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Induced subgraphs of product graphs and a generalization of Huang's theorem

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Abstract

Recently, Huang showed that every $(2^{n-1}+1)$-vertex induced subgraph of the n-dimensional hypercube has maximum degree at least $\sqrt{n}$. In this paper, we discuss the induced subgraphs of Cartesian product graphs and semistrong product graphs to generalize Huang's result. Let Γ_1 be a connected signed bipartite graph of order n and Γ_2 be a connected signed graph of order m. By defining two kinds of signed product of Γ_1 and Γ_2, denoted by $\Gamma_1\widetilde{\square}\Gamma_2$ and $\Gamma_1\widetilde{\bowtie}\Gamma_2$, we show that if Γ_1 and Γ_2 have exactly two distinct adjacency eigenvalues $\pm\theta_1$ and $\pm\theta_2$, respectively, then every $(\frac{1}{2}mn+1)$-vertex induced subgraph of $\Gamma_1\widetilde{\square}\Gamma_2$ (resp., $\Gamma_1\widetilde{\bowtie}\Gamma_2$) has maximum degree at least $\sqrt{\theta_1^2+\theta_2^2}$ (resp., $\sqrt{\left(\theta_1^2+1\right)\theta_2^2}$). Moreover, we discuss the eigenvalues of $\Gamma_1\widetilde{\square}\Gamma_2$ and $\Gamma_1\widetilde{\bowtie}\Gamma_2$ and obtain a sufficient and necessary condition such that the spectrum of $\Gamma_1\widetilde{\square}\Gamma_2$ and $\Gamma_1\widetilde{\bowtie}\Gamma_2$ is symmetric with respect to 0, from which we obtain more general results on maximum degree of the induced subgraphs.

KEYWORDS

Cartesian product, eigenvalue, induced subgraph, semistrong product, signed graph

MATHEMATICAL SUBJECT CLASSIFICATION

05C22, 05C50, 05C76

1 | INTRODUCTION

Let Q_n be the n-dimensional hypercube, whose vertex set consists of vectors in $\{0, 1\}^n$, and two vectors are adjacent if they differ in exactly one coordinate. For a simple and undirected graph $G = (V, E)$, we use $\Delta(G)$ to denote the maximum degree of G. The adjacency matrix of G is defined to be a (0, 1)-matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise.

Recently, Huang [13] constructed a signed adjacency matrix of Q_n with exactly two distinct eigenvalues $\pm\sqrt{n}$. Using eigenvalue interlacing, Huang proceeded to prove that the spectral radius (and so, the maximum degree) of any $(2^{n-1} + 1)$-vertex induced subgraph of Q_n, is at least $\sqrt{n}$. Combining this with the combinatorial equivalent formulation discovered by Gotsman and Linial [10], Huang confirmed the Sensitivity Conjecture [17] from theoretical computer science. The main contribution of Huang is the following theorem.

Theorem 1.1 (Huang [13]). *For every integer $n \geq 1$, let H be an arbitrary $(2^{n-1} + 1)$-vertex induced subgraph of Q_n, then $\Delta(H) \geq \sqrt{n}$.*

The bound $\sqrt{n}$ (or more precisely, $\lceil\sqrt{n}\,\rceil$) is sharp, as shown by Chung, Füredi, Graham, and Seymour [5] in 1988. Tao [23] also gave a great expository of Huang's work on his blog after Huang announced the proof of the Sensitivity Conjecture.

Denote the Cartesian product of two graphs G and H by $G\square H$. It is known that the hypercube Q_n can be constructed iteratively by Cartesian product, that is, $Q_1 = K_2$ and for $n \geq 2$, $Q_n = Q_1\square Q_{n-1}$. Motivated by this fact, in this paper, we generalize Huang's theorem to Cartesian product graphs and semistrong product graphs. We introduce some necessary notations in the following.

A *signed graph* $\Gamma = (G, \sigma)$ is a graph $G = (V, E)$, together with a sign function $\sigma: E \to \{+1, -1\}$ assigning a positive or negative sign to each edge. An edge e is positive if $\sigma(e) = 1$ and negative if $\sigma(e) = -1$. The unsigned graph G is said to be the *underlying graph* of Γ, while σ is called the *signature* of G. If each edge of Γ is positive (resp., negative), then Γ is denoted by $\Gamma = (G, +)$ (resp., $\Gamma = (G, -)$). A signed graph is connected if its underlying graph is connected.

The adjacency matrix of $\Gamma = (G, \sigma)$ is denoted by $A(\Gamma) = (a_{ij}^{\sigma})$, where $a_{ij}^{\sigma} = \sigma(v_i v_j)$, if v_i and v_j are adjacent, and $a_{ij}^{\sigma} = 0$ otherwise. As G is simple and undirected, the adjacency matrix $A(\Gamma)$ is a symmetric $(-1, 0, +1)$-matrix, and $A(\Gamma) = A(G)$ if $\Gamma = (G, +)$, $A(\Gamma) = -A(G)$ if $\Gamma = (G, -)$. Let $\lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \cdots \geq \lambda_n(\Gamma)$ denote the eigenvalues of $A(\Gamma)$, which are all real since $A(\Gamma)$ is real and symmetric. If Γ contains at least one edge, then $\lambda_1(\Gamma) > 0 > \lambda_n(\Gamma)$ since the trace of $A(\Gamma)$ is 0. In general, the largest eigenvalue $\lambda_1(\Gamma)$ may not be equal to the spectral radius $\rho(\Gamma) = \max\{|\lambda_i(\Gamma)|: 1 \leq i \leq n\} = \max\{\lambda_1(\Gamma), -\lambda_n(\Gamma)\}$ because the Perron–Frobenius Theorem is valid only for nonnegative matrices. The eigenvalues of the adjacency matrix of signed graph Γ are called adjacency eigenvalues of Γ. The spectrum of $A(\Gamma)$ is called the (adjacency) spectrum of Γ and $A(\Gamma)$ is also called a signed adjacency matrix of G. The spectrum of Γ is *symmetric with respect to 0* if its adjacency eigenvalues are symmetric with respect to the origin. In this paper, all eigenvalues considered are adjacency eigenvalues.

For basic results in the theory of signed graphs, the reader is referred to Zaslavsky [24]. Recently, the spectra of signed graphs have attracted much attention, as found in [1,2,4,6,8,9,14,18,19,22,25], among others. In [2], the authors surveyed some general results on the adjacency spectra of signed graphs and proposed some spectral problems which are inspired by the spectral theory of unsigned graphs. In particular, the signed graphs with exactly two distinct eigenvalues have been greatly investigated in recent years, see [8,14,16,18,19,22]. In [14], Hou, Tang, and Wang characterized all simple connected signed graphs with maximum degree at most 4 and with just two distinct adjacency eigenvalues. In this paper, we construct signed graphs with exactly two distinct eigenvalues by two kinds of graph products, which generalizes Huang's result on the induced subgraph of the hypercube.

The *Kronecker product* $A \otimes B$ of matrices $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{p\times q}$ is the $mp \times nq$ matrix obtained from A by replacing each element a_{ij} with the block $a_{ij}B$. Therefore the entries of $A \otimes B$ consist of all the $mnpq$ possible products of an entry of A with an entry of B. For matrices A, B, C, and D, we have $(A \otimes B)\cdot(C \otimes D) = AC \otimes BD$ whenever the products AC and BD exist. Note that, $(A \otimes B)^T = A^T \otimes B^T$.

The *Cartesian product* of two graphs G_1 and G_2 is a graph, denoted by $G_1 \square G_2$, whose vertex set is $V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) being adjacent in $G_1 \square G_2$ if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$, or $u_1v_1 \in E(G_1)$ and $u_2 = v_2$. The *direct product* (or *Kronecker product*) of two graphs G_1 and G_2 is a graph, denoted by $G_1 \times G_2$, whose vertex set is $V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) being adjacent to each other in $G_1 \times G_2$ if and only if both $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$. The *semistrong product* (or *strong tensor product* [11]) of two graphs G_1 and G_2 is a graph, denoted by $G_1 \bowtie G_2$, whose vertex set is $V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) being adjacent to each other in $G_1 \bowtie G_2$ if and only if either $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$, or $u_1 = v_1$ and $u_2v_2 \in E(G_2)$. Then, by the definitions, the adjacency matrices of $G_1 \square G_2$, $G_1 \times G_2$, and $G_1 \bowtie G_2$ are $A(G_1 \square G_2) = A(G_1) \otimes I_m + I_n \otimes A(G_2)$, $A(G_1 \times G_2) = A(G_1) \otimes A(G_2)$, and $A(G_1 \bowtie G_2) = A(G_1) \otimes A(G_2) + I_n \otimes A(G_2)$, respectively, where $n = |V(G_1)|$, $m = |V(G_2)|$, and I_n is the identity matrix of order n. Unlike the Kronecker product, the semistrong product operation is neither commutative nor associative.

Let $\Gamma_1 = (G_1, \sigma_1)$ be a connected signed bipartite graph of order n with bipartition (V_1, V_2), where $|V_1| = s$ and $|V_2| = n - s$, and $\Gamma_2 = (G_2, \sigma_2)$ be a connected signed graph of order m. With suitable labeling of vertices, the adjacency matrix of Γ_1 can be represented as

$$A(\Gamma_1) = \begin{bmatrix} O_s & P \\ P^T & O_{n-s} \end{bmatrix}.$$

The *signed Cartesian product* of signed bipartite graph Γ_1 and signed graph Γ_2, denoted by $\Gamma_1 \widetilde{\square} \Gamma_2$, is the signed graph with adjacency matrix

$$A(\Gamma_1 \widetilde{\square} \Gamma_2) = A(\Gamma_1) \otimes I_m + \begin{bmatrix} I_s & O \\ O & -I_{n-s} \end{bmatrix} \otimes A(\Gamma_2) = \begin{bmatrix} I_s \otimes A(\Gamma_2) & P \otimes I_m \\ P^T \otimes I_m & -I_{n-s} \otimes A(\Gamma_2) \end{bmatrix}. \tag{1}$$

The *signed semistrong product* of signed bipartite graph Γ_1 and signed graph Γ_2, denoted by $\Gamma_1 \widetilde{\bowtie} \Gamma_2$, is the signed graph with adjacency matrix

$$A(\Gamma_1 \widetilde{\bowtie} \Gamma_2) = A(\Gamma_1) \otimes A(\Gamma_2) + \begin{bmatrix} I_s & O \\ O & -I_{n-s} \end{bmatrix} \otimes A(\Gamma_2) = \begin{bmatrix} I_s & P \\ P^T & -I_{n-s} \end{bmatrix} \otimes A(\Gamma_2). \tag{2}$$

A (signed) bipartite graph with bipartition (V_1, V_2) is called *balanced* if $|V_1| = |V_2|$. As a generalization of Theorem 1.1, in this paper we obtain the following theorem.

Theorem 1.2. *Let $\Gamma_1 = (G_1, \sigma_1)$ be a signed bipartite graph of order n and $\Gamma_2 = (G_2, \sigma_2)$ be a signed graph of order m, and let λ^2 and μ^2 be the minimum eigenvalues of $A(\Gamma_1)^2$ and $A(\Gamma_2)^2$, respectively. Let H and H' be any $(\lfloor \frac{mn}{2} \rfloor + 1)$-vertex induced subgraph of $\Gamma_1 \widetilde{\Box} \Gamma_2$ and $\Gamma_1 \widetilde{\bowtie} \Gamma_2$, respectively. If Γ_1 is a balanced bipartite graph or the spectrum of Γ_2 is symmetric with respect to 0, then*

$$\Delta(H) \geq \sqrt{\lambda^2 + \mu^2}, \quad \Delta(H') \geq \sqrt{(\lambda^2 + 1)\mu^2}.$$

In particular, if both Γ_1 and Γ_2 have exactly two distinct eigenvalues, we have the following corollary.

