Discrete Mathematics 344 (2021) 112584

Contents lists available at ScienceDirect

Discrete Mathematics

www.elsevier.com/locate/disc

Symmetric core and spanning trails in directed networks

Juan Liu [a], Hong Yang [b], Hong-Jian Lai [c], Xindong Zhang [d,*]

[a] College of Big Data Statistics, Guizhou University of Finance and Economics, Guiyang, Guizhou, 550025, China
[b] College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang, 830046, China
[c] Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA
[d] College of Mathematics Sciences, Xinjiang Normal University, Urumqi, Xinjiang, 830017, China

ARTICLE INFO

Article history:
Received 19 November 2020
Received in revised form 26 July 2021
Accepted 30 July 2021
Available online xxxx

Keywords:
Symmetric core
Supereulerian digraph
Strongly trail-connected
Contraction

ABSTRACT

A digraph D is supereulerian if D has a spanning closed trail, and is strongly trail-connected if for any pair of vertices $u, v \in V(D)$, D has a spanning (u, v)-trail and a spanning (v, u)-trail. The symmetric core $J = J(D)$ of a digraph D is a spanning subdigraph of D with $A(J)$ consisting of all symmetric arcs in D. Let $J_1, J_2, \cdots, J_{k(D)}$ be the connected symmetric components of J and define $\lambda_0(D) = \min\{\lambda(J_i) : 1 \le i \le k(D)\}$. We prove that the contraction $D' = D/J$ can be used to predict the existence of spanning trails in D. It is known that if $k(D) \le 2$, then D has a spanning closed trail. In particular, each of the following holds for a strong digraph D with $k(D) \ge 3$.
(i) If $\lambda_0(D) \ge k(D) - 2$, then D has a spanning trail if and only if D' has a spanning trail.
(ii) If $\lambda_0(D) \ge k(D) - 1$, then D is supereulerian if and only if D' is supereulerian.
(iii) If $\lambda_0(D) \ge k(D)$, then D is strongly trail-connected if and only if D' is strongly trail-connected.

1. Introduction

The undirected networks are also called simple networks, which are simplified networks that ignore the difference of direction of the links between the individuals and only consider whether the links exist. In many real systems, the interactions between nodes are usually not merely binary entities (either present or not). As a pivotal type of network, the directed network distinguishes the direction of the links between individuals, and thus can more accurately describe the individuals and their relationships. The underlying topology of an interconnection network can be modeled by a digraph.

Throughout this paper, we use G to denote a graph and D a digraph. Graphs and digraphs considered are finite, and the undefined terms and notation will follow [10] for graphs and [5] for digraphs. As in [5], we use (u, v) to denote an arc oriented from a vertex u to a vertex v. A digraph D is one that does not have loops (arcs whose head and tail coincide) nor parallel arcs (pair of arcs with the same tail and same head), and $\lambda(D)$ denotes the arc-strong connectivity of D. We often use $G(D)$ for the **underlying graph** of D, the graph obtained from D by erasing all orientation on the arcs of D. A digraph D is **strong** if $\lambda(D) > 0$ and is **weakly connected** if $G(D)$ is connected. If X is a vertex subset or an arc subset of D, we use $D[X]$ to denote the subdigraph of D induced by X. As in [5], for subsets $X, Y \subseteq V(D)$, define

* Corresponding author.
E-mail addresses: liujuan1999@126.com (J. Liu), 1070556334@qq.com (H. Yang), hjlai@math.wvu.edu (H.-J. Lai), liaoyuan1126@163.com (X. Zhang).

https://doi.org/10.1016/j.disc.2021.112584

$$(X, Y)_D = \{(x, y) \in A(D) : x \in X, y \in Y\}, \text{ and } [X, Y]_D = (X, Y)_D \cup (Y, X)_D.$$

If $X = \{x\}$ or $Y = \{y\}$, we often use $(x, Y)_D$ for $(X, Y)_D$ or $(X, y)_D$ for $(X, Y)_D$, respectively. Hence $(x, y)_D = (\{x\}, \{y\})_D$. For a vertex $v \in V(D)$, let $\partial_D^+(v) = (v, V(D) - v)_D$ and $\partial_D^-(v) = (V(D) - v, v)_D$. Thus $d_D^+(v) = |\partial_D^+(v)|$ and $d_D^-(v) = |\partial_D^-(v)|$. Let $\Delta^+(D) = \max\{d_D^+(v) : v \in V(D)\}$, $\Delta^-(D) = \max\{d_D^-(v) : v \in V(D)\}$ and $\Delta^0(D) = \max\{\Delta^+(D), \Delta^-(D)\}$. For a vertex subset $X \subseteq V(D)$, let $\partial_D^+(X) = (X, V(D) - X)_D$, $\partial_D^-(X) = (V(D) - X, X)_D$, $d_D^+(X) = |\partial_D^+(X)|$ and $d_D^-(X) = |\partial_D^-(X)|$.

Throughout this paper, we use paths, cycles and trails as defined in [10] when the discussion is on an undirected graph G, and to denote directed paths, directed cycles and directed trails when the discussion is on a digraph D. A directed trail (or path, respectively) from a vertex u to a vertex v in a digraph D is often referred as to a (u, v)-trail (a (u, v)-path, respectively).

The supereulerian problem was introduced by Boesch, Suffel, and Tindell in [9], seeking to characterize graphs that have spanning eulerian subgraphs. Pulleyblank in [19] proved that determining whether a graph is supereulerian, even within planar graphs, is NP-complete. There have been lots of researches on this topic. For more literatures on supereulerian graphs, see Catlin's informative survey [11], as well as the later updates in [12] and [17]. The supereulerian problem in digraphs was considered by Gutin ([13,14]). A strong digraph D is **eulerian** if for any $v \in V(D)$, $d_D^+(v) = d_D^-(v)$. A digraph D is **supereulerian** if D contains a spanning eulerian subdigraph, or equivalently, a spanning closed trail. Thus supereulerian digraphs must be strong, and every hamiltonian digraph is also a supereulerian digraph.

A digraph D is **strongly trail-connected** if for any two vertices u and v of D, D possesses both a spanning (u, v)-trail and a spanning (v, u)-trail. As the case when $u = v$ is possible, every strongly trail-connected digraph is also supereulerian. Thus strongly trail-connected digraphs can be viewed as an extension of supereulerian digraphs.

The supereulerian digraph problem is to characterize the strong digraphs that contains a spanning closed trail. Other than the researches on hamiltonian digraphs, a number of studies on supereulerian digraphs have been conducted recently. In particular, Hong et al. in [15,16] and Bang-Jensen and Maddaloni in [6] presented several best possible sufficient degree conditions for supereulerian digraphs. Additional researches on various conditions of supereulerian digraphs can be found in [1–4,6–8,18,21], among others.

Let $D = (V(D), A(D))$ be a digraph. An arc $(u, v) \in A(D)$ is **symmetric** in D if $(u, v), (v, u) \in A(D)$, and asymmetric otherwise. Notice that a symmetric arc (u, v) together with the arc (v, u) form a pair of symmetric arcs of D. A digraph D is **symmetric** if every arc of D is symmetric. Let G be a graph. Define G^* to be the digraph with $V(G^*) = V(G)$, where $(u, v) \in A(G^*)$ if and only if $uv \in E(G)$. Thus G^* is always symmetric, and a digraph D is symmetric if and only if for some graph G, $D = G^*$. Especially, if $G = P$ is a path, then P^* is called a symmetric path. For any two vertices $u, v \in A(D)$ of D, if D contains a symmetric path from u to v, then D is a **symmetrically connected digraph**. Let L be a maximal subdigraph of D such that L is connected and symmetric, then L is called a **connected symmetric component** of D. Let $S(D) = \{e \in A(D) : e \text{ is symmetric in } D\}$. If $A(D) = S(D)$, then D is symmetric. The **symmetric core** of D, denoted by $J(D)$, has vertex set $V(D)$ and arc set $S(D)$. When D is understood from the context, we often use J for $J(D)$.

Let $e = (v_1, v_2) \in A(D)$ be an arc of D. Define D/e to be the digraph obtained from D by identifying v_1 and v_2 into a new vertex v_e, and deleting the possible resulting loop(s). If $W \subseteq A(D)$ is a symmetric arc subset, then define the **contraction** D/W to be the digraph obtained from D by contracting each arc $e \in W$, and deleting any resulting loops. Thus even D does not have parallel arcs, a contraction D/W is loopless but may have parallel arcs, with $A(D/W) \subseteq A(D) - W$. If H is a subdigraph of D, then we often use D/H for $D/A(H)$. If L is a connected symmetric component of H and v_L is the vertex in D/H onto which L is contracted, then L is the **contraction preimage** of v_L. We adopt the convention to define $D/\emptyset = D$, and define a vertex $v \in V(D/W)$ to be a **trivial vertex** if the preimage of v is a single vertex (also denoted by v) in D. Hence we often view trivial vertices in a contraction D/W as vertices in D. We use $\mathbb{Z}_k$ to denote the (additive) group of integers modulo k.

