Graphs and Combinatorics
https://doi.org/10.1007/s00373-020-02229-x

ORIGINAL PAPER

Supereulerian Graphs with Constraints on the Matching Number and Minimum Degree

Mansour J. Algefari[1] · Hong-Jian Lai[2]

Received: 20 January 2020 / Revised: 20 January 2020 / Accepted: 17 August 2020

Abstract

A graph is supereulerian if it has a spanning eulerian subgraph. We show that a connected simple graph G with $n = |V(G)| \geq 2$ and $\delta(G) \geq \alpha'(G)$ is supereulerian if and only if $G \neq K_{1,n-1}$ if n is even or $G \neq K_{2,n-2}$ if n is odd. Consequently, every connected simple graph G with $\delta(G) \geq \alpha'(G)$ has a hamiltonian line graph.

Keywords Supereulerian graph · Hamiltonian line graph · Matching · Minimum degree · Contraction

1 Introduction

We follow Bondy and Murty [5] for terms and notation, unless otherwise stated. Graphs considered in this paper are finite and loopless, but multiple edges are allowed. As in [5], $\kappa(G), \kappa'(G)$, $\delta(G)$, $\alpha(G)$ and $\alpha'(G)$ denote the connectivity, the edge connectivity, the minimum degree, the stability number (also called the independence number), and the matching number of a graph G, respectively. Let G be a graph and let $O(G)$ denote the set of odd degree vertices of G. If G is connected and $O(G) = \emptyset$, then G is an **Eulerian** graph. Thus a graph G is Eulerian if and only if G has a closed trail traversing every edge exactly once. A graph is **supereulerian** if it has a spanning eulerian subgraph. In particular, K_1 is both eulerian and supereulerian. The study of supereulerian graphs was initiated in [4]. Pulleyblank [15] showed that, even within planar graphs, it is NP-complete to determine if a

✉ Hong-Jian Lai
hjlai@math.wvu.edu

Mansour J. Algefari
mans3333@gmail.com

1 Department of Management and Humanities Sciences, Community College, Qassim University, Buraydah, Kingdom of Saudi Arabia

2 Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA

Published online: 03 September 2020

graph is supereulerian. For the literature on supereulerian graphs, see the survey of Catlin [7] and its updates in [8, 14].

If $X \subseteq E(G)$, the **contraction** G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. If H is a subgraph of G, we write G/H for $G/E(H)$. We define $G/\emptyset = G/K_1 = G$.

Chvátal and Erdös proved a classic result on hamiltonian graph, revealing an interesting property of using a relationship between connectivity and independence number of a graph to predict the hamiltonicity of the graph.

Theorem 1.1 (*Chvátal and Erdös*, [9]) *If $\kappa(G) \geq \alpha(G)$, then G is hamiltonian.*

There have been results following Theorem 1.1 and using similar relations involving edge-connectivity, stability number or matching number to predict supereulerianicity of a graph, as seen in [1, 10, 13, 16] and [17], among others. The theorem below presents some of these results.

Theorem 1.2 *Each of the following holds.*

(i) ([13]) *Let G be a graph with $\kappa'(G) \geq 2$ and $\alpha'(G) \leq 2$. Then G is supereulerian if and only if G is not contractible to a $K_{2,t}$. for some odd integer $t \geq 3$.*

(ii) (*Bang-Jensen and Maddaloni* [2]) *Let G be a graph on at least three vertices. If $\kappa'(G) \geq \alpha(G)$, then G is supereulerian.*

(iii) (*An and Xiong*, [1]) *Let G be a graph with $\kappa'(G) \geq 2$ and $\alpha'(G) \leq 2$. Then either G is supereulerian or G has a connected subgraph H such that for some odd integer $t \geq 3$, $G/H \cong K_{2,t}$.*

(iv) (*An and Xiong*, [1]) *If $\kappa'(G) \geq 3$ and $\alpha'(G) \leq 5$, then G is supereulerian if and only if G is not contractible to the Petersen graph.*

A natural question is whether $\kappa'(G) \geq \alpha'(G)$ would warrant G to be supereulerian. The following useful theorem was first proved by Jaeger in [12] and extended by Catlin in [6].

Theorem 1.3 (*Jaeger* [12] *and Catlin* [6]) *Every 4-edge-connected graph is supereulerian.*

Thus by Theorem 1.3 and Theorem 1.2(i) and (iv), it is known that for a connected graph G, if $\kappa'(G) \geq \alpha'(G)$, then G is supereulerian if and only if for any integer $t \geq 0$, G is not isomorphic to a $K_{2,2t+1}$.

These motivates the current research. As it is well known that in a graph G, we always have $\delta(G) \geq \kappa'(G) \geq \kappa(G)$, it is natural to consider whether we can use $\delta(G) \geq \alpha'(G)$ to replace $\kappa'(G) \geq \alpha'(G)$ in predicting supereulerianicity of a graph. The main result of this paper is the following.

Theorem 1.4 *Let G be a connected simple graph with $n = |V(G)| \geq 2$ and $\delta(G) \geq \alpha'(G)$. Then G is supereulerian if and only if $G \neq K_{1,n-1}$ if n is even or $G \neq K_{2,n-2}$ if n is odd.*

Our result implies a new condition for hamiltonian line graphs. The **line graph** of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have at least one vertex in common. Harary and Nash-Williams showed that there is a close relationship between a graph and its line graph concerning Hamilton cycles.