Corollary 1.3. *Let $\Gamma_1 = (G_1, \sigma_1)$ be a signed bipartite graph of order n with exactly two distinct eigenvalues $\pm\theta_1$ and $\Gamma_2 = (G_2, \sigma_2)$ be a signed graph of order m with exactly two distinct eigenvalues $\pm\theta_2$. If H and H' are arbitrary $(\frac{mn}{2} + 1)$-vertex induced subgraphs of $\Gamma_1 \widetilde{\Box} \Gamma_2$ and $\Gamma_1 \widetilde{\bowtie} \Gamma_2$, respectively, then*

$$\Delta(H) \geq \sqrt{\theta_1^2 + \theta_2^2}, \quad \Delta(H') \geq \sqrt{\left(\theta_1^2 + 1\right)\theta_2^2}.$$

A direct proof of Corollary 1.3 is presented in Section 2. From the proof we will see that Γ_1 and Γ_2 in Corollary 1.3 are regular. In Section 3, we display some preliminaries and examples. In Section 4, we give a characterization of the eigenvalues of $\Gamma_1 \widetilde{\Box} \Gamma_2$ and $\Gamma_1 \widetilde{\bowtie} \Gamma_2$ and obtain a sufficient and necessary condition such that the spectrum of $\Gamma_1 \widetilde{\Box} \Gamma_2$ and $\Gamma_1 \widetilde{\bowtie} \Gamma_2$ are symmetric with respect to 0. In Section 5, we present the proof of Theorem 1.2 and generalize the signed Cartesian product and signed semistrong product of two signed graphs to the products of n signed graphs. In Section 6, we give some concluding remarks.

2 | A DIRECT PROOF OF COROLLARY 1.3

Using the idea that Shalev Ben-David contributed on July 3, 2019 to Scott Aaronson's blog, Knuth [15] gave a direct and nice proof of Huang's theorem in one page. Here, arising from their ideas, we give a direct proof of Corollary 1.3.

Proof of Corollary 1.3. For simplicity, let $A_1 := A(\Gamma_1)$ and $A_2 := A(\Gamma_2)$. Since Γ_i has exactly two distinct eigenvalues $\pm\theta_i$ ($\neq 0$) for $i = 1, 2$, we have each eigenvalue of A_i^2 equals to θ_i^2 and so there exist orthogonal matrices Q_1 and Q_2 such that $A_1^2 = Q_1(\theta_1^2 I_n)Q_1^T = \theta_1^2 I_n$ and $A_2^2 = Q_2(\theta_2^2 I_m)Q_2^T = \theta_2^2 I_m$. The diagonal entries of A_i^2 are the degrees of vertices in Γ_i, so Γ_i is a θ_i^2-regular graph for $i = 1, 2$. Moreover, $|V_1| = s = \frac{n}{2}$ and $PP^T = P^T P = \theta_1^2 I_{n/2}$.

(a) Let $\mathcal{A} := A(\Gamma_1 \widetilde{\Box} \Gamma_2)$ and define

$$\mathcal{B} = \begin{bmatrix} P \otimes \left(A_2 + \sqrt{\theta_1^2 + \theta_2^2} I_m\right) \\ \theta_1^2 I_{n/2} \otimes I_m \end{bmatrix}$$

to be an $mn \times \frac{mn}{2}$ matrix. Since $\theta_1 \neq 0$, the rank of $\mathcal{B}$ is $\frac{mn}{2}$, and we have

$$\mathcal{A}\cdot\mathcal{B}=\begin{bmatrix} I_{n/2}\otimes A_2 & P\otimes I_m \\ P^T\otimes I_m & -I_{n/2}\otimes A_2\end{bmatrix}\cdot\begin{bmatrix} P\otimes\left(A_2+\sqrt{\theta_1^2+\theta_2^2}I_m\right) \\ \theta_1^2 I_{n/2}\otimes I_m\end{bmatrix}$$

$$=\begin{bmatrix} P\otimes\left(A_2^2+\sqrt{\theta_1^2+\theta_2^2}A_2+\theta_1^2 I_m\right) \\ P^TP\otimes\left(A_2+\sqrt{\theta_1^2+\theta_2^2}I_m\right)-\theta_1^2 I_{n/2}\otimes A_2\end{bmatrix}$$

$$=\begin{bmatrix} P\otimes\left(\theta_2^2 I_m+\sqrt{\theta_1^2+\theta_2^2}A_2+\theta_1^2 I_m\right) \\ \theta_1^2 I_{n/2}\otimes\left(A_2+\sqrt{\theta_1^2+\theta_2^2}I_m\right)-\theta_1^2 I_{n/2}\otimes A_2\end{bmatrix}$$

$$=\sqrt{\theta_1^2+\theta_2^2}\begin{bmatrix} P\otimes\left(A_2+\sqrt{\theta_1^2+\theta_2^2}I_m\right) \\ \theta_1^2 I_{n/2}\otimes I_m\end{bmatrix}=\sqrt{\theta_1^2+\theta_2^2}\mathcal{B}.$$

Let H be an arbitrary $(\frac{mn}{2}+1)$-vertex induced subgraph of $\Gamma_1\widetilde{\square}\Gamma_2$. Suppose $\mathcal{B}_0$ is the $(\frac{mn}{2}-1)\times\frac{mn}{2}$ submatrix of $\mathcal{B}$ whose rows corresponding to vertices not in H. Then there exists a unit $\frac{mn}{2}\times 1$ vector x such that $\mathcal{B}_0 x=0$, since $\mathcal{B}_0 x=0$ is a homogeneous system of $\frac{mn}{2}-1$ linear equations with $\frac{mn}{2}$ variables. As $\operatorname{rank}(\mathcal{B})=\frac{mn}{2}$, $y=\mathcal{B}x$ is an $mn\times 1$ nonzero vector such that $y_v=0$ for any vertex $v\notin H$, and $\mathcal{A}y=\sqrt{\theta_1^2+\theta_2^2}y$.

Let u be a vertex such that $|y_u|=\max\{|y_1|,\ldots,|y_{mn}|\}$. Then $|y_u|>0$, $u\in V(H)$ and

$$\sqrt{\theta_1^2+\theta_2^2}|y_u|=|(\mathcal{A}y)_u|=\left|\sum_{v=1}^{mn}\mathcal{A}_{uv}y_v\right|=\left|\sum_{v\in H}\mathcal{A}_{uv}y_v\right|\le\sum_{v\in H}|\mathcal{A}_{uv}||y_u|\le\Delta(H)|y_u|.$$

Therefore, $\Delta(H)\ge\sqrt{\theta_1^2+\theta_2^2}$.

(b) Let $\mathcal{A}:=A(\Gamma_1\widetilde{\bowtie}\Gamma_2)$ and define

$$\mathcal{B}=\begin{bmatrix} P\otimes\left(\sqrt{\theta_1^2+1}A_2+\theta_2 I_m\right) \\ \theta_1^2\theta_2 I_{n/2}\otimes I_m\end{bmatrix}$$

to be an $mn\times\frac{mn}{2}$ matrix. Since $\theta_1\neq 0$ and $\theta_2\neq 0$, the rank of $\mathcal{B}$ is $\frac{mn}{2}$, and we have

$$\mathcal{A}\cdot\mathcal{B}=\begin{bmatrix} I_{n/2}\otimes A_2 & P\otimes A_2 \\ P^T\otimes A_2 & -I_{n/2}\otimes A_2\end{bmatrix}\cdot\begin{bmatrix} P\otimes\left(\sqrt{\theta_1^2+1}A_2+\theta_2 I_m\right) \\ \theta_1^2\theta_2 I_{n/2}\otimes I_m\end{bmatrix}$$

$$=\begin{bmatrix} P\otimes\left(\sqrt{\theta_1^2+1}A_2^2+\theta_2 A_2+\theta_1^2\theta_2 A_2\right) \\ P^TP\otimes\left(\sqrt{\theta_1^2+1}A_2^2+\theta_2 A_2\right)-\theta_1^2\theta_2 I_{n/2}\otimes A_2\end{bmatrix}$$

$$=\begin{bmatrix} P\otimes\left(\sqrt{\theta_1^2+1}\theta_2^2 I_m+\theta_2\left(\theta_1^2+1\right)A_2\right) \\ \theta_1^2 I_{n/2}\otimes\left(\sqrt{\theta_1^2+1}\theta_2^2 I_m+\theta_2 A_2\right)-\theta_1^2\theta_2 I_{n/2}\otimes A_2\end{bmatrix}$$

$$=\sqrt{\left(\theta_1^2+1\right)\theta_2^2}\begin{bmatrix} P\otimes\left(\sqrt{\theta_1^2+1}A_2+\theta_2 I_m\right) \\ \theta_1^2\theta_2 I_{n/2}\otimes I_m\end{bmatrix}=\sqrt{\left(\theta_1^2+1\right)\theta_2^2}\mathcal{B}.$$

Let H' be an arbitrary $(\frac{mn}{2}+1)$-vertex induced subgraph of $\Gamma_1\widetilde{\bowtie}\Gamma_2$. Suppose $\mathcal{B}_0$ is the $(\frac{mn}{2}-1)\times\frac{mn}{2}$ submatrix of $\mathcal{B}$ whose rows corresponding to vertices not in H'. Then there exists a unit $\frac{mn}{2}\times 1$ vector x such that $\mathcal{B}_0x=0$, since $\mathcal{B}_0x=0$ is a homogeneous system of $\frac{mn}{2}-1$ linear equations with $\frac{mn}{2}$ variables. As $\mathrm{rank}(\mathcal{B})=\frac{mn}{2}$, $y=\mathcal{B}x$ is an $mn\times 1$ nonzero vector such that $y_v=0$ for any vertex $v\notin H'$, and $\mathcal{A}y=\sqrt{\left(\theta_1^2+1\right)\theta_2^2}\,y$.

Let u be a vertex such that $|y_u|=\max\{|y_1|,\ldots,|y_{mn}|\}$. Then $|y_u|>0$, $u\in V(H')$ and

$$\sqrt{\left(\theta_1^2+1\right)\theta_2^2}\,|y_u|=|(\mathcal{A}y)_u|=\left|\sum_{v=1}^{mn}\mathcal{A}_{uv}y_v\right|=\left|\sum_{v\in H'}\mathcal{A}_{uv}y_v\right|\le\sum_{v\in H'}|\mathcal{A}_{uv}||y_u|\le\Delta(H')|y_u|.$$

Therefore, $\Delta(H')\ge\sqrt{\left(\theta_1^2+1\right)\theta_2^2}$. □

3 | PRELIMINARIES

In this section, we present some useful lemmas and examples.

Lemma 3.1 (Hammack, Imrich, and Klavžar [12]). *Let G_1 and G_2 be nontrivial graphs. Then*

(i) $G_1\square G_2$ is connected if and only if G_1 and G_2 are connected, and $G_1\square G_2$ is bipartite if and only if G_1 and G_2 are bipartite.
(ii) $G_1\times G_2$ is connected if and only if G_1 and G_2 are connected and at most one of them is bipartite, and $G_1\times G_2$ is bipartite if and only if at least one of G_1 and G_2 is bipartite.

Lemma 3.2 (Garman, Ringeisen, and White [11]). *Let G_1 and G_2 be nontrivial graphs. Then*

(i) *$G_1\bowtie G_2$ is connected if and only if G_1 and G_2 are connected.*
(ii) *$G_1\bowtie G_2$ is bipartite if and only if G_2 is bipartite.*
(iii) *The semistrong product operation is neither associative nor commutative.*
(iv) *If G_1 is bipartite, then $G_1\bowtie K_2\cong G_1\square K_2$.*

By Lemma 3.2 (iv), the following corollary can be obtained easily.

Corollary 3.3 (Garman, Ringeisen, and White [11]).

(i) *Let $G_1=K_2$, and for $n\ge 2$, $G_n=G_{n-1}\bowtie K_2$, then $G_n\cong Q_n$.*
(ii) *Let $G'_1=K_2$, and for $n\ge 2$, $G'_n=K_2\bowtie G'_{n-1}$, then $G'_n\cong K_{2^{n-1},2^{n-1}}$.*

Proof. By Lemma 3.2(iv), $Q_{n-1}\bowtie K_2\cong Q_{n-1}\square K_2=Q_n$. By induction, $G_n\cong Q_n$. Let $V(K_2)=\{u,v\}$ and (V_1,V_2) be the bipartition of $K_{2^{n-2},2^{n-2}}$. Then there is an edge connecting any two vertices between $\{u,v\}\times V_1$ and $\{u,v\}\times V_2$ in $K_2\bowtie K_{2^{n-2},2^{n-2}}$. Hence, $K_2\bowtie K_{2^{n-2},2^{n-2}}=K_{2^{n-1},2^{n-1}}$. By induction, $G'_n\cong K_{2^{n-1},2^{n-1}}$. □

By the definitions of Cartesian product, direct product, and semistrong product of graphs, we can define the product of signed graphs Γ_1 and Γ_2 by their adjacency matrices. That is, $A(\Gamma_1 \square \Gamma_2) = A(\Gamma_1) \otimes I_m + I_n \otimes A(\Gamma_2)$, where $n = |V(\Gamma_1)|$ and $m = |V(\Gamma_2)|$, $A(\Gamma_1 \times \Gamma_2) = A(\Gamma_1) \otimes A(\Gamma_2)$, and $A(\Gamma_1 \bowtie \Gamma_2) = (A(\Gamma_1) + I_n) \otimes A(\Gamma_2)$. If X and Y are eigenvectors of $A_1 = A(\Gamma_1)$ and $A_2 = A(\Gamma_2)$ corresponding to eigenvalues λ and μ, respectively, then direct computation yields the following.

$$\begin{aligned} A(\Gamma_1 \square \Gamma_2)(X \otimes Y) &= (A_1 \otimes I_m + I_n \otimes A_2)(X \otimes Y) = (\lambda + \mu) X \otimes Y, \\ A(\Gamma_1 \times \Gamma_2)(X \otimes Y) &= (A_1 \otimes A_2)(X \otimes Y) = A_1 X \otimes A_2 Y = \lambda\mu X \otimes Y, \\ A(\Gamma_1 \bowtie \Gamma_2)(X \otimes Y) &= [(A_1 + I_n) \otimes A_2](X \otimes Y) = (A_1 + I_n) X \otimes A_2 Y \\ &= (\lambda + 1)\mu X \otimes Y. \end{aligned}$$

Thus, we can obtain the following theorem.

