that

$$T'' = \begin{cases} T' + \{(w_{j_1}, u'), (v', w_{j_3})\} & \text{if } d^+_{T'}(u') = 1 \text{ and } d^-_{T'}(v') = 1, \\ (T' - \{(u', w_{3-j_2})\}) + \{(v', w_{j_3})\} & \text{if } d^+_{T'}(u') = 2 \text{ and } d^-_{T'}(v') = 1, \\ (T' - \{(w_{3-j_4}, v')\}) + \{(w_{j_1}, u')\} & \text{if } d^+_{T'}(u') = 1 \text{ and } d^-_{T'}(v') = 2, \\ T' - \{(u', w_{3-j_2}), (w_{3-j_4}, v')\} & \text{if } d^+_{T'}(u') = 2 \text{ and } d^-_{T'}(v') = 2, \end{cases}$$

is a spanning (w', w'')-trail of D, for some $w', w'' \in \{w_1, w_2\}$. By Case 1, we are done. ■

Assume that (12) holds. Then $t_1 \ge 1$ and so $D - \{x_1\}$ satisfies the inequalities in Observation 3.2 (i). By the definition of D in Example 3.1, $\lambda(D - \{x_1\}) > 0$ if and only if either $t_3 > 0$, or both $(t_1 - 1) + t_1' + t_1'' > 0$ and $t_2 + t_2' + t_2'' > 0$. As $\lambda(D) > 0$, if $t_3 = 0$, then $t_2 + t_2' + t_2'' > 0$. Therefore, if $\lambda(D - \{x_1\}) = 0$, then $t_3 = 0$ and $t_2 + t_2' + t_2'' > 0$, and so by (12), we must have $t_1 = 1$ and $t_1' + t_1'' = 0$. These, together with (12), imply that D itself satisfies the inequalities in Observation 3.2 (i), and so D is supereulerian, a contradiction. Hence we must have $\lambda(D - \{x_1\}) > 0$. By Observation 3.2 (i), $D - \{x_1\}$ has a spanning closed trail Q. It follows that $Q + \{(x_1, w_2)\}$ is a spanning (x_1, w_2)-trail of D. With a similar argument, if (13) holds, then D also has a spanning trail.

Conversely, assume that D has a spanning trail. If D has a spanning closed trail, then by Observation 3.2 (i), each of (12) and (13) is satisfied. Hence we assume that D is not supereulerian. By Claim 3.3, we assume by symmetry that D has a spanning (x_1, w_2)-trail. Then $D - x_1$ has a spanning closed trail, and so (12) follows from Observation 3.2 (i). ■

Definition 3.4. Using the notation used in Example 3.1, we introduce a digraph family $\mathcal{D}(n)$ for each $n \geq 4$. Define a digraph $D \in \mathcal{D}(n)$ if and only if each of the following holds.

(F1) D has a subdigraph D', (called the **corresponding digraph of** D), such that there exist nonnegative integers $t_1, t_1', t_1'', t_2, t_2', t_2'', t_3$ satisfying $|V(D')| = 2 + t_1 + t_1' + t_1'' + t_2 + t_2' + t_2'' + t_3 \geq 4$ and $D' = D(t_1, t_1', t_1'', t_2, t_2', t_2'', t_3)$ (as defined in Example 3.1) such that both (12) and (13) are violated.

(F2) For each $i \in \{1, 2\}$, let s_i be a nonnegative integer and D_i be digraph with $V(D_i) = \{w_i, w_1^i, \ldots, w_{s_i}^i\}$ and $A(D_i) = \{(w_i, w_j^i), (w_j^i, w_i) : 1 \leq j \leq s_i\}$, such that $V(D_1) \cap V(D_2) = \emptyset$ and $V(D_i) \cap V(D') = \{w_i\}$. When $s_i = 0$, then D_i consists of a single vertex w_i.

(F3) Define D to be the digraph with $V(D) = V(D') \cup V(D_1) \cup V(D_2)$ and $A(D) = A(D') \cup A(D_1) \cup A(D_2)$, and let $n = |V(D)|$.