Throughout this paper, we will use the following notation. For a digraph D with symmetric core $J = J(D)$, let $D' = D/J$, $k(D)$ be the number of connected symmetric components of J, and $J_1, J_2, \ldots, J_{k(D)}$ denote the connected symmetric components of J. Define

$$\lambda_0(D) = \min\{\lambda(J_i) : 1 \le i \le k(D)\}.$$

It follows by the definition of contraction that $k(D) = |V(D')|$. It will be shown in Section 3 below that every strong digraph D with $k(D) \le 2$ is supereulerian. Thus we focus our study on digraph D with $k(D) \ge 3$. The following is our main result.

Theorem 1.1. *Let D be a strong digraph with $k(D) \ge 3$, D' is the contraction of D. Then each of the following holds.*
(i) If $\lambda_0(D) \ge k(D) - 2$, then D has a spanning trail if and only if D' has a spanning trail.
(ii) If $\lambda_0(D) \ge k(D) - 1$, then D is supereulerian if and only if D' is supereulerian.
(iii) If $\lambda_0(D) \ge k(D)$, then D is strongly trail-connected if and only if D' is strongly trail-connected.

In the next section, we present some examples that would play useful roles in our arguments. The properties of the symmetric core of a digraph will be investigated, which will then be applied to prove Theorem 1.1 in Section 3.

2. Examples

Following [5], a **trail** in D is an alternating sequence $T = v_1a_1v_2a_2v_3...v_{k-1}a_{k-1}v_k$ of vertices v_i and arcs a_i from D such that $a_i = (v_i, v_{i+1})$ for each i with $1 \le i \le k-1$, and such that all the arcs are mutually distinct. We observe that in a connected symmetric digraph D, the following property holds.

$$\text{For any } u, v \in V(D), D \text{ has a spanning } (u, v)\text{-trail.} \tag{1}$$

This property (1) will be extended (and be justified) in Lemma 3.2 of the next section to assist the arguments in the proof of Theorem 1.1. We display property (1) here as we are to use it to construct examples in this section to show that each of the conclusions in Theorem 1.1 is best possible in some sense. More precisely, we are to present, for any integer $k > 0$, each of the following:

(a) an infinite digraph family $\mathcal{D}_1(k)$ such that for any digraph $D \in \mathcal{D}_1(k)$, D is strong, satisfies $\lambda_0(D) = k(D) - 3$, and does not have a spanning trail,

(b) an infinite family $\mathcal{D}_2(k)$ of nonsupereulerian strong digraphs such that for any $D \in \mathcal{D}_2(k)$ satisfies $\lambda_0(D) = k(D) - 2$, and

(c) an infinite family $\mathcal{D}_3(k)$ of non strongly trail-connected digraphs such that for any $D \in \mathcal{D}_3(k)$ satisfies $\lambda_0(D) = k(D) - 1$.

Let D be a digraph and $U \subseteq V(D)$. We call a collection of trails $T_1, T_2, \ldots, T_t$ of the induced subdigraph $D[U]$ a **trail cover** of $D[U]$ if $\cup_{i=1}^{t} V(T_i) = U$ and $A(T_i) \cap A(T_j) = \emptyset$, whenever $1 \le i \ne j \le t$. The minimum value of such t, among all trail covers of $D[U]$, is denoted by $t_D(U)$. Thus, $t_D(U) = 1$ if and only if $D[U]$ has a spanning trail.

For any subset $A \subseteq V(D) - U$, let $B = V(D) - U - A$, and define

$$h(U, A) = \min\{|\partial_D^+(A)|, |\partial_D^-(A)|\} + \min\{|(U, B)_D|, |(B, U)_D|\} - t_D(U), \text{ and}$$
$$h(U) = \min\{h(U, A) : A \cap U = \emptyset\}.$$

Then we have the following propositions.

Proposition 2.1. *(Hong et al., Proposition 2.1 of [15]) If D has a spanning eulerian subdigraph, then for any $U \subseteq V(D)$, we have $h(U) \ge 0$.*

Let D be a digraph and X be an arc set such that for every $(u, v) \in X$, $u, v \in V(D)$. Define $D \cup X$ to be the digraph with vertex set $V(D)$ and arc set $A(D) \cup X$. If $X = \{e\}$, we often use $D + e$ to denote $D \cup \{e\}$.

Proposition 2.2. *If D has a spanning trail, then for any $U \subseteq V(D)$, we have $h(U) + 1 \ge 0$.*

Proof. Let H be a spanning (u, v)-trail of D. If $u = v$, then by Proposition 2.1, for any $U \subseteq V(D)$, we have $h(U) \ge 0$, and so $h(U) + 1 \ge 0$. Hence we assume that $u \ne v$.

Define $D' = D + (v, u)$ and $H' = H + (v, u)$. Then H' is a spanning eulerian subdigraph of D'. For any $U \subseteq V(D') = V(D)$ and any $A \subseteq V(D') - U = V(D) - U$, let $B = V(D') - U - A$. By Proposition 2.1,

$$\min\{|\partial_{D'}^+(A)|, |\partial_{D'}^-(A)|\} + \min\{|(U, B)_{D'}|, |(B, U)_{D'}|\} - t_{D'}(U) \ge 0.$$

We have the following observations.

(i) If $u, v \in A$, or $u, v \in B$, or $u, v \in U$, then $|\partial_D^+(A)| = |\partial_{D'}^+(A)|$, $|\partial_D^-(A)| = |\partial_{D'}^-(A)|$, $|(U, B)_D| = |(U, B)_{D'}|$ and $|(B, U)_D| = |(B, U)_{D'}|$, which imply that $t_{D'}(U) + 1 \ge t_D(U) \ge t_{D'}(U)$.

(ii) If both $u \in A$ and $v \in B$, or both $u \in B$ and $v \in A$, or both $u \in A$ and $v \in U$, or both $v \in A$ and $u \in U$, then

$$\min\{|\partial_D^+(A)|, |\partial_D^-(A)|\} \ge \min\{|\partial_{D'}^+(A)|, |\partial_{D'}^-(A)|\} - 1,$$

$|(U, B)_D| = |(U, B)_{D'}|$ and $|(B, U)_D| = |(B, U)_{D'}|$, which imply that $t_D(U) = t_{D'}(U)$.

(iii) If both $u \in B$ and $v \in U$, or both $v \in B$ and $u \in U$, then $|\partial_D^+(A)| = |\partial_{D'}^+(A)|$, $|\partial_D^-(A)| = |\partial_{D'}^-(A)|$,

$$\min\{|(U, B)_D|, |(B, U)_D|\} \ge \min\{|(U, B)_{D'}|, |(B, U)_{D'}|\} - 1,$$

which imply that $t_D(U) = t_{D'}(U)$.

By Observations (i), (ii) and (iii) above, we conclude that

$$\begin{aligned} h(U, A) + 1 &= \min\{|\partial_D^+(A)|, |\partial_D^-(A)|\} + \min\{|(U, B)_D|, |(B, U)_D|\} - t_D(U) + 1 \\ &\ge \min\{|\partial_{D'}^+(A)|, |\partial_{D'}^-(A)|\} + \min\{|(U, B)_{D'}|, |(B, U)_{D'}|\} - t_{D'}(U) \ge 0, \end{aligned}$$

and so $h(U) + 1 \ge 0$. Thus, if D has a spanning trail, then for any $U \subseteq V(D)$, we have $h(U) + 1 \ge 0$. $\square$

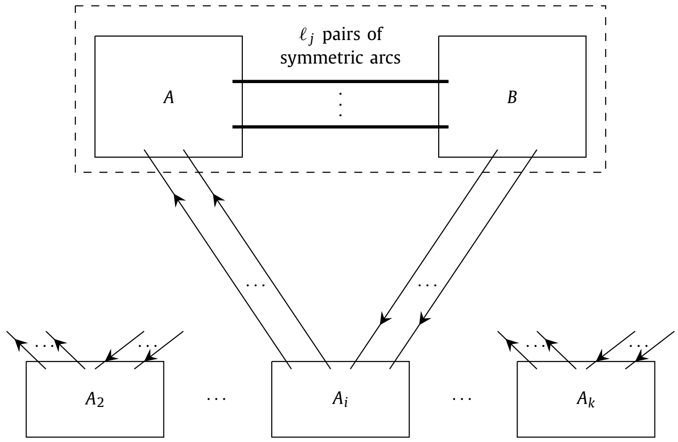

Fig. 1. For $j \in \{1, 2, 3\}$, the digraph in $\mathcal{D}_j(k)$ with $\ell_j = k - 4 + j$.