Theorem 3.4. *If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ are the adjacency eigenvalues of the signed graphs Γ_1 and Γ_2, respectively, then, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$,*

(i) *(Germina, Hameed K, and Zaslavsky* [9]*) $\lambda_i + \mu_j$ are the adjacency eigenvalues of $\Gamma_1 \square \Gamma_2$;*
(ii) *(Germina, Hameed K, and Zaslavsky* [9]*) $\lambda_i \mu_j$ are the adjacency eigenvalues of $\Gamma_1 \times \Gamma_2$;*
(iii) *$(\lambda_i + 1)\mu_j$ are the adjacency eigenvalues of $\Gamma_1 \bowtie \Gamma_2$.*

By Lemma 3.1 (ii) and Theorem 3.4 (ii), we have the following result immediately.

Corollary 3.5. *For $i = 1, 2$, let $\Gamma_i = (G_i, \sigma_i)$ be a connected signed graph with exactly two distinct eigenvalues $\pm\theta_i$, respectively. If at least one of G_1 and G_2 is non-bipartite, then $\Gamma_1 \times \Gamma_2$ is a connected signed graph with exactly two distinct eigenvalues $\pm\theta_1\theta_2$.*

In the following, we introduce some known results and examples which can be used to construct signed graphs with exactly two distinct eigenvalues. First we give some definitions. A *weighing matrix* of order n and weight k is an $n \times n$ matrix $W = W(n, k)$ with entries $0, +1$, and -1 such that $WW^T = W^TW = kI_n$. A weighing matrix $W(n, n)$ is a Hadamard matrix H_n of order n. A *conference matrix* C of order n is an $n \times n$ matrix with 0's on the diagonal, $+1$ or -1 in all other positions and with the property $CC^T = (n-1)I_n$. Thus, a conference matrix of order n is a weighing matrix of order n and weight $n - 1$, and a permutation matrix of order n is a weighing matrix of order n and weight 1.

Lemma 3.6. *For $n \geq 1$, let*

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad H_{2^{n+1}} = H_2 \otimes H_{2^n}, \quad A_n = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes H_{2^n}.$$

Then A_n is a signed adjacency matrix of $K_{2^n,2^n}$ and its eigenvalues are $\pm\sqrt{2^n}$, each with multiplicity 2^n.

Proof. Since H_{2^n} is a symmetric matrix with entries ± 1, A_n is a signed adjacency matrix of $K_{2^n,2^n}$. Note that H_{2^n} is a Hadamard matrix of order 2^n with eigenvalues $\pm\sqrt{2^n}$. By the property of Kronecker product, the eigenvalues of A_n are $\pm\sqrt{2^n}$, each with multiplicity 2^n. □

Lemma 3.7 (McKee and Smyth [16]). *Let P be a permutation matrix of order n such that $P + P^T$ is the adjacency matrix of the cycle C_n and*

$$A_n = \begin{bmatrix} P + P^T & P - P^T \\ P^T - P & -(P + P^T) \end{bmatrix}.$$

Then A_n is the adjacency matrix of the $2n$-vertex toroidal tessellation T_{2n} (see Figure 1), whose eigenvalues are ± 2, each with multiplicity n.

Lemma 3.8 (McKee and Smyth [16]). *Let $W(7, 4) = (w_{ij})$ be the weighing matrix of order 7 and weight 4, where $w_{ij} = w_{1,\ell}$ for $\ell \equiv j - i + 1 \pmod 7$ and $(w_{11}, w_{12}, w_{13}, w_{14}, w_{15}, w_{16}, w_{17}) = (-1, 1, 1, 0, 1, 0, 0)$. Let*

$$W(14, 4) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes W(7, 4).$$

Then $W(14, 4)$ is the adjacency matrix of the 14-vertex signed graph S_{14} (see Figure 1) and its eigenvalues are ± 2 with the same multiplicity 7.

Example 3.9 (Stinson [21]). For each $n \in \{2, 6, 10, 14, 18, 26, 30\}$, there exists a symmetric conference matrices $W(n, n-1)$. Then, $W(n, n-1)$ is a signed adjacency matrix of K_n and its eigenvalues are $\pm\sqrt{n-1}$, each with multiplicity $n/2$.

By the property of Kronecker product of matrices, we have the following examples.

Example 3.10. Let $W(k, k-1)$ be a symmetric conference matrix of order k and H_n be a symmetric Hadamard matrix of order n. Then $W(k, k-1) \otimes H_n$ is a signed adjacency matrix of the complete k-partite graph $K_{n,n,\ldots,n}$ and its eigenvalues are $\pm\sqrt{(k-1)n}$ with the same multiplicity. In particular, $W(6, 5) \otimes H_2$ is a signed adjacency matrix of the complete 6-partite graph $K_{2,2,2,2,2,2}$.

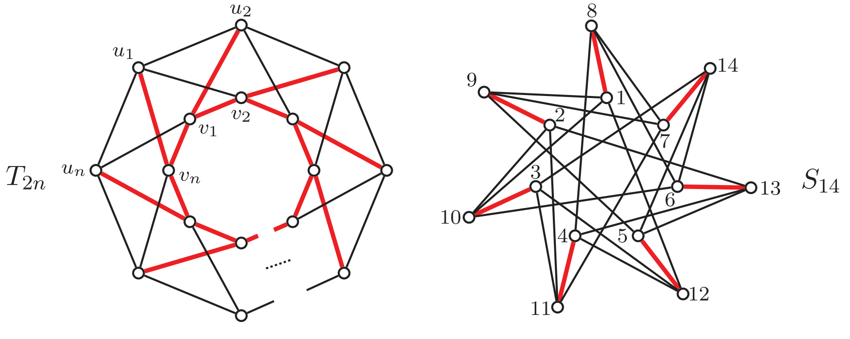

FIGURE 1 The graphs T_{2n} in Lemma 3.7 and S_{14} in Lemma 3.8, where red edges represent negative edges [Color figure can be viewed at wileyonlinelibrary.com]

Example 3.11. Let Γ be a signed graph of order m and H_n be a symmetric Hadamard matrix of order n. Then $H_n \otimes A(\Gamma)$ is an adjacency matrix of the signed graph $\Gamma^{(n)}$ of order mn obtained from Γ. If Γ has exactly two distinct eigenvalues $\pm\theta$, then $\Gamma^{(n)}$ has exactly two distinct eigenvalues $\pm\theta\sqrt{n}$.

4 | EIGENVALUES OF SIGNED CARTESIAN PRODUCT AND SIGNED SEMISTRONG PRODUCT GRAPHS

In this section, we discuss the adjacency eigenvalues of $\Gamma_1 \widetilde{\square} \Gamma_2$ and $\Gamma_1 \widetilde{\bowtie} \Gamma_2$ and obtain a sufficient and necessary condition such that the spectrums of $\Gamma_1 \widetilde{\square} \Gamma_2$ and $\Gamma_1 \widetilde{\bowtie} \Gamma_2$ are symmetric with respect to 0.

Theorem 4.1. *Let $\Gamma_1 = (G_1, \sigma_1)$ be a signed bipartite graph of order n with bipartition (V_1, V_2) and $\Gamma_2 = (G_2, \sigma_2)$ be a signed graph of order m. If λ^2 is an eigenvalue of $A(\Gamma_1)^2$ with multiplicity p and μ^2 is an eigenvalue of $A(\Gamma_2)^2$ with multiplicity q, then each of the following holds.*

(i) *$\lambda^2 + \mu^2$ (resp., $(\lambda^2 + 1)\mu^2$) is an eigenvalue of $A(\Gamma_1 \widetilde{\square} \Gamma_2)^2$ (resp., $A(\Gamma_1 \widetilde{\bowtie} \Gamma_2)^2$) with multiplicity pq.*

(ii) *If $\lambda = 0$ and $\mu \neq 0$, then $\pm\mu$ are eigenvalues of $\Gamma_1 \widetilde{\square} \Gamma_2$ (also $\Gamma_1 \widetilde{\bowtie} \Gamma_2$) with multiplicities $\frac{1}{2}pq \pm \frac{1}{2}(n - 2|V_1|)(q - 2t)$, respectively, where t is the multiplicity of eigenvalue μ of $A(\Gamma_2)$.*

(iii) *If $\lambda \neq 0$, then $\pm\sqrt{\lambda^2 + \mu^2}$ are eigenvalues of $\Gamma_1 \widetilde{\square} \Gamma_2$, each with multiplicity $pq/2$.*

(iv) *If $\lambda\mu \neq 0$, then $\pm\sqrt{(\lambda^2 + 1)\mu^2}$ are eigenvalues of $\Gamma_1 \widetilde{\bowtie} \Gamma_2$, each with multiplicity $pq/2$.*

Corollary 4.2. *For $i = 1, 2$, let Γ_i be a signed graph with exactly two distinct eigenvalues $\pm\theta_i$, where Γ_1 is bipartite. Then $\Gamma_1 \widetilde{\square} \Gamma_2$ and $\Gamma_1 \widetilde{\bowtie} \Gamma_2$ have exactly two distinct eigenvalues $\pm\sqrt{\theta_1^2 + \theta_2^2}$ and $\pm\sqrt{\left(\theta_1^2 + 1\right)\theta_2^2}$, respectively.*

The following theorem gives a sufficient and necessary condition such that the spectrums of $\Gamma_1 \widetilde{\square} \Gamma_2$ and $\Gamma_1 \widetilde{\bowtie} \Gamma_2$ are symmetric with respect to 0.

Theorem 4.3. *Let Γ_1 be a signed bipartite graph and Γ_2 be a signed graph. The spectrum of $\Gamma_1 \widetilde{\square} \Gamma_2$ (resp., $\Gamma_1 \widetilde{\bowtie} \Gamma_2$) is symmetric with respect to 0 if and only if Γ_1 is balanced or the spectrum of Γ_2 is symmetric with respect to 0.*

In the following proofs of Theorems 4.1 and 4.3, we always assume that $A_1 := A(\Gamma_1) = \begin{bmatrix} O_s & P \\ P^T & O_{n-s} \end{bmatrix}$ and $A_2 := A(\Gamma_2)$, where $|V_1| = s$ and P is an $s \times (n - s)$ matrix.

Proof of Theorem 4.1 (i). By (1), we have

$$
\begin{aligned}
A(\Gamma_1\widetilde{\Box}\Gamma_2)^2 &= \left(A_1 \otimes I_m + \begin{bmatrix} I_s & O \\ O & -I_{n-s} \end{bmatrix} \otimes A_2\right)^2 \\
&= A_1^2 \otimes I_m + \begin{bmatrix} I_s & O \\ O & I_{n-s} \end{bmatrix} \otimes A_2^2 \\
&\quad + \left(\begin{bmatrix} O & P \\ P^T & O \end{bmatrix} \otimes I_m\right)\left(\begin{bmatrix} I_s & O \\ O & -I_{n-s} \end{bmatrix} \otimes A_2\right) \\
&\quad + \left(\begin{bmatrix} I_s & O \\ O & -I_{n-s} \end{bmatrix} \otimes A_2\right)\left(\begin{bmatrix} O & P \\ P^T & O \end{bmatrix} \otimes I_m\right) \\
&= A_1^2 \otimes I_m + I_n \otimes A_2^2 + \begin{bmatrix} O & -P \\ P^T & O \end{bmatrix} \otimes A_2 + \begin{bmatrix} O & P \\ -P^T & O \end{bmatrix} \otimes A_2 \\
&= A_1^2 \otimes I_m + I_n \otimes A_2^2.
\end{aligned} \tag{3}
$$

For each $i = 1, \ldots, p$ and $j = 1, \ldots, q$, let X_i and Y_j be eigenvectors of A_1^2 and A_2^2 with respect to eigenvalues λ^2 and μ^2, respectively. Thus, by (3), we have

$$A(\Gamma_1\widetilde{\Box}\Gamma_2)^2(X_i \otimes Y_j) = \left(A_1^2 \otimes I_m + I_n \otimes A_2^2\right)(X_i \otimes Y_j) = (\lambda^2 + \mu^2)(X_i \otimes Y_j).$$

Therefore, $\lambda^2 + \mu^2$ is an eigenvalue of $A(\Gamma_1\widetilde{\Box}\Gamma_2)^2$ with multiplicity pq.