By Lemma 2.2 (vi) and using the notation in Definition 3.4, a digraph $D \in \mathcal{D}(n)$ has a spanning trail if and only if the corresponding D' of D has a spanning trail. The following follows from Example 3.1.

$$\text{For any digraph } D \in \mathcal{D}(n),\ D \text{ does not have a spanning trail.} \tag{16}$$

Corollary 3.5. *Let D be a digraph obtained from a digraph $D' = D(t_1, t_1', t_1'', t_2, t_2', t_2'', t_3)$ (as defined in Example 3.1) with $4 = |V(D')| = 2 + t_1 + t_1' + t_1'' + t_2 + t_2' + t_2'' + t_3$ by attaching a number of 2-cycles to each vertex of $V(D')$. Then D is supereulerian if and only if D is strong.*

Proof. By Lemma 2.2 (vii), it suffices to examine these properties for D'. Since D is strong, by the way we form D from D', D' is also strong. By Example 3.1, D' is strong if and only if both $t_1 + t_1' + t_1'' + t_3 > 0$ and $t_2 + t_2' + t_2'' + t_3 > 0$. As $2 = t_1 + t_1' + t_1'' + t_2 + t_2' + t_2'' + t_3$, we have both $t_1 \leq t_2 + t_2' + t_2'' + t_3$ and $t_2 \leq t_1 + t_1' + t_1'' + t_3$. Thus Corollary 3.5 follows from Observation 3.2 (i). ■

Lemma 3.6. *Let D be a digraph with $|V(D)| = 5$ such that $G(D)$ has a hamiltonian cycle. If D is strongly connected, then D has a spanning trail.*

Proof. If D is supereulerian, then D has a spanning trail. Hence we assume that D is not supereulerian to show that D has a spanning trail. Let C be the longest cycle in D with arcs $A(C) = \{(z_1, z_2), (z_2, z_3), \ldots, (z_{c-1}, z_c)\}$. As D is not supereulerian, we have $3 \leq c \leq 4$. Suppose first that $c = 4$, let $x \in V(D) - V(C)$. Since D is strongly connected, hence there exists a vertex $z_i \in V(C)$ such that $(x, z_i) \in A(D)$. Without loss of generality, assume that $(x, z_1) \in A(D)$. Thus D has a spanning trail induced by the arcs $\{(x, z_1), (z_1, z_2), (z_2, z_3), (z_3, z_4)\}$. Suppose now that $c = 3$. Fix a vertex $x \in V(D) - V(C)$. Since D is strong, there exist vertices $z_x', z_x'' \in \{z_1, z_2, z_3\}$ such that D contains an (x, z_x')-path P_x' satisfying P_x' is a shortest path of D from vertex x to C and a (z_x'', x)-path P_x'' satisfying P_x'' is a shortest path of D from C to vertex x. If $|V(P_x')| \geq 3$, since $|V(D)| = 5$, hence $|V(P_x')| = 3$ and $V(D) - V(C) \subset V(P_x')$. Assume that $z_x' = z_1$, thus D has a spanning trail induced by the arcs $A(P_x') \cup \{(z_1, z_2), (z_2, z_3)\}$. Likewise, if $|V(P_x'')| \geq 3$, then we can obtain a spanning trail of D. Hence assume that $P_x' = (x, z_x')$ and $P_x'' = (z_x'', x)$. If for any $x \in V(D) - V(C)$, we always have $z_x' = z_x''$, then D would be supereulerian, a contradiction. Hence there exists a vertex x_1 such that $z_{x_1}' \neq z_{x_1}''$. By symmetry, we assume that $z_2 = z_{x_1}'$ and $z_3 = z_{x_1}''$. Since $c = 3$, D does not have a 4-cycle and so we must have $(x_1, z_2), (z_3, x_1) \in A(D)$. Let x_2 denote the only vertex in $V(D) - \{z_1, z_2, z_3, x_1\}$. If $z_{x_2}' = z_{x_2}''$, then we must have $(x_2, z_{x_2}'), (z_{x_2}', x_2) \in A(D)$, and so D has a spanning trail induced by the arcs $\{(z_1, z_2), (z_2, z_3), (z_3, x_1), (x_2, z_{x_2}'), (z_{x_2}', x_2)\}$. Therefore, we assume that $z_{x_2}' \neq z_{x_2}''$. If $z_1 \in \{z_{x_2}', z_{x_2}''\}$, then we may assume by symmetry that $\{z_1, z_3\} = \{z_{x_2}', z_{x_2}''\}$. It follows by $c = 3$ that $(z_1, x_2), (x_2, z_3) \in A(D)$, and so D has a spanning closed trail induced by the arcs $\{(x_1, z_2), (z_2, z_3), (z_3, x_1), (z_1, x_2), (x_2, z_3), (z_3, z_1)\}$. If $z_1 \notin \{z_{x_2}', z_{x_2}''\}$, then by $c = 3$ and as D is not supereulerian, we must have that $(x_2, z_2), (z_3, x_2) \in A(D)$. Since $G(D)$ has a 5-cycle, there must be an arc $e \in A(D)$ incident with two vertices in $\{z_1, x_1, x_2\}$. By symmetry, assume that $(x_1, x_2) \in A(D)$, then D has a spanning trail induced by the arcs $\{(x_1, x_2), (x_2, z_2), (z_2, z_3), (z_3, z_1)\}$. This completes the proof of the lemma. ■