In each of the Examples 2.3, 2.5 and 2.7 below, we assume that a, b, k are integers and A, B, and $A_2, \ldots, A_k$ are mutually disjoint vertex sets satisfying the following:

$$k \geq 3, |A| = a \geq k \text{ and } |B| = b \geq k. \tag{2}$$

Example 2.3. For $2 \leq i \leq k$, assume that $|A_i| \geq k$. We construct a digraph family $\mathcal{D}_1(k)$ such that a digraph $D \in \mathcal{D}_1(k)$ if and only if $V(D) = A \cup B \cup \bigcup_{i=2}^{k} A_i$ and $A(D)$ consists of exactly the arcs as described in (D1-1), (D1-2) and (D1-3) below. (See Fig. 1 for an illustration.)
(D1-1) $D[A]$, $D[B]$ are connected symmetric digraphs with $\lambda(D[A]) \geq k-3$ and $\lambda(D[B]) \geq k-3$. There are exactly $k-3$ pairs of symmetric arcs between A and B. Let $J_1 = D[A \cup B]$.
(D1-2) For any $2 \leq i \leq k$, $J_i = D[A_i]$ is a connected symmetric digraph with $\lambda(J_i) \geq k-3$.
(D1-3) For any $2 \leq i \leq k$, there are exactly $k-3$ arcs in $(B, A_i)_D$ and exactly $k-3$ arcs in $(A_i, A)_D$.

Proposition 2.4. *Let $D \in \mathcal{D}_1(k)$ for given parameter k be defined as in Example 2.3, and let $J = \bigcup_{i=1}^{k} J_i$. Each of the following holds.*
(i) $D \in \mathcal{D}_1(k)$ is a strong digraph with $k(D) = k$.
(ii) $\lambda_0(D) = k-3$.
(iii) D/J has a spanning trail, but D does not have a spanning trail.

Proof. As (i) and (ii) follow from the definition of $D \in \mathcal{D}_1(k)$, it remains to justify (iii). By (D1-1), (D1-2) and (D1-3), D/J is spanned by a $K^*_{1,k-1}$, and so D/J has a spanning trail. Let $U = \bigcup_{i=2}^{k} A_i$. We apply Proposition 2.2 to show that D does not have a spanning trail. By (D1-1) and (D1-3), we have $\min\{|\partial_D^+(A)|, |\partial_D^-(A)|\} = |\partial_D^+(A)| = k-3$, as well as $\min\{|(U, B)_D|, |(B, U)_D|\} = |(U, B)_D| = 0$. By (1), each J_i has a spanning trail and $[A_i, A_j]_D = \emptyset$ with $2 \leq i \neq j \leq k$, and so $t_D(U) = k-1$. It follows that

$$h(U, A) + 1 = |\partial_D^+(A)| + |(U, B)_D| - t_D(U) + 1 = (k-3) - (k-1) + 1 < 0.$$

Thus by Proposition 2.2, D does not have a spanning trail. This proves (iii). $\square$

Example 2.5. For $2 \leq i \leq k$, assume that $|A_i| \geq k$. We construct a digraph family $\mathcal{D}_2(k)$ such that a digraph $D \in \mathcal{D}_2(k)$ if and only if $V(D) = A \cup B \cup \bigcup_{i=2}^{k} A_i$ and $A(D)$ consists of exactly the arcs as described in (D2-1), (D2-2) and (D2-3) below. (See Fig. 1 for an illustration.)
(D2-1) $D[A]$, $D[B]$ are connected symmetric digraphs with $\lambda(D[A]) \geq k-2$ and $\lambda(D[B]) \geq k-2$. There are exactly $k-2$ pairs of symmetric arcs between A and B. Let $J_1 = D[A \cup B]$.
(D2-2) For any $2 \leq i \leq k$, $J_i = D[A_i]$ is a connected symmetric digraph with $\lambda(J_i) \geq k-2$.
(D2-3) For any $2 \leq i \leq k$, there are exactly $k-2$ arcs in $(B, A_i)_D$ and exactly $k-2$ arcs in $(A_i, A)_D$.

Proposition 2.6. *Let $k \geq 3$ be an integer, $D \in \mathcal{D}_2(k)$ be the digraph defined in Example 2.5 and let $J = \bigcup_{i=1}^{k} J_i$. Each of the following holds.*
(i) $D \in \mathcal{D}_2(k)$ is a strong digraph with $k(D) = k$.
(ii) $\lambda_0(D) = k-2$.
(iii) D/J is supereulerian, but D is not supereulerian.

Proof. By definition of $D \in \mathcal{D}_2(k)$, (i) and (ii) hold. By (D2-1), (D2-2) and (D2-3), D/J is spanned by a $K^*_{1,k-1}$, and so D/J is supereulerian. Let $U = \bigcup_{i=2}^k A_i$. We apply Proposition 2.1 to show that D is not supereulerian. By (D2-1) and (D2-3) in Example 2.5, $\min\{|\partial_D^+(A)|, |\partial_D^-(A)|\} = |\partial_D^+(A)| = k-2$. By definition of $D \in \mathcal{D}_2(k)$, $\min\{|(U,B)_D|, |(B,U)_D|\} = |(U,B)_D| = 0$. By (1), each J_i has a spanning trail and $[A_i, A_j]_D = \emptyset$ with $2 \le i \ne j \le k$, and so $t_D(U) = k-1$. It follows that

$$h(U,A) = |\partial_D^+(A)| + |(U,B)_D| - t_D(U) = (k-2) - (k-1) < 0,$$

and so by Proposition 2.1, D is not supereulerian. This proves (iii). $\square$

Example 2.7. Assume that $|A_i| \ge k$ for any i with $2 \le i \le k$. We construct a digraph family $\mathcal{D}_3(k)$ such that a digraph $D \in \mathcal{D}_3(k)$ if and only if $V(D) = A \cup B \cup \bigcup_{i=2}^k A_i$ and $A(D)$ consists of exactly the arcs as described in (D3-1), (D3-2) and (D3-3) below. (See Fig. 1 for an illustration.)
(D3-1) $D[A]$, $D[B]$ are connected symmetric digraphs with $\lambda(D[A]) \ge k-1$ and $\lambda(D[B]) \ge k-1$. There are exactly $k-1$ pairs of symmetric arcs between A and B. Let $J_1 = D[A \cup B]$.
(D3-2) For any $2 \le i \le k$, $J_i = D[A_i]$ is a connected symmetric digraph with $\lambda(J_i) \ge k-1$.
(D3-3) For any $2 \le i \le k$, there are exactly $k-1$ arcs in $(B, A_i)_D$ and exactly $k-1$ arcs in $(A_i, A)_D$.

Proposition 2.8. *Let $D \in \mathcal{D}_3(k)$ for given parameter k be defined as in Example 2.7, and let $J = \bigcup_{i=1}^k J_i$. Each of the following holds.*
(i) $D \in \mathcal{D}_3(k)$ is a strong digraph with $k(D) = k$.
(ii) $\lambda_0(D) = k-1$.
(iii) D/J is a strongly trail-connected digraph, but D is not strongly trail-connected digraph.

Proof. By definition of $D \in \mathcal{D}_3(k)$, (i) and (ii) hold. By (D3-1), (D3-2) and (D3-3), D/J is spanned by a $K^*_{1,k-1}$, and so by (1), D/J is a strongly trail-connected digraph. Let $x \in A$ and $y \in B$ be two vertices, and T be an (x,y)-trail in D that contains all vertices in $A_2 \cup A_3 \cup \ldots \cup A_k$. As $x \in A$ and $y \in B$, T must traverse from A to B for the first time via an arc $e_0 \in (A,B)_D$. By the definition of $D \in \mathcal{D}_3(k)$, each time T traverses vertices in an A_i, T must use at least one arc $e_i \in (A,B)_D$. As there are $k-1$ subsets $A_2, \ldots, A_k$, it forces that $|(A,B)_D| \ge |\{e_0, e_1, \ldots, e_k\}| = k$. However, by (D3-1), we have $|(A,B)_D| = k-1$, a contradiction. This implies that D does not have a spanning (x,y)-trail, and so D is not strongly trail-connected. This proves (iii). $\square$

3. Main results

In this section, we investigate some properties on the symmetric core of a digraph for future applications in our arguments. These properties will then be applied to prove Theorem 1.1 at the end of this section.