By (2), we have

$$
\begin{aligned}
A(\Gamma_1\widetilde{\bowtie}\Gamma_2)^2 &= \left(\begin{bmatrix} I_s & P \\ P^T & -I_{n-s} \end{bmatrix} \otimes A_2\right)^2 \\
&= \left(\begin{bmatrix} O_s & P \\ P^T & O_{n-s} \end{bmatrix} + \begin{bmatrix} I_s & O \\ O & -I_{n-s} \end{bmatrix}\right)^2 \otimes A_2^2 \\
&= \left(A_1^2 + I_n\right) \otimes A_2^2 + \begin{bmatrix} O_s & -P + P \\ P^T - P^T & O_{n-s} \end{bmatrix} \otimes A_2^2 \\
&= \left(A_1^2 + I_n\right) \otimes A_2^2.
\end{aligned} \tag{4}
$$

For each $i = 1, \ldots, p$ and $j = 1, \ldots, q$, let X_i and Y_j be eigenvectors of A_1^2 and A_2^2 with respect to eigenvalues λ^2 and μ^2, respectively. Thus, by (4), we have

$$A(\Gamma_1\widetilde{\bowtie}\Gamma_2)^2(X_i \otimes Y_j) = \left[\left(A_1^2 + I_n\right) \otimes A_2^2\right](X_i \otimes Y_j) = (\lambda^2 + 1)\mu^2(X_i \otimes Y_j).$$

Therefore, $(\lambda^2 + 1)\mu^2$ is an eigenvalue of $A(\Gamma_1\widetilde{\bowtie}\Gamma_2)^2$ with multiplicity pq. □

Lemma 4.4. *Let Γ be a signed bipartite graph of order n with bipartition (V_1, V_2), where $|V_1| = s$, and $A = \begin{bmatrix} O_s & P \\ P^T & O_{n-s} \end{bmatrix}$ be the adjacency matrix of Γ. Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_a\}$ be a basis of null space of P^T and $\{\mathbf{z}_1, \ldots, \mathbf{z}_b\}$ be a basis of null space of P. The following $a + b$ vectors of length n*

$$\left\{\begin{bmatrix}\mathbf{w}_1\\ \mathbf{0}\end{bmatrix}, \ldots, \begin{bmatrix}\mathbf{w}_a\\ \mathbf{0}\end{bmatrix}, \begin{bmatrix}\mathbf{0}\\ \mathbf{z}_1\end{bmatrix}, \ldots, \begin{bmatrix}\mathbf{0}\\ \mathbf{z}_b\end{bmatrix}\right\},$$

is a basis of null space of A.

Proof. Since $\text{rank}(A) = \text{rank}(P) + \text{rank}(P^T)$, by Rank–Nullity Theorem

$$n - \text{rank}(A) = (n - s - \text{rank}(P)) + (s - \text{rank}(P^T)) = a + b.$$

The result follows. □

Proof of Theorem 4.1 (ii). Since the multiplicity of eigenvalue μ of A_2 is t, the multiplicity of eigenvalue $-\mu$ of A_2 is $q - t$. Assume that $A_2 Y_j = \mu Y_j$ for each $1 \le j \le t$ and $A_2 Y_k' = -\mu Y_k'$ for each $1 \le k \le q - t$. In particular, if $t = 0$, then $1 \le k \le q$ and there exists no such Y_j; if $t = q$, then $1 \le j \le q$ and there exists no such Y_k'.

By the assumption, $\lambda = 0$ is an eigenvalue of A_1^2 (and so A_1) with multiplicity p. Hence, the rank of A_1 is $\text{rank}(A_1) = n - p$ and $\text{rank}(P) = \text{rank}(P^T) = (n - p)/2$. Thus, the nullity of P^T is

$$r := s - \text{rank}(P^T) = p/2 - (n - 2s)/2$$

and the nullity of P is $p - r = p/2 + (n - 2s)/2$. Suppose that $\{X_{11}, \ldots, X_{r1}\}$ is a basis of null space of P^T and $\{X_{12}, \ldots, X_{(p-r)2}\}$ is a basis of null space of P. Let

$$Z_i := \begin{bmatrix} X_{i1}\\ \mathbf{0}\end{bmatrix}, Z_\ell' := \begin{bmatrix}\mathbf{0}\\ X_{\ell 2}\end{bmatrix}$$

be column vectors of length n for each $1 \le i \le r$ and $1 \le \ell \le p - r$. In particular, if $r = 0$, then $1 \le \ell \le p = n - 2s$ and there is no such Z_i; if $r = p$, then $1 \le i \le p = 2s - n$ and there is no such Z_ℓ'. By Lemma 4.4, $\{Z_1, \ldots, Z_r\} \cup \{Z_1', \ldots, Z_{p-r}'\}$ is a basis of null space of A_1. Therefore, $A_1 Z_i = A_1 Z_\ell' = \mathbf{0}$ and for every $1 \le i \le r$, $1 \le j \le t$, and $1 \le k \le q - t$,

$$A(\Gamma_1 \widetilde{\square} \Gamma_2)(Z_i \otimes Y_j) = \mu (Z_i \otimes Y_j) = A(\Gamma_1 \widetilde{\boxtimes} \Gamma_2)(Z_i \otimes Y_j),$$
$$A(\Gamma_1 \widetilde{\square} \Gamma_2)(Z_i \otimes Y_k') = -\mu (Z_i \otimes Y_k') = A(\Gamma_1 \widetilde{\boxtimes} \Gamma_2)(Z_i \otimes Y_k').$$

For every $1 \le \ell \le p - r$, $1 \le j \le t$, and $1 \le k \le q - t$,

$$A(\Gamma_1 \widetilde{\square} \Gamma_2)(Z_\ell' \otimes Y_k') = \mu (Z_\ell' \otimes Y_k') = A(\Gamma_1 \widetilde{\boxtimes} \Gamma_2)(Z_\ell' \otimes Y'_k),$$
$$A(\Gamma_1 \widetilde{\square} \Gamma_2)(Z_\ell' \otimes Y_j) = -\mu (Z_\ell' \otimes Y_j) = A(\Gamma_1 \widetilde{\boxtimes} \Gamma_2)(Z_\ell' \otimes Y_j).$$

Note that all of Z_i, Z_ℓ', Y_j, and Y_k' are nonzero vectors for each $1 \le i \le r, 1 \le \ell \le p - r$, $1 \le j \le t$, and $1 \le k \le q - t$. Hence, the Kronecker products of them are also nonzero vectors. By $(Z_i \otimes Y_j)^T (Z_\ell' \otimes Y_k') = 0$, we have $Z_i \otimes Y_j$ and $Z_\ell' \otimes Y_k'$ are

$$rt + (p - r)(q - t) = pq/2 + (n - 2s)(q - 2t)/2$$

eigenvectors of $A(\Gamma_1\widetilde{\square}\Gamma_2)$ (resp., $A(\Gamma_1\widetilde{\bowtie}\Gamma_2)$) with respect to eigenvalue μ. By $(Z_i \otimes Y_k')^T(Z_\ell' \otimes Y_j) = 0$, we know that $Z_i \otimes Y_k'$ and $Z_\ell' \otimes Y_j$ are

$$r(q-t) + (p-r)t = pq/2 - (n-2s)(q-2t)/2$$

eigenvectors of $A(\Gamma_1\widetilde{\square}\Gamma_2)$ (resp., $A(\Gamma_1\widetilde{\bowtie}\Gamma_2)$) with respect to eigenvalue $-\mu$. Thus, $\pm\mu$ are eigenvalues of $\Gamma_1\widetilde{\square}\Gamma_2$ (resp., $\Gamma_1\widetilde{\bowtie}\Gamma_2$) with multiplicities $\frac{1}{2}pq \pm \frac{1}{2}(n-2s)(q-2t)$, respectively. □

Proof of Theorem 4.1(iii). Suppose that $\lambda \neq 0$. Since Γ_1 is bipartite, λ and $-\lambda$ are eigenvalues of Γ_1, each with multiplicity $p/2$. Without loss of generality, assume that $\mu \geq 0$, $A_2Y_j = \mu Y_j$ for each $j = 1, \ldots, t$ and $A_2Y_k' = -\mu Y_k'$ for each $k = 1, \ldots, q-t$. In particular, if $t = 0$, then $1 \leq k \leq q$ and there exists no such Y_j; if $t = q$, then $1 \leq j \leq q$ and there exists no such Y'_k. Note that if $\mu = 0$, then $t = q$. Now, for $i = 1, \ldots, p/2$, suppose that $X_i = \begin{bmatrix} X_{i1} \\ X_{i2} \end{bmatrix}$ is the unit vector such that $A_1X_i = \lambda X_i$, where X_{i1} and X_{i2} are column vectors of length s and $n-s$, respectively. Then $PX_{i2} = \lambda X_{i1}$ and $P^TX_{i1} = \lambda X_{i2}$. For each $i = 1, \ldots, p/2$, let $X_i' = \begin{bmatrix} X_{i1} \\ -X_{i2} \end{bmatrix}$, then $A_1X'_i = -\lambda X'_i$. Since $\lambda \neq 0$, we have $X_i^TX_i' = 0$ and so $X_{i1}^TX_{i1} = X_{i2}^TX_{i2} = \frac{1}{2}$, which implies that X_{i1} and X_{i2} are nonzero vectors. On the basis of eigenvalues $\pm\lambda$, $\pm\mu$ and the corresponding eigenvectors, we construct $pq/2$ vectors as follows:

$$Z_i \otimes Y_j = \begin{bmatrix} (\sqrt{\lambda^2+\mu^2}+\mu)X_{i1} \\ \lambda X_{i2} \end{bmatrix} \otimes Y_j, \quad W_i \otimes Y_k' = \begin{bmatrix} \lambda X_{i1} \\ (\sqrt{\lambda^2+\mu^2}+\mu)X_{i2} \end{bmatrix} \otimes Y_k',$$

for each $i = 1, \ldots, p/2$, $j = 1, \ldots, t$, and $k = 1, \ldots, q-t$, and construct $pq/2$ vectors as follows:

$$Z_i' \otimes Y_j = \begin{bmatrix} -\lambda X_{i1} \\ (\sqrt{\lambda^2+\mu^2}+\mu)X_{i2} \end{bmatrix} \otimes Y_j, \quad W_i' \otimes Y_k' = \begin{bmatrix} (\sqrt{\lambda^2+\mu^2}+\mu)X_{i1} \\ -\lambda X_{i2} \end{bmatrix} \otimes Y_k',$$

for $i = 1, \ldots, p/2$, $j = 1, \ldots, t$, and $k = 1, \ldots, q-t$. Then, we have

$$\begin{aligned} A(\Gamma_1\widetilde{\square}\Gamma_2)\cdot(Z_i \otimes Y_j) &= \begin{bmatrix} I_s \otimes A_2 & P \otimes I_m \\ P^T \otimes I_m & -I_{n-s} \otimes A_2 \end{bmatrix} \cdot \begin{bmatrix} (\sqrt{\lambda^2+\mu^2}+\mu)X_{i1} \otimes Y_j \\ \lambda X_{i2} \otimes Y_j \end{bmatrix} \\ &= \begin{bmatrix} (\sqrt{\lambda^2+\mu^2}+\mu)X_{i1} \otimes \mu Y_j + \lambda^2 X_{i1} \otimes Y_j \\ (\sqrt{\lambda^2+\mu^2}+\mu)\lambda X_{i2} \otimes Y_j - \lambda X_{i2} \otimes \mu Y_j \end{bmatrix} \\ &= \sqrt{\lambda^2+\mu^2} \cdot \begin{bmatrix} (\sqrt{\lambda^2+\mu^2}+\mu)X_{i1} \otimes Y_j \\ \lambda X_{i2} \otimes Y_j \end{bmatrix} \\ &= \sqrt{\lambda^2+\mu^2} \cdot (Z_i \otimes Y_j), \end{aligned} \tag{5}$$