A **block** of a graph G is a maximal subgraph H of G such that H contains no cut vertices of itself. By definition, if B is a block of a graph G with at least 3 vertices, then B must be 2-connected. Also by definition, if D is strong, then either D is a 2-cycle, or every block of $G(D)$ must be 2-connected. The main purpose of this subsection is to prove Theorem 3.7, which implies Theorem 1.3 (i).

Theorem 3.7. *Let $n > 1$ be an integer, D be a strong digraph with $|V(D)| = n$. Then one of the following holds.*

(i) $\alpha'(D) = 1$ and D is strongly trail-connected.

(ii) $\alpha'(D) = 2$ and the following are equivalent.

(ii-1) D has a spanning trail.

(ii-2) $D \notin \mathcal{D}(n)$.

Proof. Suppose first that $\alpha'(D) = 1$. Then $G(D)$ is spanned by a $K_{1,n-1}$. As (i) holds trivially if $n = 2$, we assume that $n \geq 3$. Let v_0 be the vertex of degree $n - 1$ in this $K_{1,n-1}$. If $G(D)$ does not have a cycle of length longer than 2, then v_0 is incident with every arc in $A(D)$. As D is strong, every arc of D is symmetric, and so D is the symmetric core of itself. It follows from Lemma 2.2 (iii) that D is strongly trail-connected. Hence we assume that $G(D)$ contains a cycle of length at least 3. Then D has an arc that is not incident with v_0. By $\alpha'(D) = 1$, we must have $n = 3$ and so D is spanned by a directed 3-cycle. Once again we have that D is strongly trail-connected. This proves (i).

To prove (ii), we assume that $\alpha'(D) = 2$. By (16), every member $D \in \mathcal{D}(n)$ does not have a spanning trail, and so ($ii - 1$) implies ($ii - 2$). Hence we assume that $D \notin \mathcal{D}(n)$ to show that D has a spanning trail. As it is routine to verify that every strong digraph with at most 3 vertices is supereulerian, we assume that $n \geq 4$.

Let $c = c(G(D))$ denote the length of a longest cycle of $G(D)$. Since D is strong and $\alpha'(G(D)) = \alpha'(D) = 2$, $2 \leq c \leq 5$. If $c = 2$, then $\tilde{G}$, the simplification of $G(D)$, must be a tree and so every pair of adjacent vertices $u, v \in V(D)$ are vertices of a 2-cycle in D. It follows by Lemma 2.2 (i) that $D = J(D)$ is supereulerian. Thus we may assume that $3 \leq c \leq 5$. Let B be a block of $G(D)$ that contains a longest cycle of $G(D)$.

Claim 3.8. *Each of the following holds.*

(i) If $c = 5$, then $G(D) = B$ with $|V(G(D))| = 5$.

(ii) If $c = 4$, then either $G(D) = B$, or B is spanned by a $K \cong K_{2,t}$ for some $t \geq 2$ with w_1, w_2 being two nonadjacent vertices of degree t in K, such that every block B' of $G(D)$ other than B is a 2-cycle in D and contains exactly one vertex $v_{B'} \in V(K)$. Furthermore, if $t \geq 3$, then $v_{B'} \in \{w_1, w_2\}$.