Let H and H' denote two digraphs. Define $H \cup H'$ to be the digraph with $V(H \cup H') = V(H) \cup V(H')$ and $A(H \cup H') = A(H) \cup A(H')$. If T is a (v,w)-trail of a digraph D and $(u,v), (w,z) \in A(D) - A(T)$, then we use $(u,v)T(w,z)$ to denote the (u,z)-trail $D[A(T) \cup \{(u,v),(w,z)\}]$. The subdigraphs $(u,v)T$ and $T(w,z)$ are similarly defined.

Let $x_1, x_2, \ldots, x_s$ and $y_1, y_2, \ldots, y_s$ be two sequences of (not necessarily distinct) vertices of a digraph D. A **weak s-linking** from $(x_1, x_2, \ldots, x_s)$ to $(y_1, y_2, \ldots, y_s)$ in D is a system of arc-disjoint paths $P_1, P_2, \ldots, P_s$ such that P_i is an (x_i, y_i)-path in D with $i \in \{1, 2, \ldots, s\}$. A digraph $D = (V, A)$ is **weakly s-linked** if it contains a weak s-linking from $(x_1, x_2, \ldots, x_s)$ to $(y_1, y_2, \ldots, y_s)$ for every choice of (not necessarily distinct) vertices $x_1, x_2, \ldots, x_s$, $y_1, y_2, \ldots, y_s$. Shiloach [20] proved the following:

Theorem 3.1. *(Shiloach [20]) A directed multigraph D is weakly s-linked if and only if $\lambda(D) \ge s$.*

This theorem of Shiloach can be utilized to prove the following. The conclusion when $s = 1$ of Lemma 3.2 implies (1).

Lemma 3.2. *Let $s \ge 1$ be an integer, D be a connected symmetric digraph with $\lambda(D) \ge s$, and $x_1, x_2, \ldots, x_s$ and $y_1, y_2, \ldots, y_s$ be two vertex sequences of D. Then there exists a connected spanning subdigraph T'_D of D such that T'_D is an arc-disjoint union of trails $T_1, \ldots, T_s$ and for each i with $1 \le i \le s$, T_i is an (x_i, y_i)-trail.*

Proof. By Theorem 3.1, there is a system of arc-disjoint paths $P_1, P_2, \ldots, P_s$ such that each P_i is an (x_i, y_i)-path in D with $i \in \{1, 2, \ldots, s\}$. For each i, define $A_i = \{(u,v) \in A(D) : \ \{(u,v),(v,u)\} \cap A(P_i) \ne \emptyset\}$. Since D is a symmetric digraph, for any arc $(u,v) \in A(P_i) \subseteq A(D)$, we also have $(v,u) \in A(D)$, and so if $(u,v) \in A_i$, then $(v,u) \in A_i$ also. Thus $D[A_i]$ is a symmetric digraph. Let $H = D - \bigcup_{i=1}^s A_i$. Since D is a symmetric digraph and since each $D[A_i]$ is a symmetric digraph, it follows that H is also a symmetric digraph, and so every component of H is eulerian. Hence $T'_D = \bigcup_{i=1}^s P_i \cup H$ is a connected spanning subdigraph of D.

Let $H_1, H_2, \ldots, H_c$ be the connected components of H. Since D is connected, for any j with $1 \le j \le c$, there exists an i_j with $1 \le i_j \le s$ such that $V(H_j) \cap V(P_{i_j}) \ne \emptyset$. Hence the collection of connected components $\mathcal{F} = \{H_1, H_2, \ldots, H_c\}$ has a partition into s mutually disjoint sub-collections $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_s$ such that for each i with $1 \le i \le s$, either $\mathcal{F}_i$ is empty or every component of H in $\mathcal{F}_i$ has at least one vertex in common with the path P_i. Let $T_i = D[A(P_i) \cup \{A(H_{i_t}) : H_{i_t} \in \mathcal{F}_i\}]$. As H is symmetric, every component H_{i_t} in $\mathcal{F}_i$ is also symmetric, and so it is eulerian. It follows that T_i is an (x_i, y_i)-trail in D, and the collection $\{T_1, T_2, \ldots, T_s\}$ satisfies Lemma 3.2. $\square$

Lemma 3.3. *Let D be a digraph and k be a integer with $k \geq 2$. Then each of the following holds.*
(i) If D has a closed trail T with k vertices, then D contains a closed trail T' with $V(T') = V(T)$ and $\Delta^0(T') \leq k - 1$.
(ii) If D has an (x, y)-trail T with k vertices and $x \neq y$, then D contains an (x, y)-trail T' with $V(T') = V(T)$ and for any vertex $z \in V(T') - \{x, y\}$, $d^+_{T'}(z) = d^-_{T'}(z) \leq k - 2$, $d^+_{T'}(x) = d^-_{T'}(x) + 1 \leq k - 1$ and $d^-_{T'}(y) = d^+_{T'}(y) + 1 \leq k - 1$.

Proof. Let T' be a closed trail of D with $V(T') = V(T)$ and $|A(T')|$ be minimized. By contradiction, we assume that there is a vertex $z \in V(T')$ such that $d^+_{T'}(z) = d^-_{T'}(z) = k' \geq k$. Then T' has a family $\mathcal{C} = \{C_1, C_2, \ldots, C_{k'}\}$ of k' arc-disjoint cycles with $z \in V(C_i)$ and $|V(C_i)| \geq 2$ for any index i with $1 \leq i \leq k'$, and as $|V(T')| = k$, $k' \geq k$, thus, there is a cycle $C_\ell \in \mathcal{C}$ such that

$$V(C_\ell) \subseteq \bigcup_{C \in \mathcal{C} - \{C_\ell\}} V(C). \tag{3}$$

Otherwise, $|V(T')| \geq 1 + k' > k$, a contradiction. Let $T'' = T' - A(C_\ell)$. By (3) and $z \in V(C_i)$ for any index i with $1 \leq i \leq k'$, then T'' is connected and $V(T'') = V(T') = V(T)$. Thus T'' is a closed trail with $V(T'') = V(T)$ and $|A(T'')| < |A(T')|$, a contradiction to the assumption that $|A(T')|$ is minimum. Hence T' is a closed trail with $V(T') = V(T)$ and $\Delta^0(T') \leq k - 1$. This proves (i).

If $k = 2$, then the arc (x, y) is desired. Assume that $k \geq 3$. Let T' be an (x, y)-trail $(x \neq y)$ with $V(T') = V(T)$ and $|A(T')|$ be minimized. By contradiction, we assume that there is a vertex $z \in V(T') - \{x, y\}$ such that $d^+_{T'}(z) = d^-_{T'}(z) = k' \geq k - 1$, or $d^+_{T'}(x) = d^-_{T'}(x) + 1 = k' \geq k$, or $d^-_{T'}(y) = d^+_{T'}(y) + 1 = k' \geq k$.

If there is a vertex $z \in V(T') - \{x, y\}$ such that $d^+_{T'}(z) = d^-_{T'}(z) = k' \geq k - 1$, then we have

(a). T' has a family $\mathcal{C} = \{C_1, C_2, \ldots, C_{k'}\}$ of k' arc-disjoint cycles with $z \in V(C_i)$ and $|V(C_i)| \geq 2$ for any index i with $1 \leq i \leq k'$, or

(b). T' has an (x, y)-path P_0 with $z \in V(P_0)$ and a family $\mathcal{C} = \{C_1, C_2, \ldots, C_{k'-1}\}$ of $k' - 1$ arc-disjoint cycles with $z \in V(C_i)$ and $|V(C_i)| \geq 2$ for any index i with $1 \leq i \leq k' - 1$.

Since T' is an (x, y)-trail with $x \neq y$, we can claim that if (a) holds, then there is a cycle $C_\ell \in \mathcal{C}$ with $|V(C_\ell)| \geq 3$. Otherwise, if (a) holds and for any $C_i \in \mathcal{C}$ with $|V(C_i)| = 2$, then T' is a close trail, a contradiction. As $|V(T')| = k$ and $k' \geq k - 1$, thus, there is a cycle $C_{\ell'} \in \mathcal{C}$ such that

$$V(C_{\ell'}) \subseteq \bigcup_{C \in \mathcal{C} - \{C_{\ell'}\}} V(C). \tag{4}$$

Otherwise, $|V(T')| \geq 3 + (k' - 1) > k$, a contradiction. Let $T'' = T' - A(C_{\ell'})$. By (4) and $z \in V(C_i)$ for any index i with $1 \leq i \leq k' - 1$, then T'' is connected and $V(T'') = V(T') = V(T)$. Thus T'' is an (x, y)-trail with $V(T'') = V(T)$ and $|A(T'')| < |A(T')|$, a contradiction to the assumption that $|A(T')|$ is minimum.