$$A(\Gamma_1\widetilde{\square}\Gamma_2)\cdot(W_i\otimes Y'_k)=\begin{bmatrix} I_s\otimes A_2 & P\otimes I_m \\ P^T\otimes I_m & -I_{n-s}\otimes A_2\end{bmatrix}\cdot\begin{bmatrix}\lambda X_{i1}\otimes Y'_k \\ (\sqrt{\lambda^2+\mu^2}+\mu)X_{i2}\otimes Y'_k\end{bmatrix}$$
$$=\begin{bmatrix}-\lambda X_{i1}\otimes\mu Y'_k+(\sqrt{\lambda^2+\mu^2}+\mu)\lambda X_{i1}\otimes Y'_k \\ \lambda^2X_{i2}\otimes Y'_k+(\sqrt{\lambda^2+\mu^2}+\mu)X_{i2}\otimes\mu Y'_k\end{bmatrix} \quad (6)$$
$$=\sqrt{\lambda^2+\mu^2}\cdot\begin{bmatrix}\lambda X_{i1}\otimes Y'_k \\ (\sqrt{\lambda^2+\mu^2}+\mu)X_{i2}\otimes Y'_k\end{bmatrix}$$
$$=\sqrt{\lambda^2+\mu^2}\cdot(W_i\otimes Y'_k),$$

$$A(\Gamma_1\widetilde{\square}\Gamma_2)\cdot(Z'_i\otimes Y_j)=\begin{bmatrix} I_s\otimes A_2 & P\otimes I_m \\ P^T\otimes I_m & -I_{n-s}\otimes A_2\end{bmatrix}\cdot\begin{bmatrix}-\lambda X_{i1}\otimes Y_j \\ (\sqrt{\lambda^2+\mu^2}+\mu)X_{i2}\otimes Y_j\end{bmatrix}$$
$$=\begin{bmatrix}-\lambda X_{i1}\otimes\mu Y_j+(\sqrt{\lambda^2+\mu^2}+\mu)\lambda X_{i1}\otimes Y_j \\ -\lambda^2X_{i2}\otimes Y_j-(\sqrt{\lambda^2+\mu^2}+\mu)X_{i2}\otimes\mu Y_j\end{bmatrix} \quad (7)$$
$$=-\sqrt{\lambda^2+\mu^2}\cdot\begin{bmatrix}-\lambda X_{i1}\otimes Y_j \\ (\sqrt{\lambda^2+\mu^2}+\mu)X_{i2}\otimes Y_j\end{bmatrix}$$
$$=-\sqrt{\lambda^2+\mu^2}\cdot(Z'_i\otimes Y_j),$$

$$A(\Gamma_1\widetilde{\square}\Gamma_2)\cdot(W'_i\otimes Y'_k)=\begin{bmatrix} I_s\otimes A_2 & P\otimes I_m \\ P^T\otimes I_m & -I_{n-s}\otimes A_2\end{bmatrix}\cdot\begin{bmatrix}(\sqrt{\lambda^2+\mu^2}+\mu)X_{i1}\otimes Y'_k \\ -\lambda X_{i2}\otimes Y'_k\end{bmatrix}$$
$$=\begin{bmatrix}-(\sqrt{\lambda^2+\mu^2}+\mu)X_{i1}\otimes\mu Y'_k-\lambda^2X_{i1}\otimes Y'_k \\ (\sqrt{\lambda^2+\mu^2}+\mu)\lambda X_{i2}\otimes Y'_k-\lambda X_{i2}\otimes\mu Y'_k\end{bmatrix} \quad (8)$$
$$=-\sqrt{\lambda^2+\mu^2}\cdot\begin{bmatrix}(\sqrt{\lambda^2+\mu^2}+\mu)X_{i1}\otimes Y'_k \\ -\lambda X_{i2}\otimes Y'_k\end{bmatrix}$$
$$=-\sqrt{\lambda^2+\mu^2}\cdot(W'_i\otimes Y'_k).$$

Since $\lambda\neq 0$, and X_{i1} and X_{i2} are nonzero, we know that all of $Z_i, W_i, Z'_i, W'_i, Y_j, Y'_k$ are nonzero vectors for each $i\in\{1,\ldots,p/2\}$, $j\in\{1,\ldots,t\}$, and $k\in\{1,\ldots,q-t\}$, and the Kronecker products of them are also nonzero vectors. As $(Z_i\otimes Y_j)^T(W_i\otimes Y'_k)=0$, by (5) and (6), we have $Z_i\otimes Y_j$ and $W_i\otimes Y'_k$ are $pq/2$ eigenvectors of $A(\Gamma_1\widetilde{\square}\Gamma_2)$ with respect to eigenvalue $\sqrt{\lambda^2+\mu^2}$. As $(Z'_i\otimes Y_j)^T(W'_i\otimes Y'_k)=0$, by (7) and (8), we have $Z'_i\otimes Y_j$ and $W'_i\otimes Y'_k$ are $pq/2$ eigenvectors of $A(\Gamma_1\widetilde{\square}\Gamma_2)$ with respect to eigenvalue $-\sqrt{\lambda^2+\mu^2}$. Therefore, $\pm\sqrt{\lambda^2+\mu^2}$ are adjacency eigenvalues of $\Gamma_1\widetilde{\square}\Gamma_2$, each with multiplicity $pq/2$. □

Proof of Theorem 4.1 (iv). Suppose that $\lambda\mu \neq 0$. Since Γ_1 is bipartite, λ and $-\lambda$ are eigenvalues of Γ_1, each with multiplicity $p/2$. Without loss of generality, assume that $A_2 Y_j = \mu Y_j$ for each $j = 1, \ldots, t$ and $A_2 Y_k' = -\mu Y_k'$ for each $k = 1, \ldots, q - t$. In particular, if $t = 0$, then $1 \leq k \leq q$ and there exists no such Y_j; if $t = q$, then $1 \leq j \leq q$ and there exists no such Y_k'. Now, for each $i = 1, \ldots, p/2$, suppose that $X_i = \begin{bmatrix} X_{i1} \\ X_{i2} \end{bmatrix}$ is the unit vector such that $A_1 X_i = \lambda X_i$, where X_{i1} and X_{i2} are column vectors of length s and $n - s$, respectively. Then $PX_{i2} = \lambda X_{i1}$ and $P^T X_{i1} = \lambda X_{i2}$. Let $X_t' = \begin{bmatrix} X_{i1} \\ -X_{i2} \end{bmatrix}$, then $A_1 X'_i = -\lambda X'_i$ and so $X_i^T X_t' = 0$. Thus $(X_{i1})^T X_{i1} = (X_{i2})^T X_{i2} = \frac{1}{2}$, and so X_{i1} and X_{i2} are nonzero vectors. On the basis of eigenvalues $\pm\lambda, \pm\mu$ and the corresponding eigenvectors, we construct $pq/2$ vectors as follows:

$$Z_i \otimes Y_j = \begin{bmatrix} (\sqrt{\lambda^2 + 1} + 1)X_{i1} \\ \lambda X_{i2} \end{bmatrix} \otimes Y_j, \quad Z'_i \otimes Y'_k = \begin{bmatrix} -\lambda X_{i1} \\ (\sqrt{\lambda^2 + 1} + 1)X_{i2} \end{bmatrix} \otimes Y'_k,$$

for each $i = 1, \ldots, p/2$, $j = 1, \ldots, t$, and $k = 1, \ldots, q - t$, and construct $pq/2$ vectors as follows:

$$Z'_i \otimes Y_j = \begin{bmatrix} -\lambda X_{i1} \\ (\sqrt{\lambda^2 + 1} + 1)X_{i2} \end{bmatrix} \otimes Y_j, \quad Z_i \otimes Y'_k = \begin{bmatrix} (\sqrt{\lambda^2 + 1} + 1)X_{i1} \\ \lambda X_{i2} \end{bmatrix} \otimes Y'_k,$$

for each $i = 1, \ldots, p/2$, $j = 1, \ldots, t$, and $k = 1, \ldots, q - t$. Since

$$\begin{aligned}
\begin{bmatrix} I_s & P \\ P^T & -I_{n-s} \end{bmatrix} Z_i &= \begin{bmatrix} I_s & P \\ P^T & -I_{n-s} \end{bmatrix} \begin{bmatrix} (\sqrt{\lambda^2 + 1} + 1)X_{i1} \\ \lambda X_{i2} \end{bmatrix} \\
&= \begin{bmatrix} (\sqrt{\lambda^2 + 1} + 1)X_{i1} + \lambda^2 X_{i1} \\ (\sqrt{\lambda^2 + 1} + 1)\lambda X_{i2} - \lambda X_{i2} \end{bmatrix} \\
&= \sqrt{\lambda^2 + 1}\, Z_i, \\
\begin{bmatrix} I_s & P \\ P^T & -I_{n-s} \end{bmatrix} Z_t' &= \begin{bmatrix} I_s & P \\ P^T & -I_{n-s} \end{bmatrix} \begin{bmatrix} -\lambda X_{i1} \\ (\sqrt{\lambda^2 + 1} + 1)X_{i2} \end{bmatrix} \\
&= \begin{bmatrix} -\lambda X_{i1} + (\sqrt{\lambda^2 + 1} + 1)\lambda X_{i1} \\ -\lambda^2 X_{i2} - (\sqrt{\lambda^2 + 1} + 1)X_{i2} \end{bmatrix} \\
&= -\sqrt{\lambda^2 + 1}\, Z_t',
\end{aligned}$$

we can obtain the following equations:

$$A(\Gamma_1 \widetilde{\bowtie} \Gamma_2) \cdot (Z_i \otimes Y_j) = \sqrt{\lambda^2 + 1}\, Z_i \otimes A_2 Y_j = \mu\sqrt{\lambda^2 + 1} \cdot (Z_i \otimes Y_j), \tag{9}$$

$$A(\Gamma_1\widetilde{\bowtie}\Gamma_2)\cdot(Z'_i\otimes Y'_k) = -\sqrt{\lambda^2+1}\,Z'_i\otimes A_2Y'_k = \mu\sqrt{\lambda^2+1}\cdot(Z'_i\otimes Y'_k), \quad (10)$$

$$A(\Gamma_1\widetilde{\bowtie}\Gamma_2)\cdot(Z'_i\otimes Y_j) = -\sqrt{\lambda^2+1}\,Z'_i\otimes A_2Y_j = -\mu\sqrt{\lambda^2+1}\cdot(Z'_i\otimes Y_j), \quad (11)$$

$$A(\Gamma_1\widetilde{\bowtie}\Gamma_2)\cdot(Z_i\otimes Y'_k) = \sqrt{\lambda^2+1}\,Z_i\otimes A_2Y'_k = -\mu\sqrt{\lambda^2+1}\cdot(Z_i\otimes Y'_k). \quad (12)$$

Since $\lambda\mu\neq 0$, X_{i1} and X_{i2} are nonzero, we know that all of Z_i, Z'_i, Y_j, Y'_k are nonzero vectors for each $i\in\{1,\ldots,p/2\}$, $j\in\{1,\ldots,t\}$, and $k\in\{1,\ldots,q-t\}$, and the Kronecker products of them are also nonzero. As $(Z_i\otimes Y_j)^T(Z'_i\otimes Y'_k)=0$, by (9) and (10), we have $Z_i\otimes Y_j$ and $Z'_i\otimes Y'_k$ are $pq/2$ eigenvectors of $A(\Gamma_1\widetilde{\bowtie}\Gamma_2)$ with respect to eigenvalue $\mu\sqrt{\lambda^2+1}$. As $(Z'_i\otimes Y_j)^T(Z_i\otimes Y'_k)=0$, by (11) and (12), we have $Z'_i\otimes Y_j$ and $Z_i\otimes Y'_k$ are $pq/2$ eigenvectors of $A(\Gamma_1\widetilde{\bowtie}\Gamma_2)$ with respect to eigenvalue $-\mu\sqrt{\lambda^2+1}$. Therefore, $\pm\mu\sqrt{\lambda^2+1}$ are adjacency eigenvalues of $\Gamma_1\widetilde{\bowtie}\Gamma_2$, each with multiplicity $pq/2$. □

Proof of Theorem 4.3. Let λ^2 be any eigenvalue of $A(\Gamma_1)^2$ with multiplicity p and μ^2 be any eigenvalue of $A(\Gamma_2)^2$ with multiplicity q, where μ is the eigenvalue of Γ_2 with multiplicity t.