Suppose that $c = 5$ and let C be a cycle of length 5. If $|V(B)| > 5$, then as B is connected, an edge $e \in E(B) - E(C)$ together with a matching of size 2 not adjacent with e forms a matching of sizes 3 in B, leading to a contradiction that $2 = \alpha'(G(D)) \geq \alpha'(B) \geq 3$. Hence we must have $|V(B)| = 5$. Assume that $G(D)$ has a block B_1 other than B. Then there must be an edge $e' \in E(B_1)$. By definition of block, $|V(B) \cap V(B_1)| \leq 1$. Since C contains a matching M' of size 2. It follows that $2 = \alpha'(G(D)) \geq |M' \cup \{e'\}| = 3$, a contradiction. Hence we must have $G(D) = B$.

Now we assume that $c = 4$, and so B contains a $K_{2,2}$ as a subgraph. Choose a maximum value t such that B contains a subgraph K isomorphic to a $K_{2,t}$. Let w_1, w_2 denote two nonadjacent vertices of degree t in K and let $V(K) - \{w_1, w_2\} = \{v_1, v_2, \ldots, v_t\}$. If there exists a vertex $z \in V(B) - V(K)$, then since $\kappa(B) \geq 2$, there will be two internally disjoint shortest paths from z to two distinct vertices z', z'' in $V(K)$, implying that either B has a cycle of length at least 5, or $G(D)$ has a subgraph isomorphic to a $K_{2,t+1}$. As either case leads to a contradiction, we conclude that B is spanned by K.

Assume that $G(D) \neq B$. Let B' be an arbitrary block of $G(D)$ other than B. If $V(B') \cap V(B) = \emptyset$, then an edge in B' together with a 2-matching in B would lead to the contradiction $2 = \alpha'(D) \geq 3$. Hence every block B' other than B in $G(D)$ must contain a vertex $v_{B'}$ such that $V(B') \cap V(K) = V(B') \cap V(B) = \{v_{B'}\}$, and every edge in B' is incident with the vertex $v_{B'} \in V(K)$. Again by $\alpha'(D) = 2$, if $t \geq 3$, then we must have $v_{B'} \in \{w_1, w_2\}$ for any block B' other than B in $G(D)$. As D is strong, $G(D)$ is 2-edge-connected and so $\kappa'(B') \geq 2$. This implies that B' is a 2-cycle containing $v_{B'}$. Since D is strong, this 2-cycle in B' is a 2-cycle in D. This justifies Claim 3.8.

By Claim 3.8 and Lemma 3.6, if $c = 5$, then D has a spanning trail. Hence it suffices to assume that $3 \leq c \leq 4$ to prove Theorem 3.7 (ii).

Claim 3.9. *Suppose that $c = 3$. Each of the following holds.*

(i) Every block of $G(D)$ has 2 or 3 vertices.

(ii) There are at most two blocks of order 3, and if $G(D)$ has two blocks B', B'' of order 3, then $|V(B') \cap V(B'')| = 1$.

(iii) D has a spanning closed trail.

Assume that $c = 3$. Let $B_1, B_2, \ldots, B_b$ be all the blocks of $G(D)$ such that for some b' with $1 \leq b' \leq b$, $|V(B_1)| \geq \cdots \geq |V(B_{b'})| \geq 3$ and $|V(B_{b'+1})| = \cdots = |V(B_b)| = 2$. For each $B \in \{B_1, \ldots, B_{b'}\}$, as $c = 3$, B contains a 3-cycle C. If there exists a vertex $v \in V(B) - V(C)$, then as $\kappa(B) \geq 2$, there will be two internally disjoint shortest paths from v to two distinct vertices in $V(C)$, implying the B has a cycle of length at least 4. Hence we must have $V(B) = V(C)$, and so Claim 3.9 (i) follows.

Since two distinct blocks B', B'' of $G(D)$ must satisfy $|V(B') \cap V(B'')| \leq 1$, it follows that $b' \leq \alpha'(D) = 2$. Furthermore, assume that $|V(B') \cap V(B'')| = 0$, then as $G(D)$ is connected, there must be an additional block B''' of $G(D)$. It follows by $|V(B')| = |V(B'')| = 3$ and $|V(B''')| = 2$ that $G(D)$ has a matching of size 3, contrary to $\alpha'(D) = 2$. This justifies Claim 3.9 (ii).