If (b) holds, as $|V(T')| = k$ and $k' \geq k - 1$, then there is a cycle $C_{\ell''} \in \mathcal{C}$ such that

$$V(C_{\ell''}) \subseteq V(P_0) \cup \bigcup_{C \in \mathcal{C} - \{C_{\ell''}\}} V(C) \tag{5}$$

Otherwise, $|V(T')| \geq 3 + (k' - 1) > k$, a contradiction. Let $T'' = T' - A(C_{\ell''})$. By (5) and $z \in V(C_i)$ for any index i with $1 \leq i \leq k' - 1$, then T'' is connected and $V(T'') = V(T') = V(T)$. Thus T'' is an (x, y)-trail with $V(T'') = V(T)$ and $|A(T'')| < |A(T')|$, a contradiction to the assumption that $|A(T')|$ is minimum.

If $d^+_{T'}(x) = d^-_{T'}(x) + 1 = k' \geq k$, then T' has an (x, y)-path P_0 and a family $\mathcal{C} = \{C_1, C_2, \ldots, C_{k'-1}\}$ of $k' - 1$ arc-disjoint cycles with $x \in V(C_i)$ and $|V(C_i)| \geq 2$ for any index i with $1 \leq i \leq k' - 1$, and as $|V(T')| = k$ and $k' \geq k$, there is a cycle $C_{\ell'''} \in \mathcal{C}$ such that

$$V(C_{\ell'''}) \subseteq V(P_0) \cup \bigcup_{C \in \mathcal{C} - \{C_{\ell'''}\}} V(C). \tag{6}$$

Otherwise, $|V(T')| \geq 2 + (k' - 1) > k$, a contradiction. Let $T'' = T' - A(C_{\ell'''})$. By (6) and $x \in V(C_i)$ for any index i with $1 \leq i \leq k' - 1$, then T'' is connected and $V(T'') = V(T') = V(T)$. Thus T'' is an (x, y)-trail with $V(T'') = V(T)$ and $|A(T'')| < |A(T')|$, a contradiction to the assumption that $|A(T')|$ is minimum. Likewise, if $d^-_{T'}(y) = d^+_{T'}(y) + 1 = k' \geq k$, then a contradiction will be obtained similarly.

Hence, T' is an (x, y)-trail $(x \neq y)$ with $V(T') = V(T)$ and for any vertex $z \in V(T') - \{x, y\}$, $d^+_{T'}(z) = d^-_{T'}(z) \leq k - 2$, $d^+_{T'}(x) = d^-_{T'}(x) + 1 \leq k - 1$ and $d^-_{T'}(y) = d^+_{T'}(y) + 1 \leq k - 1$. This proves (ii). $\square$

Throughout the rest of this section, we assume that D is a digraph, $J = J(D)$ is the symmetric core of D with $k = k(D)$ and $J_1, J_2, \ldots, J_k$ are the connected symmetric components of J. Let $D' = D/J$ and denote $V(D') = \{v_{J_i} : 1 \le i \le k\}$ such that for each $i \in \{1, 2, \ldots, k\}$, J_i is the contraction preimage of the vertex $v_{J_i} \in V(D')$.

Lemma 3.4. *Let D be a digraph. Each of the following holds.*
(i) Let $t_i \le \lambda(J_i)$ be an integer for $1 \le i \le k$. If $\{v_{J_{i_\theta}} a_{(i_\theta,i)} v_{J_i} a_{(i,i'_\theta)} v_{J_{i'_\theta}} : 1 \le \theta \le t_i\}$ is a collection of t_i arc-disjoint paths in D', then D has a collection $\{T_{J_{i_\theta}} : 1 \le \theta \le t_i\}$ of t_i arc-disjoint trails with $V(J_i) \subseteq \bigcup_{\theta=1}^{t_i} V(T_{J_{i_\theta}})$.
(ii) If $\lambda_0(D) \ge k-1$ and T^0 is a $(v_{J_{j_1}}, v_{J_{jm}})$-trail of D' on vertices set $\{v_{J_{j_1}}, \ldots, v_{J_{jm}}\}$ with $v_{J_{j_1}} \neq v_{J_{jm}}$, then for any vertices $x \in V(J_{j_1})$ and $y \in V(J_{jm})$, D has an (x, y)-trail T with $\bigcup_{\ell=1}^{m} V(J_{j_\ell}) \subseteq V(T)$.
(iii) If $\lambda_0(D) \ge k-1$ and D' has a spanning closed trail, then D has a spanning closed trail.
(iv) Suppose that D' has a spanning closed trail. For a fixed index i and for any index i' with $1 \le i' \neq i \le k$, if $\lambda(J_i) \ge k$ and $\lambda(J_{i'}) \ge k-1$, then for any two distinct vertices $x, y \in V(J_i)$, D has a spanning (x, y)-trail.

Proof. Let $\{v_{J_{i_\theta}} a_{(i_\theta,i)} v_{J_i} a_{(i,i'_\theta)} v_{J_{i'_\theta}} : 1 \le \theta \le t_i\}$ be a collection of t_i arc-disjoint paths in D'. For each θ with $1 \le \theta \le t_i$, by the definition of contraction, the arcs $a_{(i_\theta,i)}, a_{(i,i'_\theta)} \in A(D') \subseteq A(D)$. Thus there exist vertices $x_{i_\theta}, y_{i_\theta} \in V(J_i)$, and $z_{i_\theta}, z_{i'_\theta} \in V(D) - V(J_i)$ such that as arcs in D, $a_{(i_\theta,i)} = (z_{i_\theta}, x_{i_\theta})$ and $a_{(i,i'_\theta)} = (y_{i_\theta}, z_{i'_\theta})$. Hence $x_{i_1}, x_{i_2}, \ldots, x_{i_{t_i}}$ and $y_{i_1}, y_{i_2}, \ldots, y_{i_{t_i}}$ are two vertex sequences of J_i. As $\lambda(J_i) \ge t_i$, by Lemma 3.2, J_i has t_i arc-disjoint $(x_{i_\theta}, y_{i_\theta})$-trails $T_{(x_{i_\theta}, y_{i_\theta})}$ and $V(J_i) \subseteq \bigcup_{\theta=1}^{t_i} V(T_{(x_{i_\theta}, y_{i_\theta})})$. Thus,

$$\{T_{J_{i_\theta}} = (z_{i_\theta}, x_{i_\theta}) T_{(x_{i_\theta}, y_{i_\theta})} (y_{i_\theta}, z_{i'_\theta}) : 1 \le \theta \le t_i\}$$

is a collection of t_i arc-disjoint trails with $V(J_i) \subseteq \bigcup_{\theta=1}^{t_i} V(T_{J_{i_\theta}})$, and

$$T_{J_i} = \bigcup_{\theta=1}^{t_i} T_{J_{i_\theta}} \tag{7}$$

is the connected arc-disjoint union of t_i trails in D as described in Lemma 3.4 (i).

To prove (ii), let T^0 be a $(v_{J_{j_1}}, v_{J_{jm}})$-trail in D' on vertex set $\{v_{J_{j_1}}, \ldots, v_{J_{jm}}\}$ with $v_{J_{j_1}} \neq v_{J_{jm}}$. By Lemma 3.3 (ii), D' has a $(v_{J_{j_1}}, v_{J_{jm}})$-trail T' with $V(T') = V(T^0)$ and

$$\begin{aligned} &\text{for any vertex } z \in V(T') - \{v_{J_{j_1}}, v_{J_{jm}}\}, d^+_{T'}(z) = d^-_{T'}(z) \le |V(T^0)| - 2, \\ &d^+_{T'}(v_{J_{j_1}}) = d^-_{T'}(v_{J_{j_1}}) + 1 \le |V(T^0)| - 1 \text{ and } d^-_{T'}(v_{J_{jm}}) = d^+_{T'}(v_{J_{jm}}) + 1 \le |V(T^0)| - 1. \end{aligned} \tag{8}$$

Let $T' = v_{J_{j_1}} a_{(j_1,j_2)} v_{J_{j_2}} \cdots v_{J_{j_{m-1}}} a_{(j_{m-1},j_m)} v_{J_{jm}}$. Since T' is a trail, for notational convenience, we assume that $v_{J_{j_1}} = v_{J_1}$ and $v_{J_{jm}} = v_{J_k}$. Define $Int(T') = \{v_{J_{j_\ell}} : 2 \le \ell \le m-1\}$. For $1 \le i \le k$, let

$$t_i = t_i(v_{J_i}) = |\{v_{J_{j_\ell}} : v_{J_{j_\ell}} = v_{J_i}\}| \text{ for each } \ell \text{ with } 2 \le \ell \le m-1. \tag{9}$$