(a) Consider $\Gamma_1\widetilde{\square}\Gamma_2$. By Theorem 4.1 (i), $\lambda^2+\mu^2$ is an eigenvalue of $A(\Gamma_1\widetilde{\square}\Gamma_2)^2$ with multiplicity pq.

Assume that Γ_1 is balanced or the spectrum of Γ_2 is symmetric with respect to 0. Then the bipartition (V_1, V_2) of Γ_1 satisfies $|V_1|=\frac{n}{2}$ or the multiplicity of eigenvalue $\mu(\neq 0)$ of Γ_2 is equal to $t=\frac{q}{2}$, and so $(n-2|V_1|)(q-2t)=0$. It suffices to prove that the multiplicities of eigenvalues $\pm\sqrt{\lambda^2+\mu^2}$ of $\Gamma_1\widetilde{\square}\Gamma_2$ are equal to $\frac{1}{2}pq$ when $\lambda^2+\mu^2\neq 0$. If $\lambda\neq 0$, then by Theorem 4.1 (iii), the multiplicities of eigenvalues $\pm\sqrt{\lambda^2+\mu^2}$ of $\Gamma_1\widetilde{\square}\Gamma_2$ are equal to $\frac{1}{2}pq$. If $\lambda=0$ and $\mu\neq 0$, then by Theorem 4.1 (ii), the multiplicities of eigenvalues $\pm\mu$ of $\Gamma_1\widetilde{\square}\Gamma_2$ are equal to $\frac{1}{2}pq$. Thus, the spectrum of $\Gamma_1\widetilde{\square}\Gamma_2$ is symmetric with respect to 0.

Conversely, assume that the spectrum of $\Gamma_1\widetilde{\square}\Gamma_2$ is symmetric with respect to 0. If all the eigenvalues of Γ_1 are nonzero, then the rank of $A(\Gamma_1)$ is n and so $\mathrm{rank}(P)=\mathrm{rank}(P^T)=\frac{n}{2}$. This implies $|V_1|=|V_2|$ and so Γ_1 is balanced. If $\lambda=0$ and $\mu\neq 0$, then the multiplicities of eigenvalues $\pm\mu$ of $\Gamma_1\widetilde{\square}\Gamma_2$ must be equal. By Theorem 4.1 (ii), we have

$$pq+(n-2|V_1|)(q-2t)=pq-(n-2|V_1|)(q-2t),$$

that is, $(n-2|V_1|)(q-2t)=0$ and so Γ_1 is balanced or the spectrum of Γ_2 is symmetric with respect to 0.

(b) Consider $\Gamma_1\widetilde{\bowtie}\Gamma_2$. By Theorem 4.1 (i), $(\lambda^2+1)\mu^2$ is an eigenvalue of $A(\Gamma_1\widetilde{\bowtie}\Gamma_2)^2$ with multiplicity pq.

Assume that Γ_1 is balanced or the spectrum of Γ_2 is symmetric with respect to 0. Then the bipartition (V_1, V_2) of Γ_1 satisfies $|V_1|=\frac{n}{2}$ or the multiplicity of eigenvalue $\mu(\neq 0)$ of Γ_2 is equal to $t=\frac{q}{2}$, and so $(n-2|V_1|)(q-2t)=0$. It suffices to prove that the multiplicities of eigenvalues $\pm\sqrt{(\lambda^2+1)\mu^2}$ of $\Gamma_1\widetilde{\bowtie}\Gamma_2$ are equal to $\frac{1}{2}pq$ when $\mu^2 \neq 0$. If $\lambda \neq 0$ and $\mu \neq 0$, then by Theorem 4.1(iv), the multiplicities of eigenvalues $\pm\sqrt{(\lambda^2+1)\mu^2}$ of $\Gamma_1\widetilde{\bowtie}\Gamma_2$ are equal to $\frac{1}{2}pq$. If $\lambda = 0$ and $\mu \neq 0$, then by Theorem 4.1(ii), the multiplicities of eigenvalues $\pm\mu$ of $\Gamma_1\widetilde{\bowtie}\Gamma_2$ are equal to $\frac{1}{2}pq$. Thus, the spectrum of $\Gamma_1\widetilde{\bowtie}\Gamma_2$ is symmetric with respect to 0.

Conversely, assume that the spectrum of $\Gamma_1\widetilde{\bowtie}\Gamma_2$ is symmetric with respect to 0. If all the eigenvalues of Γ_1 are nonzero, then the rank of $A(\Gamma_1)$ is n and so $\text{rank}(P) = \text{rank}(P^T) = \frac{n}{2}$. This implies $|V_1| = |V_2|$ and so Γ_1 is balanced. If $\lambda = 0$ and $\mu \neq 0$, then the multiplicities of eigenvalues $\pm\mu$ of $\Gamma_1\widetilde{\bowtie}\Gamma_2$ must be equal. By Theorem 4.1(ii), we have

$$pq + (n - 2|V_1|)(q - 2t) = pq - (n - 2|V_1|)(q - 2t),$$

that is, $(n - 2|V_1|)(q - 2t) = 0$ and so Γ_1 is balanced or the spectrum of Γ_2 is symmetric with respect to 0. □

5 | INDUCED SUBGRAPHS OF THE SIGNED PRODUCT GRAPHS

In this section, we mainly give the proof of Theorem 1.2 and generalize it to signed product of n ($n \geq 3$) graphs. To establish Theorem 1.2, we need the following lemmas.

Lemma 5.1 (Cauchy's Interlacing Theorem [3]). *Let A be an $n \times n$ symmetric matrix, and B be an $m \times m$ principle submatrix of A, where $m < n$. If the eigenvalues of A are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and the eigenvalues of B are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$, then for all $1 \leq i \leq m$,*

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i}.$$

Lemma 5.2. *Suppose $\Gamma = (G, \sigma)$ is a signed graph of order n, and $A = \left(a_{ij}^{\sigma}\right)$ is the adjacency matrix of Γ. Let $\widetilde{A} = (\tilde{a}_{ij})$ be an $n \times n$ symmetric matrix with $|\tilde{a}_{ij}| \leq |a_{ij}^{\sigma}|$ for any $1 \leq i, j \leq n$. Then*

$$\Delta(\Gamma) \geq \lambda_1(\widetilde{A}).$$

In particular, $\Delta(\Gamma) \geq \lambda_1(\Gamma)$ when $\widetilde{A} = A$.

Proof. It suffices to consider that $\widetilde{A}$ is not an all zero matrix. Thus, $\lambda_1(\widetilde{A}) > 0$. Suppose $X = (x_1, x_2, \ldots, x_n)^T$ is an eigenvector corresponding to $\lambda_1(\widetilde{A})$. Then $\lambda_1(\widetilde{A})X = \widetilde{A}X$. Assume that $|x_u| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$. Then $|x_u| > 0$ and

$$|\lambda_1(\widetilde{A})x_u| = \left|\sum_{j=1}^{n} \tilde{a}_{uj}x_j\right| = \left|\sum_{j\sim u} \tilde{a}_{uj}x_j\right| \leq \sum_{j\sim u} \left|\tilde{a}_{uj}\right||x_u| \leq \sum_{j\sim u} \left|a_{uj}^{\sigma}\right||x_u| \leq \Delta(\Gamma)|x_u|.$$

Hence, $\Delta(\Gamma) \geq \lambda_1(\widetilde{A})$. □

Lemma 5.3. *Let Γ be a connected signed graph of order n with k nonnegative adjacency eigenvalues $\lambda_1(\Gamma) \geq \cdots \geq \lambda_k(\Gamma) \geq 0$. If H is an $(n-k+1)$-vertex induced subgraph of Γ, then*

$$\Delta(H) \geq \lceil \lambda_k(\Gamma) \rceil.$$

Proof. Note that $A(H)$ is an $(n-k+1) \times (n-k+1)$ submatrix of $A(\Gamma)$. By Lemma 5.1, $\lambda_1(H) \geq \lambda_k(\Gamma)$. By Lemma 5.2, $\Delta(H) \geq \lambda_1(H) \geq \lambda_k(\Gamma)$. Hence, $\Delta(H) \geq \lceil \lambda_k(\Gamma) \rceil$. □

Example 5.4. The Petersen graph $(PG, +)$ has spectrum $3^{(1)}, 1^{(5)}, -2^{(4)}$. If H is a 5-vertex induced subgraph of $(PG, +)$, then by Lemma 5.3, $\Delta(H) \geq 1$ and there exists an induced subgraph H_1 in Figure 2 such that the bound is tight.

The signed Petersen graph $(PG, -)$ has spectrum $2^{(4)}, -1^{(5)}, -3^{(1)}$. If H is a 7-vertex induced subgraph of $(PG, -)$, then by Lemma 5.3, $\Delta(H) \geq 2$ and there exists an induced subgraph H_2 in Figure 2 such that the bound is tight.

Proof of Theorem 1.2. Denote $N = mn$. Let $\Gamma = \Gamma_1 \widetilde{\square} \Gamma_2$ and $\Gamma' = \Gamma_1 \widetilde{\boxtimes} \Gamma_2$. Then H (resp., H') is a $(\lfloor \frac{N}{2} \rfloor + 1)$-vertex induced subgraph of Γ (resp., Γ'). By Lemma 5.3,

$$\Delta(H) \geq \lambda_{\lceil \frac{N}{2} \rceil}(\Gamma) \quad \text{and} \quad \Delta(H') \geq \lambda_{\lceil \frac{N}{2} \rceil}(\Gamma').$$

By Theorem 4.1 (i), $\lambda^2 + \mu^2$ is the minimum eigenvalue of $A(\Gamma)^2$ and $(\lambda^2 + 1)\mu^2$ is the minimum eigenvalue of $A(\Gamma')^2$. Thus, by Theorem 4.3, the adjacency spectrums of Γ and Γ' are symmetric with respect to 0 and so

$$\lambda_{\lceil \frac{N}{2} \rceil}(\Gamma) = \sqrt{\lambda^2 + \mu^2} \quad \text{and} \quad \lambda_{\lceil \frac{N}{2} \rceil}(\Gamma') = \sqrt{(\lambda^2 + 1)\mu^2}.$$

Combining these (in)equalities, the results follow. □

Remark 1. Since the spectrum of Γ_2 in Corollary 1.3 is symmetric with respect to 0, Corollary 1.3 follows immediately from Theorem 1.2. Moreover, Corollary 1.3 also could be deserved from Corollary 4.2 and Lemma 5.3.

The lower bounds on $\Delta(H)$ and $\Delta(H')$ in Theorem 1.2 are tight for some graphs.

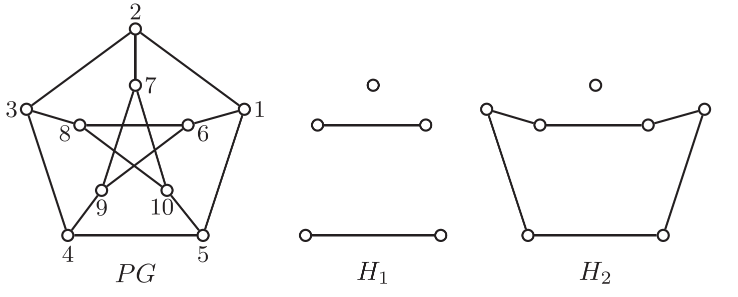

FIGURE 2 The 5-vertex induced subgraph H_1 and 7-vertex induced subgraph H_2 of the Petersen graph that have maximum degrees 1 and 2, respectively, in Example 5.4

(a) Let Γ_i be a signed graph of Q_{n_i} whose eigenvalues are $\pm\sqrt{n_i}$ for $i = 1, 2$, and let H be an arbitrary $(\frac{1}{2}n_1n_2 + 1)$-vertex induced subgraph of $\Gamma_1 \widetilde{\square} \Gamma_2$. By Theorem 1.2, $\Delta(H) \geq \lceil \sqrt{n_1 + n_2} \rceil$. Since $Q_{n_1} \square Q_{n_2} = Q_{n_1+n_2}$, the lower bound on $\Delta(H)$ is tight, as shown by Chung, Füredi, Graham, and Seymour [5] in 1988.