Since D is strong, every block B of $G(D)$ induces a strong subdigraph $D[V(B)]$ of D. It follows by $|V(B)| \leq 3$ that every $D[V(B)]$ is supereulerian. Thus D has a spanning closed trail. This completes the proof of Claim 3.9.

By Claims 3.8 and 3.9 and by Lemma 3.6, we may assume that $c = 4$. By Claim 3.8 (ii), for some integer $t \geq 2$, $G(D)$ has a unique block B spanned by a $K_{2,t}$. If $t = 2$, then B is a 4-cycle. By Claim 3.8 (ii) and Corollary 3.5, D is supereulerian, and so D has a spanning trail.

Hence we assume that $t \geq 3$. Let w_1, w_2 denote the two vertices of degree t in this $K_{2,t}$ such that every block of $G(D)$ other than B is a 2-cycle of D containing w_1 or w_2. By Example 3.1 (and using the notation in Example 3.1), $B = D(t_1, t_1', t_1'', t_2, t_2', t_2'', t_3)$ for some non negative integers $t_1, t_1', t_1'', t_2, t_2', t_2'', t_3$ satisfying $|V(B)| = 2 + t_1 + t_1' + t_1'' + t_2 + t_2' + t_2'' + t_3$. As $D \notin \mathcal{D}(n)$, we conclude that either (12) or (13) must hold. By Observation 3.2 (ii), D has a spanning trail. This completes the proof for Theorem 3.7 (ii). ■

3.2. Supereulerian digraphs and strongly trail-connected digraphs

The main result of this subsection is to prove Theorem 1.3 (*iii*) and (*iv*), restated in Theorem 3.10. Recall that D_0 denotes the vertex disjoint union of three complete digraphs of order 3.

Theorem 3.10. *Let D be a strong digraph on n vertices with* $\alpha'(D) \geq 3$*, and* $n \geq 2\alpha'(D)+3$*, and let* $J = J(D)$ *be a symmetric core of D. Each of the following holds.*
(i) If $\lambda(D) \geq \alpha'(D) - 1$*, then D is supereulerian.*
(ii) If $\lambda(D) \geq \alpha'(D) \geq 4$*, then J is a connected spanning subdigraph of D.*

Proof. Let $k = \alpha'(D) \geq 3$ and $n = |V(D)| \geq 2k + 3$. By Corollary 2.8, Theorem 3.10 (*ii*) holds. It suffices to prove Theorem 3.10(i). As $\lambda(D) \geq k - 1 \geq 2$, $D \ncong D_0$ and for any vertex $v \in V(D)$, $d_D(v) \geq 2k - 2$.

Suppose first that there exists a vertex $x_1 \in X$ such that $d_D(x_1) \geq 2k - 1$. If $k_1(x_1) > 0$, then by Lemma 2.5 (*iv*), D is supereulerian; if $k_1(x_1) = 0$, then by Lemma 2.9 (*iv*) and as $\lambda(D) \geq 2$, D is supereulerian. Therefore, we assume that for any vertex $x \in X$, $d_D(x) = 2k - 2$. If there exists a vertex $x_1 \in X$ with $k_1(x_1) > 0$, then by Lemma 2.13 (*ii*), D is supereulerian. Now assume that for any vertex $x \in X$, $k_1(x) = 0$. By Lemmas 2.10 (*iii*) and 2.12 (*iii*), D must also be supereulerian. This completes the proof of Theorem 3.10. ■

3.3. Spanning trails in digraphs

The purpose of this subsection is to prove Theorem 1.3 (*ii*). Throughout this subsection, D denotes a strong digraph with $|V(D)| = n \geq 6$ and $\alpha'(D) = k \geq 3$, and let $\delta^+(D)$, $\delta^-(D)$ denote the minimum out-degree and the minimum in-degree of D, respectively. The following example was first presented in [15].