By $k = |V(D')|$ and (8), we observe that $0 \le t_i \le k-2$. Hence, for any $2 \le i \le k-1$, we may assume that there are $x_{i_1}, x_{i_2}, \ldots, x_{i_{t_i}}$, $y_{i_1}, y_{i_2}, \ldots, y_{i_{t_i}} \in V(J_i)$, $z_{i_\theta} \in V(J_{i_\theta})$ and $z_{i'_\theta} \in V(J_{i'_\theta})$ with $1 \le \theta \le t_i$ and $i_\theta, i'_\theta \in \{j_1, j_2, \ldots, j_m\}$ such that, as arcs in $A(D)$, $(z_{i_\theta}, x_{i_\theta}) = a_{(i_\theta,i)} \in \partial^-_{T'}(v_{J_i})$ and $(y_{i_\theta}, z_{i'_\theta}) = a_{(i,i'_\theta)} \in \partial^+_{T'}(v_{J_i})$. Since $\lambda_0(D) \ge k-1$ and $t_i \le k-2$, we have $t_i < \lambda(J_i)$. By Lemma 3.4 (i), for any $v_{J_i} \in Int(T') - \{v_{J_1}, v_{J_k}\}$, T_{J_i} (as defined in (7)) is a connected arc-disjoint union of t_i trails in D with $V(J_i) \subseteq V(T_{J_i})$.

By (9), $d^-_{T'}(v_{J_1}) = t_1 \le k-2$ and $d^+_{T'}(v_{J_1}) = t_1 + 1$. Denote $a_{(j_1,j_2)} = (y_1, z_1)$ with $y_1 \in V(J_1)$ and $z_1 \in V(J_{j_2})$, as an arc in $A(D)$. We may assume that there exist vertices $x_{1_1}, x_{1_2}, \ldots, x_{1_{t_1}}$ and $y_{1_1}, y_{1_2}, \ldots, y_{1_{t_1}}$ in J_1, $z_{1_\theta} \in V(J_{1_\theta})$ and $z_{1'_\theta} \in V(J_{1'_\theta})$ with $1 \le \theta \le t_1$ and $1_\theta, 1'_\theta \in \{j_1, j_2, \ldots, j_m\}$ such that, as arcs in $A(D)$, $(z_{1_\theta}, x_{1_\theta}) = a_{(1_\theta,1)} \in \partial^-_{T'}(v_{J_1})$ and $(y_{1_\theta}, z_{1'_\theta}) = a_{(1,1'_\theta)} \in \partial^+_{T'}(v_{J_1})$. Since $x, y_1 \in V(J_1)$, it follows by $\lambda(J_1) \ge \lambda_0(D) \ge k-1 \ge t_1 + 1$ and by Lemma 3.2 that J_1 has an (x, y_1)-trail $T_{(x,y_1)}$, and for each θ, there exists an $(x_{1_\theta}, y_{1_\theta})$-trail $T_{(x_{1_\theta}, y_{1_\theta})}$ such that

$$T_{J_1} = T_{(x,y_1)}(y_1, z_1) \cup \bigcup_{\theta=1}^{t_1} (z_{1_\theta}, x_{1_\theta}) T_{(x_{1_\theta}, y_{1_\theta})} (y_{1_\theta}, z_{1'_\theta})$$

is a connected arc-disjoint union of $t_1 + 1$ trails in D with $V(J_1) \subseteq V(T_{J_1})$.

Likewise, by (9), $d^+_{T'}(v_{J_k}) = t_k \le k-2$ and $d^-_{T'}(v_{J_k}) = t_k + 1$. As arcs in $A(D)$, we denote $a_{(j_{m-1},j_m)} = (z_k, x_k)$ with $z_k \in V(J_{j_{m-1}})$ and $x_k \in V(J_k)$; and assume that there exist vertices $x_{k_1}, x_{k_2}, \ldots, x_{k_{t_k}}$, $y_{k_1}, y_{k_2}, \ldots, y_{k_{t_k}} \in V(J_k)$, $z_{k_\theta} \in V(J_{k_\theta})$ and $z_{k'_\theta} \in V(J_{k'_\theta})$ with $1 \le \theta \le t_k$ such that $(z_{k_\theta}, x_{k_\theta}) = a_{(k_\theta,k)} \in \partial^-_{T'}(v_{J_k})$ and $(y_{k_\theta}, z_{k'_\theta}) = a_{(k,k'_\theta)} \in \partial^+_{T'}(v_{J_k})$. As $x_k, y \in V(J_k)$, it follows by $\lambda(J_k) \ge \lambda_0(D) \ge k-1 \ge t_k + 1$ and by Lemma 3.2 that J_k has an (x_k, y)-trail $T_{(x_k,y)}$, and for each θ, there exists an $(x_{k_\theta}, y_{k_\theta})$-trail $T_{(x_{k_\theta}, y_{k_\theta})}$ such that

$$T_{J_k} = \bigcup_{\theta=1}^{t_k} (z_{k_\theta}, x_{k_\theta}) T_{(x_{k_\theta}, y_{k_\theta})} (y_{k_\theta}, z_{k'_\theta}) \cup (z_k, x_k) T_{(x_k, y)}$$

is a connected arc-disjoint union of $t_k + 1$ trails in D with $V(J_k) \subseteq V(T_{J_k})$.

Let $\mathcal{J}$ be the set with $\mathcal{J} = \{J_j : t_j(v_{J_j}) \geq 1 \text{ for } 2 \leq j \leq k-1\}$. Then,

$$T := T_{J_1} \cup (\bigcup_{J_j \in \mathcal{J}} T_{J_j}) \cup T_{J_k}$$

is a spanning (x, y)-trail of D with $\bigcup_{\ell=1}^{m} V(J_{j_\ell}) \subseteq V(T)$. This proves (ii).

Since D' has a spanning closed trail, by Lemma 3.3 (i), we assume that D' has a spanning closed trail $T' = v_{J_{j_1}} a_{(j_1,j_2)} v_{J_{j_2}} \cdots v_{J_{j_{m-1}}} a_{(j_{m-1},j_m)} v_{J_{jm}} a_{(j_m,j_1)} v_{J_{j_1}}$ such that for any vertex $v_{J_i} \in V(T')$, $d^+_{T'}(v_{J_i}) = d^-_{T'}(v_{J_i}) \leq k-1$. As $\lambda(J_i) \geq k-1$ and by (i) and (7), $\bigcup_{i=1}^{k} T_{J_i}$ is a spanning closed trail of D. This proves (iii).

To prove (iv), by Lemma 3.3 (i), we assume that D' has a spanning closed trail $T' = v_{J_{j_1}} a_{(j_1,j_2)} v_{J_{j_2}} \cdots v_{J_{j_{m-1}}} a_{(j_{m-1},j_m)} v_{J_{jm}} a_{(j_m,j_1)} v_{J_{j_1}}$ with $d^+_{T'}(v_{J_i}) = d^-_{T'}(v_{J_i}) \leq k-1$ for any $v_{J_i} \in V(T')$. For any $1 \leq i \leq k$, let $x, y \in V(J_i)$ be two distinct vertices. By symmetry, we may assume that $x, y \in V(J_1)$ with $J_i = J_1$. As $d^+_{T'}(v_{J_1}) = d^-_{T'}(v_{J_1}) \leq k-1$, let $d^+_{T'}(v_{J_1}) = d^-_{T'}(v_{J_1}) = t_1$, we assume that there are vertices $x_{1_1}, x_{1_2}, \ldots, x_{1_{t_1}}$, $y_{1_1}, y_{1_2}, \ldots, y_{1_{t_1}} \in V(J_1)$, and for each θ with $1 \leq \theta \leq t_1$, vertices $z_{1_\theta} \in V(J_{1_\theta})$ and $z_{1'_\theta} \in V(J_{1'_\theta})$ such that, as arcs in $A(D)$, $(z_{1_\theta}, x_{1_\theta}) = a_{(1_\theta,1)} \in \partial^-_{T'}(v_{J_1})$ and $(y_{1_\theta}, z_{1'_\theta}) = a_{(1,1'_\theta)} \in \partial^+_{T'}(v_{J_1})$ for $2 \leq \theta \leq t_1$, $(z_{1_1}, x_{1_1}) = a_{(j_m,j_1)} \in \partial^-_{T'}(v_{J_1})$ and $(y_{1_1}, z_{1'_1}) = a_{(j_1,j_2)} \in \partial^+_{T'}(v_{J_1})$. Since $x, y \in V(J_1)$, it follows by Lemma 3.2 and $\lambda(J_1) \geq k$ that J_1 contains an (x, y_{1_1})-trail $T_{(x,y_{1_1})}$, an (x_{1_1}, y)-trail $T_{(x_{1_1},y)}$ and for each θ with $2 \leq \theta \leq t_1$, there exists an $(x_{1_\theta}, y_{1_\theta})$-trail $T_{(x_{1_\theta},y_{1_\theta})}$ such that

$$T_{J_1} = T_{(x,y_{1_1})}(y_{1_1}, z_{1'_1}) \cup \bigcup_{\theta=2}^{t_1} (z_{1_\theta}, x_{1_\theta}) T_{(x_{1_\theta},y_{1_\theta})} (y_{1_\theta}, z_{1'_\theta}) \cup (z_{1_1}, x_{1_1}) T_{(x_{1_1},y)}$$

is a connected arc-disjoint union of $t_1 + 1$ trails in D with $V(J_1) \subseteq V(T_{J_1})$. By $\lambda(J_1) \geq k$, $\lambda(J_j) \geq k-1$ for $j \neq 1$ and (ii), $T = \bigcup_{i=1}^{k} T_{J_i}$ is a spanning (x, y)-trail of D. This proves (iv). □