(b) Let $\Gamma_1 = (PG, +)$ be the Petersen graph in Figure 2 and $\Gamma_2 = (K_2, +)$, and let H be an arbitrary 11-vertex induced subgraph of $\Gamma_1 \widetilde{\square} \Gamma_2$. By Theorem 1.2, $\Delta(H) \geq \lceil \sqrt{1^2 + 1^2} \rceil = 2$. Let $V_1 = \{2, 4, 5, 6, 7, 8\} \subset V(\Gamma_1)$ and H_1 be a subgraph of $\Gamma_1 \widetilde{\square} \Gamma_2$ induced by $V_1 \times V(K_2)$. Then for any 11-vertex induced subgraph H_0 of H_1, $\Delta(H_0) = 2$ and so the lower bound on $\Delta(H)$ is tight for H_0.

(c) Let $\Gamma_1 = T_{16}$ be the signed graph in Figure 1 and $\Gamma_2 = (K_2, +)$, and let H be an arbitrary 17-vertex induced subgraph of $\Gamma_1 \widetilde{\square} \Gamma_2$. By Theorem 1.2, $\Delta(H) \geq \lceil \sqrt{2^2 + 1^2} \rceil = 3$. Let $V_1 = \{u_1, v_1, u_2, v_2\} \cup \{u_4, v_4, u_5, v_5\} \cup \{u_7, v_7\}$ and H_1 be a subgraph of $T_{16} \widetilde{\square} K_2$ induced by $V_1 \times V(K_2)$. Then for any 17-vertex induced subgraph H_0 of H_1, $\Delta(H_0) = 3$ and so the lower bound on $\Delta(H)$ is tight for H_0. Since T_{16} is bipartite, $T_{16} \widetilde{\square} K_2 \cong T_{16} \widetilde{\bowtie} K_2$ by Lemma 3.2. Thus, the lower bound on $\Delta(H')$ in Theorem 1.2 is also tight for H_0.

(d) Let $\Gamma_1 = S_{14}$ be the signed graph in Figure 1 and $\Gamma_2 = (K_2, +)$, and let H be an arbitrary 15-vertex induced subgraph of $\Gamma_1 \widetilde{\square} \Gamma_2$. By Theorem 1.2, $\Delta(H) \geq \lceil \sqrt{2^2 + 1^2} \rceil = 3$. Let $V_1 = \{1, 3, 5, 6, 8, 9, 11, 13\} \subset V(S_{14})$ and H_1 be a subgraph of $S_{14} \widetilde{\square} K_2$ induced by $V_1 \times V(K_2)$. Then for any 15-vertex induced subgraph H_0 of H_1, $\Delta(H_0) = 3$ and so the lower bound on $\Delta(H)$ is tight for H_0. Since S_{14} is bipartite, $S_{14} \widetilde{\square} K_2 \cong S_{14} \widetilde{\bowtie} K_2$ by Lemma 3.2.

Now, we generalize the signed Cartesian product and signed semistrong product of two signed graphs to the product of n signed graphs.

Thus, the lower bound on $\Delta(H')$ in Theorem 1.2 is also tight for H_0.

Definition 5.5. For $i = 1, 2, \ldots, n - 1$, let Γ_i be a signed bipartite graph and Γ_n be a signed graph. Let $\Gamma^1_{\widetilde{\square},R} = \Gamma^1_{\widetilde{\bowtie},R} = \Gamma_1$ and $\Gamma^1_{\widetilde{\square},L} = \Gamma^1_{\widetilde{\bowtie},L} = \Gamma_n$. For $2 \leq k \leq n$, we define

(i) $\Gamma^k_{\widetilde{\square},R} = \Gamma^{k-1}_{\widetilde{\square},R} \widetilde{\square} \Gamma_k$ and $\Gamma^k_{\widetilde{\bowtie},R} = \Gamma^{k-1}_{\widetilde{\bowtie},R} \widetilde{\bowtie} \Gamma_k$;

(ii) $\Gamma^k_{\widetilde{\square},L} = \Gamma_{n-k+1} \widetilde{\square} \Gamma^{k-1}_{\widetilde{\square},L}$ and $\Gamma^k_{\widetilde{\bowtie},L} = \Gamma_{n-k+1} \widetilde{\bowtie} \Gamma^{k-1}_{\widetilde{\bowtie},L}$.

To illustrate Definition 5.5, one can consider $n = 3$, that is,

$$\Gamma^3_{\widetilde{\square},R} = ((\Gamma_1 \widetilde{\square} \Gamma_2) \widetilde{\square} \Gamma_3) \text{ and } \Gamma^3_{\widetilde{\bowtie},R} = ((\Gamma_1 \widetilde{\bowtie} \Gamma_2) \widetilde{\bowtie} \Gamma_3), \quad \Gamma^3_{\widetilde{\square},L} = (\Gamma_1 \widetilde{\square} (\Gamma_2 \widetilde{\square} \Gamma_3)) \text{ and } \Gamma^3_{\widetilde{\bowtie},L} = (\Gamma_1 \widetilde{\bowtie} (\Gamma_2 \widetilde{\bowtie} \Gamma_3)).$$

By Lemmas 3.1 and 3.2, the Cartesian product and semistrong product of two bipartite graphs are still bipartite. Therefore, Definition 5.5 (i) is well-defined. Since the Kronecker product of matrices is an associative operation, the underling graphs of $\Gamma^n_{\widetilde{\square},R}$ and $\Gamma^n_{\widetilde{\square},L}$ are isomorphic. However, by Lemma 3.2 (iii) and Corollary 3.3, the underling graphs of $\Gamma^{n}_{\widetilde{\bowtie},R}$ and $\Gamma^{n}_{\widetilde{\bowtie},L}$ are not always isomorphic.

By Definition 5.5, Theorem 4.1 (i) can be easily generalized to the following theorem.

Theorem 5.6. *For $i = 1, 2, \ldots, n$, let $\Gamma_i = (G_i, \sigma_i)$ be a signed graph and θ_i^2 be an eigenvalue of $A(\Gamma_i)^2$ with multiplicity p_i, where $\Gamma_1, \ldots, \Gamma_{n-1}$ are bipartite. Then $\sum_{i=1}^{n}\theta_i^2$, $\sum_{i=1}^{n}\theta_i^2$, $\theta_n^2\prod_{i=1}^{n-1}(\theta_i^2+1)$, and $\sum_{k=1}^{n}\prod_{i=k}^{n}\theta_i^2$ are eigenvalues of $A(\Gamma^{n}_{\widetilde{\Box},L})^2$, $A(\Gamma^{n}_{\widetilde{\Box},R})^2$, $A\left(\Gamma^{n}_{\widetilde{\bowtie},L}\right)^2$, and $A\left(\Gamma^{n}_{\widetilde{\bowtie},R}\right)^2$ with multiplicity $p_1p_2\cdots p_n$, respectively.*

By Theorem 5.6, we have the following corollary immediately.

Corollary 5.7. *For $i = 1, 2, \ldots, n$, let Γ_i be a signed graph with exactly two distinct eigenvalues $\pm\theta_i$, where $\Gamma_1, \ldots, \Gamma_{n-1}$ are bipartite. Then $\Gamma^{n}_{\widetilde{\Box},L}$, $\Gamma^{n}_{\widetilde{\Box},R}$, $\Gamma^{n}_{\widetilde{\bowtie},L}$, and $\Gamma^{n}_{\widetilde{\bowtie},R}$ have exactly two distinct eigenvalues $\pm\sqrt{\sum_{i=1}^{n}\theta_i^2}$, $\pm\sqrt{\sum_{i=1}^{n}\theta_i^2}$, $\pm\sqrt{\theta_n^2\prod_{i=1}^{n-1}\left(\theta_i^2+1\right)}$, and $\pm\sqrt{\sum_{k=1}^{n}\prod_{i=k}^{n}\theta_i^2}$, respectively.*

Example 5.8. Let $\Gamma_i = (K_2, +)$ for each $i = 1, 2, \ldots, n$. Then

(i) each of $\Gamma^{n}_{\widetilde{\Box},R}$, $\Gamma^{n}_{\widetilde{\Box},L}$ (see Figure 3 for $n = 4$) and $\Gamma^{n}_{\widetilde{\bowtie},R}$ is a signed graph of Q_n whose eigenvalues are $\pm\sqrt{n}$;
(ii) $\Gamma^{n}_{\widetilde{\bowtie},L}$ is a signed graph of $K_{2^{n-1},2^{n-1}}$ and its eigenvalues are $\pm\sqrt{2^{n-1}}$.

Example 5.9. For each $i = 1, 2, \ldots, n$, let $\Gamma_i = (K_{2,2}, \sigma)$ be the signed graph of $K_{2,2}$ with exactly one negative edge. Then

(i) $\Gamma^{n}_{\widetilde{\Box},R}$ and $\Gamma^{n}_{\widetilde{\Box},L}$ are signed graphs of Q_{2n} whose eigenvalues are $\pm\sqrt{2n}$;
(ii) the eigenvalues of $\Gamma^{n}_{\widetilde{\bowtie},L}$ are $\pm\sqrt{2\cdot 3^{n-1}}$;
(iii) the eigenvalues of $\Gamma^{n}_{\widetilde{\bowtie},R}$ are $\pm\sqrt{\sum_{k=1}^{n}2^k} = \pm\sqrt{2^{n+1}-2}$.

By Definition 5.5, Theorem 4.3 can be generalized to Theorem 5.10.

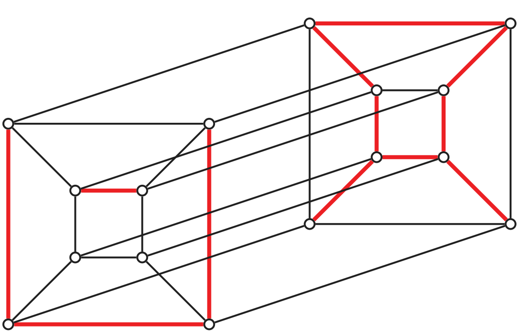

FIGURE 3 The signed graph $\Gamma^{4}_{\widetilde{\Box},L}$ of Q_4 in Example 5.8, where red edges represent negative edges. [Color figure can be viewed at wileyonlinelibrary.com]

Theorem 5.10. *For $n \geq 2$ and $i = 1, 2, \ldots, n-1$, let Γ_i be a signed bipartite graph and Γ_n be a signed graph.*

(i) *The spectrum of $\Gamma^n_{\widetilde{\bowtie},R}$ is symmetric with respect to 0 if and only if Γ_{n-1} is balanced or the spectrum of Γ_n is symmetric with respect to 0.*

(ii) *The spectrum of every graph in $\{\Gamma^n_{\widetilde{\square},R}, \Gamma^n_{\widetilde{\square},L}, \Gamma^n_{\widetilde{\bowtie},L}\}$ is symmetric with respect to 0 if and only if there exists an integer $i \in \{1, \ldots, n-1\}$ such that Γ_i is balanced or the spectrum of Γ_n is symmetric with respect to 0.*

Proof. We only need to consider $n \geq 3$.

(i) By Theorem 4.3, the spectrum of $\Gamma^n_{\widetilde{\bowtie},R} = \Gamma^{n-1}_{\widetilde{\bowtie},R} \widetilde{\bowtie} \Gamma_n$ is symmetric with respect to 0 if and only if $\Gamma^{n-1}_{\widetilde{\bowtie},R}$ is balanced or the spectrum of Γ_n is symmetric with respect to 0. Since the semistrong product of a graph G and a bipartite graph H is balanced if and only if H is balanced, we have $\Gamma^{n-1}_{\widetilde{\bowtie},R} = \Gamma^{n-2}_{\widetilde{\bowtie},R} \widetilde{\bowtie} \Gamma_{n-1}$ is balanced if and only if Γ_{n-1} is balanced. Thus, (i) is proved.

(ii) By Theorem 4.3, the spectrum of $\Gamma^n_{\widetilde{\square},R} = \Gamma^{n-1}_{\widetilde{\square},R} \widetilde{\square} \Gamma_n$ is symmetric with respect to 0 if and only if $\Gamma^{n-1}_{\widetilde{\square},R}$ is balanced or the spectrum of Γ_n is symmetric with respect to 0. Since $\Gamma^{n-1}_{\widetilde{\square},R}$ is balanced if and only if there exists an integer $i \in \{1, \ldots, n-1\}$ such that Γ_i is balanced. So the conclusion for $\Gamma^n_{\widetilde{\square},R}$ is proved.