Example 3.11. Let $k_1, k_2, \ell \geq 2$ be integers, and D_1 and D_2 be two disjoint complete digraphs of order $k_1 + 1$ and $k_2 + 1$, respectively, and let U be an independent set disjoint from $V(D_1) \cup V(D_2)$ with $|U| = \ell$. Let $\mathcal{F}(k_1, k_2, \ell)$ denote the family of digraphs such that $D \in \mathcal{F}(k_1, k_2, \ell)$ if and only if D is the digraph obtained from $D_1 \cup D_2 \cup U$ by adding all arcs directed from every vertex in U and D_2 to every vertex in D_1, and all arcs directed from every vertex in D_2 to every vertex in U, and then by adding an set of $\ell - 1$ arcs directed from some vertices in D_1 to some vertices in D_2, in such a way that U is a stable set in D.

Assume $k_1, k_2 \geq \ell - 1$. For any $D \in \mathcal{F}(k_1, k_2, \ell)$, D has $n = k_1 + k_2 + \ell + 2$ vertices, and is a strong digraph with minimum degree $\delta^+(D) = k_1$ and $\delta^-(D) = k_2$. Direct computation shows that for each $D \in \mathcal{F}(k_1, k_2, 2)$, $\delta^+(D) + \delta^-(D) = |V(D)| - 4$. Let $\mathcal{F}_0(k_1, k_2, 2)$ be the set of spanning subdigraphs D' of the digraphs in $\mathcal{F}(k_1, k_2, 2)$ which satisfy $\delta^+(D') + \delta^-(D') = |V(D')| - 4$.

In [15], Hong et al. showed that every digraph in $\mathcal{F}_0(k_1, k_2, 2)$ is a not supereulerian, and proved the following.

Theorem 3.12 (*Hong et al. Theorem 3.4 of [15]*)**.** *Let D be a strong digraph of order n and minimum out-degree* $\delta^+(D) \geq 4$ *and minimum in-degree* $\delta^-(D) \geq 4$*. If* $\delta^+(D) + \delta^-(D) \geq n - 4$*, then the following are equivalent.*
(i) D has a spanning eulerian subdigraph.
(ii) Either $\delta^+(D) + \delta^-(D) > n - 4$*, or for some integer* k_1, k_2*,* $\delta^+(D) = k_1$*,* $\delta^-(D) = k_2$ *but* $D \notin \mathcal{F}_0(k_1, k_2, 2)$*.*

Let $k \geq 3$ be an integer. It is routine to verify the following.

Observation 3.13. *Every digraph* $D \in \mathcal{F}_0(k - 1, k - 1, 2)$ *with* $\lambda(D) \geq k - 1$ *has a spanning trail.*

In fact, using the notation in Example 3.11 for the structure of D, we let $D_1 \cong D_2 \cong K_k^*$ and $U = \{u_1, u_2\}$ with an arc $(v', v'') \in (V(D_1), V(D_2))_D$, one can start with a vertex $w'' \in V(D_2) - \{v''\}$, traverses every vertices in D_2 and then passes u_2; then from u_2 to a vertex $w' \in V(D_1) - \{v'\}$ and traverses every vertex in $V(D_1)$ with the last vertex in v'; and finally completes the trail with the arcs $(v', v''), (v'', u_1)$. Thus D has a spanning trail.

Proof of Theorem 1.3 (*ii*). Assume that $n = |V(D)| \geq 12$, $\alpha'(D) = k \geq 3$ and $\lambda(D) \geq k - 1 \geq 2$. By Theorem 1.3 (*iii*), if $n = |V(D)| \geq 2k + 3$, then D is supereulerian and so has a spanning trail. Hence we assume that $2k \leq n \leq 2k + 2$. If $n \in \{2k, 2k + 1\}$, then by Theorem 3.12, D is supereulerian. Therefore we assume that $n = 2k + 2$, and so by $n \geq 12$, $\min\{\delta^+(D), \delta^-(D)\} \geq \lambda(D) \geq k - 1 \geq \frac{n-4}{2} \geq 4$ and $\delta^+(D) + \delta^-(D) \geq n - 4$. By Theorem 3.12, either D is supereulerian or $D \in \mathcal{F}_0(k - 1, k - 1, 2)$. By Observation 3.13, D has a spanning trail. This completes the proof of Theorem 1.3 (*ii*).

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