Theorem 3.5. *Let D be a strong digraph. Each of the following holds.*
(i) If $k(D) = 1$, then D is strongly trail-connected, and so D is supereulerian.
(ii) If D' has a hamiltonian cycle, then D is supereulerian. Consequently, if $k(D) = 2$, then D is supereulerian.
(iii) If $k(D) = 3$, D' is spanned by a symmetric path $P_2 = v_{J_1} v_{J_2} v_{J_3}$ and $\lambda(J_2) \geq 2$, then D is supereulerian.

Proof. If $k(D) = 1$, then D is symmetrically connected digraph, and D has a spanning connected symmetric subdigraph J, by Lemma 3.2 with $k = 1$, J is strongly trail-connected, so D is strongly trail-connected and (i) follows. To show (ii), let C be a hamiltonian cycle of D' with $V(C) = \{v_{J_1}, v_{J_2}, \ldots, v_{J_k}\}$ and $A(C) = \{a_i = (v_{J_i}, v_{J_{i+1}}) : i \in \mathbb{Z}_k\}$. Let $J_1, J_2, \ldots, J_k$ be the preimage of $v_{J_1}, v_{J_2}, \ldots, v_{J_k}$, respectively. By definition, each J_i is a connected symmetric component of D, and for each $i \in \mathbb{Z}_k$, the arc $a_i \in A(D') \subseteq A(D)$. Therefore, there exist vertices $v'_i \in V(J_i)$ and $v''_{i+1} \in V(J_{i+1})$ with $a_i = (v'_i, v''_{i+1}) \in A(D)$. Since each J_i is a connected symmetric subdigraph of D, it follows by (i) that J_i has a spanning (v''_i, v'_i)-trail T_i. Let $A_1 = \{(v'_i, v''_{i+1}) : i \in \mathbb{Z}_k\}$. Then $H = D[A_1 \cup (\bigcup_{i \in \mathbb{Z}_k} A(T_i))]$ is a spanning closed trail of D, and so D is supereulerian. When $|V(D')| = 2$, as D is strong, D' is also strong, and so D' is hamiltonian, implying that D is supereulerian. Thus (ii) follows.

If $k(D) = 3$ and D' is spanned by a symmetric path $P_2 = v_{J_1} v_{J_2} v_{J_3}$, then there are vertices $z_1, z_2 \in V(J_1), x_1, x_2, y_1, y_2 \in V(J_2)$ and $w_1, w_2 \in V(J_3)$ such that $(z_1, x_1), (x_2, z_2), (y_1, w_1), (w_2, y_2) \in A(D)$, and since J_1 and J_3 are connected symmetric digraphs, by (i), we have a spanning (z_2, z_1)-trail $T_{(z_2,z_1)}$ of J_1 and a spanning (w_1, w_2)-trail $T_{(w_1,w_2)}$ of J_3. Since $\lambda(J_2) \geq 2$, by Lemma 3.2, J_2 has two arc-disjoint (x_1, y_1)-trial T_1 and (y_2, x_2)-trial T_2 such that $V(J_2) = V(T_1) \cup V(T_2)$. Let $T' = T_1 \cup T_2$. Thus,

$$T' \cup (y_1, w_1) T_{(w_1,w_2)} (w_2, y_2) \cup (x_2, z_2) T_{(z_2,z_1)} (z_1, x_1)$$

is a spanning closed trail of D. Thus, D is supereulerian. This proves (iii). □

Lemma 3.6. *Let D be a digraph. If D has a spanning trail, then for any $e \in A(D)$, D/e also has a spanning trail.*

Proof. Let $T = v_0 e_1 v_1 e_2 \ldots v_{m-1} e_m v_m$ be a spanning trail of D, and let $e \in A(D)$ be an arc. Then it is routine to verify that $(T \cup \{e\})/e$ is a spanning trail of G/e. □

Proof of Theorem 1.1. By Lemma 3.6, it suffices to prove the sufficiencies of each of the conclusions of Theorem 1.1. Throughout the proof arguments, let $k = k(D)$ and $\ell = \lambda_0(D)$. Assume first that $\ell \geq k-2$ and D' has a spanning trail.

By Lemma 3.3 (ii), we assume that D' has a spanning $(v_{J_{j_1}}, v_{J_{jm}})$-trail $T' = v_{J_{j_1}} a_{(j_1,j_2)} v_{J_{j_2}} \cdots v_{J_{j_{m-1}}} a_{(j_{m-1},j_m)} v_{J_{jm}}$ with $v_{J_{j_1}} \neq v_{J_{jm}}$ and

$$\begin{aligned} d^+_{T'}(v_{J_i}) = d^-_{T'}(v_{J_i}) \le k-2 \text{ for any } v_{J_i} \in V(T') - \{v_{J_{j_1}}, v_{J_{jm}}\}, \\ d^+_{T'}(v_{J_{j_1}}) = d^-_{T'}(v_{J_{j_1}}) + 1 \le k-1 \text{ and } d^-_{T'}(v_{J_{jm}}) = d^+_{T'}(v_{J_{jm}}) + 1 \le k-1. \end{aligned} \tag{10}$$

By symmetry, we assume that $v_{J_{j_1}} = v_{J_1}$ and $v_{J_{jm}} = v_{J_k}$. Define t_i as in (9). Then for each vertex $v_{J_i} \in V(D')$ with $J_i \in \{J_2, J_3, \ldots, J_{k-1}\}$, we have $d^+_{T'}(v_{J_i}) = d^-_{T'}(v_{J_i}) = t_i \le k-2$, and there exist vertices $x_{i_1}, x_{i_2}, \ldots, x_{i_{t_i}}, y_{i_1}, y_{i_2}, \ldots, y_{i_{t_i}} \in V(J_i)$, for each θ with $1 \le \theta \le t_i$, $z_{i_\theta} \in V(J_{i_\theta})$, $z_{i'_\theta} \in V(J_{i'_\theta})$ and $i_\theta, i'_\theta \in \{j_1, j_2, \ldots, j_m\}$ such that, as arcs in $A(D)$, $(z_{i_\theta}, x_{i_\theta}) = a_{(i_\theta,i)} \in \partial^-_{T'}(v_{J_i})$ and $(y_{i_\theta}, z_{i'_\theta}) = a_{(i,i'_\theta)} \in \partial^+_{T'}(v_{J_i})$. By Lemma 3.4 (i), for any $J_i \in \{J_2, J_3, \ldots, J_{k-1}\}$, the subdigraph T_{J_i} as defined in (7) is a connected arc-disjoint union of t_i trails in D with $V(J_i) \subseteq V(T_{J_i})$.

Let $y_1 \in V(J_1)$ and $z_1 \in V(J_{j_2})$ be vertices such that, as an arc in $A(D)$, $(y_1, z_1) = a_{(j_1,j_2)} \in \partial^+_{T'}(v_{J_1}) \cap \partial^-_{T'}(v_{J_{j_2}})$. By (10), $d^-_{T'}(v_{J_1}) = t_1 \le k-2$ and $d^+_{T'}(v_{J_1}) = t_1 + 1 \le k-1$. By Lemma 3.4 (i), there exist vertices $x_{1_1}, x_{1_2}, \ldots, x_{1_{t_1}}, y_{1_1}, y_{1_2}, \ldots, y_{1_{t_1}} \in V(J_1)$, and for each θ with $1 \le \theta \le t_1$, $z_{1_\theta} \in V(J_{1_\theta})$ and $z_{1'_\theta} \in V(J_{1'_\theta})$ for some $1_\theta, 1'_\theta \in \{j_1, j_2, \ldots, j_m\}$, such that, as arcs in $A(D)$, $(z_{1_\theta}, x_{1_\theta}) = a_{(1_\theta,1)} \in \partial^-_{T'}(v_{J_1})$ and $(y_{1_\theta}, z_{1'_\theta}) = a_{(1,1'_\theta)} \in \partial^+_{T'}(v_{J_1})$, where $a_{(1_\theta,1)}, a_{(1,1'_\theta)} \in \{a_{(j_1,j_2)}, a_{(j_2,j_3)}, \ldots, a_{(j_{m-1},j_m)}\}$. Then T_{J_1} as defined in (7) is a connected arc-disjoint union of t_1 trails in D with $V(J_1) \subseteq V(T_{J_1})$.