By Theorem 4.3, the conclusion holds for $\Gamma^2_{\widetilde{\square},L} = \Gamma_{n-1} \widetilde{\square} \Gamma_n$ (resp., $\Gamma^2_{\widetilde{\bowtie},L} = \Gamma_{n-1} \widetilde{\bowtie} \Gamma_n$). By induction on n, assume that the spectrum of $\Gamma^{n-1}_{\widetilde{\square},L}$ (resp., $\Gamma^{n-1}_{\widetilde{\bowtie},L}$) is symmetric with respect to 0 if and only if there exists an integer $i \in \{2, \ldots, n-1\}$ such that Γ_i is balanced or the spectrum of Γ_n is symmetric with respect to 0. By Theorem 4.3, the spectrum of $\Gamma^n_{\widetilde{\square},L} = \Gamma_1 \widetilde{\square} \Gamma^{n-1}_{\widetilde{\square},L}$ (resp., $\Gamma^n_{\widetilde{\bowtie},L} = \Gamma_1 \widetilde{\bowtie} \Gamma^{n-1}_{\widetilde{\bowtie},L}$) is symmetric with respect to 0 if and only if Γ_1 is balanced or the spectrum of $\Gamma^{n-1}_{\widetilde{\square},L}$ (resp., $\Gamma^{n-1}_{\widetilde{\bowtie},L}$) is symmetric with respect to 0. By induction, the conclusion for $\Gamma^n_{\widetilde{\square},L}$ (resp., $\Gamma^n_{\widetilde{\bowtie},L}$) is proved. □

Now, Theorem 1.2 is generalized to the following theorem.

Theorem 5.11. *For $i = 1, 2, \ldots, n$, let $\Gamma_i = (G_i, \sigma_i)$ be a signed graph of order N_i and θ_i^2 be the minimum eigenvalue of $A(\Gamma_i)^2$, where $G_1, \ldots, G_{n-1}$ are bipartite. Let $H_{\square}$, $H_{\bowtie,L}$, and $H_{\bowtie,R}$ be any $\left(\lfloor \frac{1}{2} \prod_{i=1}^{n} N_i \rfloor + 1\right)$-vertex induced subgraph of $\Gamma^n_{\widetilde{\square},L}$, $\Gamma^n_{\widetilde{\bowtie},L}$, and $\Gamma^n_{\widetilde{\bowtie},R}$, respectively.*

(i) *If there exists an integer $i \in \{1, 2, \ldots, n-1\}$ such that Γ_i is balanced or the spectrum of Γ_n is symmetric with respect to 0, then $\Delta(H_{\square}) \geq \sqrt{\sum_{i=1}^{n} \theta_i^2}$ and $\Delta(H_{\bowtie,L}) \geq \sqrt{\theta_n^2 \prod_{i=1}^{n-1} \left(\theta_i^2 + 1\right)}$.*

(ii) If Γ_{n-1} is balanced or the spectrum of Γ_n is symmetric with respect to 0, then $\Delta(H_{\bowtie,R}) \geq \sqrt{\sum_{k=1}^{n} \prod_{i=k}^{n} \theta_i^2}$.

Proof. For simplicity, let $N = \prod_{i=1}^{n} N_i$. Since $H_{\square}$, $H_{\bowtie,L}$, and $H_{\bowtie,R}$ are $(\lfloor \frac{N}{2} \rfloor + 1)$-vertex induced subgraphs of $\Gamma^n_{\widetilde{\square},L}$, $\Gamma^n_{\widetilde{\bowtie},L}$, and $\Gamma^n_{\widetilde{\bowtie},R}$, respectively. By Lemma 5.3,

$$\Delta(H_{\square}) \geq \lambda_{\lceil \frac{1}{2}N \rceil}\left(\Gamma^n_{\widetilde{\square},L}\right), \quad \Delta(H_{\bowtie,L}) \geq \lambda_{\lceil \frac{1}{2}N \rceil}\left(\Gamma^n_{\widetilde{\bowtie},L}\right), \quad \text{and} \quad \Delta(H_{\bowtie,R}) \geq \lambda_{\lceil \frac{1}{2}N \rceil}\left(\Gamma^n_{\widetilde{\bowtie},R}\right).$$

By Theorem 5.6, the minimum eigenvalues of $A(\Gamma^n_{\widetilde{\square},L})^2$, $A(\Gamma^n_{\widetilde{\bowtie},L})^2$, and $A(\Gamma^n_{\widetilde{\bowtie},R})^2$ are obtained. Thus, by Theorem 5.10, the spectrums of $\Gamma^n_{\widetilde{\square},L}$, $\Gamma^n_{\widetilde{\bowtie},L}$, and $\Gamma^n_{\widetilde{\bowtie},R}$ are symmetric with respect to 0 and so $\lambda_{\lceil \frac{1}{2}N \rceil}(\Gamma^n_{\widetilde{\square},L}) = \sqrt{\sum_{i=1}^{n} \theta_i^2}$, $\lambda_{\lceil \frac{1}{2}N \rceil}(\Gamma^n_{\widetilde{\bowtie},L}) = \sqrt{\theta_n^2 \prod_{i=1}^{n-1}\left(\theta_i^2 + 1\right)}$, and $\lambda_{\lceil \frac{1}{2}N \rceil}\left(\Gamma^n_{\widetilde{\bowtie},R}\right) = \sqrt{\sum_{k=1}^{n} \prod_{i=k}^{n} \theta_i^2}$. Combining these (in)equalities, the results follow. □

Corollary 5.12. *For $i = 1, 2, \ldots, n$, let $\Gamma_i = (G_i, \sigma_i)$ be a signed graph of order N_i with exactly two distinct eigenvalues $\pm\theta_i$, where $G_1, \ldots, G_{n-1}$ are bipartite. Let $H_{\square}$, $H_{\bowtie,L}$, and $H_{\bowtie,R}$ be any $\left(\lfloor \frac{1}{2} \prod_{i=1}^{n} N_i \rfloor + 1\right)$-vertex induced subgraph of $\Gamma^n_{\widetilde{\square},L}$, $\Gamma^n_{\widetilde{\bowtie},L}$, and $\Gamma^n_{\widetilde{\bowtie},R}$, respectively. Then $\Delta(H_{\square}) \geq \sqrt{\sum_{i=1}^{n} \theta_i^2}$, $\Delta(H_{\bowtie,L}) \geq \sqrt{\theta_n^2 \prod_{i=1}^{n-1}\left(\theta_i^2 + 1\right)}$, and $\Delta(H_{\bowtie,R}) \geq \sqrt{\sum_{k=1}^{n} \prod_{i=k}^{n} \theta_i^2}$.*

When $\Gamma_i = (K_2, +)$ for each $i = 1, 2, \ldots, n$ in Corollary 5.12, $\Gamma^n_{\widetilde{\square},L}$ is the signed graph of hypercube Q_n. Therefore, Corollary 5.12 implies Huang's theorem.

Example 5.13. For $i = 1, 2, \ldots, n$, let $\Gamma_i = \left(K_{2^t,2^t}, \sigma\right)$ be the signed graph $K_{2^t,2^t}$ with exactly two distinct eigenvalues $\pm\sqrt{2^t}$. For any integer $n \geq 1$ and $t \geq 0$, let $H_{\square}$, $H_{\bowtie,L}$, and $H_{\bowtie,R}$ be any $(2^{n(t+1)-1} + 1)$-vertex induced subgraph of $\Gamma^n_{\widetilde{\square},L}$, $\Gamma^n_{\widetilde{\bowtie},L}$, and $\Gamma^n_{\widetilde{\bowtie},R}$, respectively. Then $\Delta(H_{\square}) \geq \sqrt{2^t \cdot n}$, $\Delta(H_{\bowtie,L}) \geq \sqrt{2^t(2^t + 1)^{n-1}}$, and $\Delta(H_{\bowtie,R}) \geq \sqrt{\sum_{k=1}^{n} 2^{kt}}$.

6 | CONCLUDING REMARKS

I. Corollaries 3.5, 4.2, and 5.7 provide product methods to construct signed graphs with exactly two distinct eigenvalues of opposite signatures from factor graphs Γ_1 and Γ_2. There are many options for the factor graph, such as the signed graphs of Q_n and $K_{2^n,2^n}$ in Example 5.8, T_{2n} in Lemma 3.7, S_{14} in Lemma 3.8, the signed graph of K_n in Example 3.9, signed graphs in Examples 3.10, 3.11, 5.9, and so on.

II. If the following conjecture is true, it would provide a way to construct an infinite family of d-regular Ramanujan graphs by 2-lift of graphs.

Conjecture 6.1 (Bilu–Linial [4]). *Every connected d-regular graph G has a signature σ such that $\rho(G, \sigma) \leq 2\sqrt{d-1}$.*

Gregory considered the following Conjecture 6.2 without the regularity assumption on G.

Conjecture 6.2 (Gregory [7]). *If G is a nontrivial graph with maximum degree $\Delta > 1$, then there exists a signed graph $\Gamma = (G, \sigma)$ such that $\rho(\Gamma) \leq 2\sqrt{\Delta - 1}$.*

Since the trace of the square of the signed adjacency matrix is equal to the sum of the square of the degree of every vertex, Gregory [7] proved that for any signature σ of G, $\rho(G,\sigma) \geq \sqrt{d}$, with equality if and only if G is d-regular and the adjacency matrix $A(G,\sigma)$ is a symmetric weighing matrix of weight d. Thus, the minimum spectral radius for a signed adjacency matrix of Q_n, $K_{2^n,2^n}$, T_{2n}, and S_{14} is exactly $\sqrt{n}$, $\sqrt{2^n}$, 2, and 2, respectively. Therefore, Corollary 4.2 illustrates that the signed Cartesian product and the signed semistrong product of the signed graphs above also have minimum spectral radius and Conjecture 6.1 holds for these graphs. For more general graphs, by Theorem 4.1 (i), we have the following theorem.

Theorem 6.3. *For $i = 1, 2$, let G_i be a graph with maximum degree Δ_i and $\Gamma_i = (G_i, \sigma_i)$ be a signed graph such that $\rho(\Gamma_i) \leq 2\sqrt{\Delta_i - 1}$. If G_1 is bipartite, then*

$$\rho(\Gamma_1 \widetilde{\square} \Gamma_2) \leq 2\sqrt{\Delta_1 + \Delta_2 - 2}.$$

Since $\rho(\Gamma_1 \widetilde{\square} \Gamma_2) = \sqrt{\rho(\Gamma_1)^2 + \rho(\Gamma_2)^2}$ and $\Delta(\Gamma_1 \square \Gamma_2) = \Delta(\Gamma_1) + \Delta(\Gamma_2)$, Theorem 6.3 shows that if Conjecture 6.2 holds for Γ_1 and Γ_2, then Conjecture 6.2 also holds for the signed Cartesian product of them.

III. The method which is utilized to construct a larger weighing matrix can construct a larger signed graph with exactly two distinct eigenvalues $\pm\theta$ from small graphs. Conversely, the ideas of signed Cartesian product and semistrong product in our paper can also be applied to construct a weighing matrix. If for $i = 1, 2$, W_i is a weighing matrix of order n_i and weight k_i, then we can construct weighing matrices as follows:

$$W(4n_1n_2, k_1 + k_2) = \begin{bmatrix} O_{n_1} & W_1 \\ W_1^T & O_{n_1} \end{bmatrix} \otimes I_{2n_2} + \begin{bmatrix} I_{n_1} & O_{n_1} \\ O_{n_1} & -I_{n_1} \end{bmatrix} \otimes \begin{bmatrix} O_{n_2} & W_2 \\ W_2^T & O_{n_2} \end{bmatrix},$$
$$W(4n_1n_2, (k_1 + 1)k_2) = \begin{bmatrix} I_{n_1} & W_1 \\ W_1^T & -I_{n_1} \end{bmatrix} \otimes \begin{bmatrix} O_{n_2} & W_2 \\ W_2^T & O_{n_2} \end{bmatrix},$$
$$W(2n_1n_2, (k_1 + 1)k_2) = \begin{bmatrix} I_{n_1} & W_1 \\ W_1^T & -I_{n_1} \end{bmatrix} \otimes W_2.$$

Furthermore, if W_2 is symmetric, then we can construct weighing matrix

$$W(2n_1n_2, k_1 + k_2) = \begin{bmatrix} I_{n_1} \otimes W_2 & W_1 \otimes I_{n_2} \\ W_1^T \otimes I_{n_2} & -I_{n_1} \otimes W_2 \end{bmatrix}.$$

More methods for constructing weighing matrices, one can refer to the book of Jennifer Seberry on orthogonal designs [20].

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