Similarly, let $z_k \in V(J_{j_{m-1}})$ and $x_k \in V(J_k)$ be vertices such that, as an arc in $A(D)$, $(z_k, x_k) = a_{(j_{m-1},j_m)} \in \partial^+_{T'}(v_{J_{j_{m-1}}}) \cap \partial^-_{T'}(v_{J_k})$. By (10), $d^+_{T'}(v_{J_k}) = t_k \le k-2$ and $d^-_{T'}(v_{J_k}) = t_k + 1 \le k-1$. By Lemma 3.4 (i), there exist vertices $x_{k_1}, x_{k_2}, \ldots, x_{k_{t_k}}, y_{k_1}, y_{k_2}, \ldots, y_{k_{t_k}} \in V(J_k)$, and for each θ with $1 \le \theta \le t_k$, vertices $z_{k_\theta} \in V(J_{k_\theta})$ and $z_{k'_\theta} \in V(J_{k'_\theta})$ for some $k_\theta, k'_\theta \in \{j_1, j_2, \ldots, j_m\}$, such that, as arcs in $A(D)$, $(z_{k_\theta}, x_{k_\theta}) = a_{(k_\theta,k)} \in \partial^-_{T'}(v_{J_{j_k}})$ and $(y_{k_\theta}, z_{k'_\theta}) = a_{(k,k'_\theta)} \in \partial^+_{T'}(v_{J_{j_k}})$, where $a_{(k_\theta,k)}, a_{(k,k'_\theta)} \in \{a_{(j_1,j_2)}, a_{(j_2,j_3)}, \ldots, a_{(j_{m-1},j_m)}\}$. Hence T_{J_k} as defined in (7) is a connected arc-disjoint union of t_k trails in D with $V(J_k) \subseteq V(T_{J_k})$. It follows that $T = (y_1, z_1) \bigcup_{i=1}^{k} T_{J_i}(z_k, x_k)$ is a spanning (y_1, x_k)-trail of D. This proves (i).

Next assume that $\ell \ge k-1$ and D' has a spanning closed trail, then by Lemma 3.3 (i), we assume that D' has a spanning closed trail $T' = v_{J_{j_1}} a_{(j_1,j_2)} v_{J_{j_2}} \cdots v_{J_{j_{m-1}}} a_{(j_{m-1},j_m)} v_{J_{jm}} a_{(j_m,j_1)} v_{J_{j_1}}$, with $d^+_{T'}(v_{J_i}) = d^-_{T'}(v_{J_i}) \le k-1$ for any $v_{J_i} \in V(T')$. By $\lambda(J_i) \ge \ell \ge k-1$ and by Lemma 3.4 (iii), D is supereulerian. This proves (ii).

To prove (iii), we assume that $\ell \ge k$ and D' is strongly trail-connected to show that D is strongly trail-connected. Let $x, y \in V(D)$ be given. We want to show that D has a spanning (x, y)-trail. Since D' is strongly trail-connected, D' has a spanning closed trail. By (ii) and $\ell \ge k$, D has a spanning closed trail. Thus we can assume that $x \neq y$. Furthermore, by Lemma 3.4 (iv), if for some $i \in \{1, 2, \ldots, k\}$, $x, y \in V(J_i)$, then D has a spanning (x, y)-trail. Hence we may assume that $x \in V(J_1)$ and $y \in V(J_k)$. Since D' is strongly trail-connected, D' has a spanning (v_{J_1}, v_{J_k})-trail, by Lemma 3.3 (ii), we assume that D' has a spanning (v_{J_1}, v_{J_k})-trail T' with $v_{J_1} \neq v_{J_k}$ and $d^+_{T'}(v_{J_i}) = d^-_{T'}(v_{J_i}) \le k-2$ for any $v_{J_i} \in V(T') - \{v_{J_1}, v_{J_k}\}$, $d^+_{T'}(v_{J_1}) = d^-_{T'}(v_{J_1}) + 1 \le k-1$ and $d^-_{T'}(v_{J_k}) = d^+_{T'}(v_{J_k}) + 1 \le k-1$. By Lemma 3.4 (ii), D has a spanning (x, y)-trail. Hence by the definition of strongly trail-connected digraphs, D is strongly trail-connected. This completes the proof of Theorem 1.1. $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The authors would like to thank the referees for their valuable suggestions that improve the presentation of the paper.

This research was supported by National Natural Science Foundation of China (Nos. 11761071, 11861068, 11771039, 11771443), the Youth Science and Technology Education Project of Xinjiang (No. 2017Q025) and the Natural Science Foundation of Xinjiang Uygur Autonomous Region (No. 2018D01A27).

References

[1] M. Alfegari, H.-J. Lai, Supereulerian digraphs with large arc-strong connectivity, J. Graph Theory 81 (4) (2016) 393–402.
[2] M.J. Algefari, K.A. Alsatami, H.-J. Lai, J. Liu, Supereulerian digraphs with given local structures, Inf. Process. Lett. 116 (2016) 321–326.
[3] M. Algefari, H.-J. Lai, J. Xu, Locally dense supereulerian digraphs, Discrete Appl. Math. 238 (2018) 24–31.
[4] K.A. Alsatami, X. Zhang, J. Liu, H.-J. Lai, On a class of supereulerian digraphs, Appl. Math. 7 (2016) 320–326.
[5] J. Bang-Jensen, G. Gutin, Digraphs: Theory, Algorithms and Applications, 2nd edition, Springer-Verlag, London, 2009.
[6] J. Bang-Jensen, A. Maddaloni, Sufficient conditions for a digraph to be supereulerian, J. Graph Theory 79 (2015) 8–20.
[7] J. Bang-Jensen, H. Déprés, A. Yeo, Spanning Eulerian subdigraphs avoiding k prescribed arcs in tournaments, Discrete Math. 343 (12) (2020) 112129.
[8] J. Bang-Jensen, F. Havet, A. Yeo, Spanning Eulerian subdigraphs in semicomplete digraphs, arXiv:1905.11019v1.
[9] F.T. Boesch, C. Suffel, R. Tindell, The spanning subgraphs of Eulerian graphs, J. Graph Theory 1 (1977) 79–84.
[10] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, New York, 2008.
[11] P.A. Catlin, Supereulerian graphs: a survey, J. Graph Theory 16 (1992) 177–196.

[12] Z.H. Chen, H.-J. Lai, Reduction techniques for super-Eulerian graphs and related topics-a survey, in: Combinatorics and Graph Theory '95, vol. 1, in: Hefei, World Sci. Publishing, River Edge, NJ, 1995, pp. 53–69.
[13] G. Gutin, Cycles and paths in directed graphs, PhD thesis, School of Mathematics, Tel Aviv University, 1993.
[14] G. Gutin, Connected $(g; f)$-factors and supereulerian digraphs, Ars Comb. 54 (2000) 311–317.
[15] Y. Hong, H.-J. Lai, Q. Liu, Supereulerian digraphs, Discrete Math. 330 (2014) 87–95.
[16] Y. Hong, Q. Liu, H.-J. Lai, Ore-type degree condition of supereulerian digraphs, Discrete Math. 339 (2016) 2042–2050.
[17] H.-J. Lai, Y. Shao, H. Yan, An update on supereulerian graphs, WSEAS Trans. Math. 12 (2013) 926–940.
[18] F. Liu, Z.-X. Tian, D.M. Li, Supereulerian locally semicomplete multipartite digraphs, Int. J. Math. Comb. 2 (2017) 123–128.
[19] W.R. Pulleyblank, A note on graphs spanned by Eulerian graphs, J. Graph Theory 3 (1979) 309–310.
[20] Y. Shiloach, Edge-disjoint branching in directed multigraphs, Inf. Process. Lett. 8 (1) (1979) 24–27.
[21] X.D. Zhang, J. Liu, L. Wang, H.-J. Lai, Supereulerian bipartite digraphs, J. Graph Theory 89 (2018) 64–75.