Theorem 1.5 (*Harary and Nash-Williams*, [11]) *Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian if and only if G has an eulerian subgraph H with $E(G - V(H)) = \emptyset$.*

Using Theorem 1.5, we obtain the corollary below, as an immediate application of Theorem 1.4.

Corollary 1.1 *Every connected simple graph G with $|E(G)| \geq 3$ and with $\delta(G) \geq \alpha'(G)$ has a hamiltonian line graph.*

In the next section, we present some preliminaries and prove an associated result. These will be utilized in Sect. 3 to justify the main results. Sharpness of the main result will be discussed in the last section.

2 Preliminaries

Throughout the rest of this section, G always denotes a simple graph under discussion. For subsets $X, Y \subseteq V(G)$, define

$$(X, Y)_G = \{xy \in E(G) \,:\, x \in X, y \in Y\}.$$

When $Y = V(G) - X$, we define

$$\partial_G(X) = (X, V(G) - X)_G.$$

For a vertex $v \in V(G)$, $d_G(v) = |\partial_G(\{v\})|$ is the degree of v in G. When G is understood from the context, we often use $d(v)$ for $d_G(v)$.

Let M be a matching in G. We use $V(M)$ to denote the set $V(G[M])$. A path P in G is an ***M*-augmenting path** if the edges of P are alternately in M and in $E(G) - M$, and if both end vertices of P are not in $V(M)$. We start with a fundamental theorem of Berge.

Theorem 2.1 (*Berge*, [3]) *A matching M in G is a maximum matching if and only if G does not have M-augmenting paths.*

Lemma 2.1 *Let $k > 0$ be an integer and G be a graph with a matching M such that $|M| = k$. Suppose that $V(G) - V(M)$ has a subset X with $|X| \geq 2$ such that for any $v \in X$, $d(v) \geq k$. If X has at least one vertex u such that $d(u) \geq k + 1$, then M is not a maximum matching of G.*

Proof By contradiction, we assume that M is a maximum matching in G. By Theorem 2.1, G has no M-augmenting path. Let $u, v \in X$ be distinct vertices such that $d(u) \geq k + 1$. Since M is maximum, u and v are not adjacent in G, and any vertices adjacent to u or v must be in $V(M)$. By the assumption that M is a maximum matching and by Theorem 2.1, we have the following claim.

Claim 1. For each edge $e = xy \in M$, exactly one in $\{ux, vy\}$ can be in $E(G)$, and exactly one in $\{uy, vx\}$ can be in $E(G)$.

Claim 2 below now follows immediately from Claim 1.

Claim 2. For any edge $e = xy \in M$, if $ux, uy \in E(G)$, or if $ux, vx \in E(G)$, or if $vx, vy \in E(G)$, then $|(\{u, v\}, \{x, y\})_G| \leq 2$.

By Claim 2, we conclude that $|\partial(u) \cup \partial(v)| \leq 2|M|$. As $d(v) \geq k$ and $d(u) \geq k+1$, we are lead to a contradiction: $2k+1 = k + (k+1) \leq d(v) + d(u) = |\partial(v) \cup \partial(u)| \leq 2|M| = 2k$. This proves the lemma. □

Corollary 2.1 *Let G be a graph with $\alpha'(G) = k$. If $\kappa'(G) \geq \alpha'(G)$, then*

(i) $\kappa'(G) = \alpha'(G)$ *when* $|V(G)| \geq 2k+2$, *and*
(ii) $\kappa'(G) \leq 2\alpha'(G)$ *when* $2k \leq |V(G)| \leq 2k+1$.

Proof

(i) By contradiction, we assume that $\kappa'(G) \geq k+1$. Let M denote a maximum matching of G. Then $|V(G) - V(M)| \geq 2k + 2 - 2k = 2$. Since $\kappa'(G) \geq k+1$, for every vertex $u \in V(G) - V(M)$, $d(u) \geq k+1$. It follows by Lemma 2.1 that M is not a maximum matching of G, which is a contradiction.
(ii) As $2k \leq |V(G)| \leq 2k+1$, for each vertex $v \in V(G)$, we have $2k \geq d(v) \geq \kappa'(G)$, and so $\kappa'(G) \leq 2\alpha'(G)$.

□

Lemma 2.2 *Let G be a graph with $n = |V(G)|$ and $k = \alpha'(G)$. Let M be a maximum matching of G, and let $X = V(G) - V(M)$. If $n \geq 2k+2$ and $\delta(G) \geq \alpha'(G)$, then each of the following holds.*

(i) *For every $x \in V(G) - V(M)$, we have $d_G(x) = k$.*
(ii) *If either $n \geq 2k+3$ or both $n = 2k+2$ and G is connected, then for every $e = ab \in M$ and for every $x \in X$, we have the following conclusions.*

(ii-1) There exists exactly one $v(e) \in \{a, b\}$ *such that $v(e)x$ is in $E(G)$, and the vertex $u(e) \in \{a, b\} - \{v(e)\}$ is not adjacent to any vertex in X.*

(ii-2) The set $\{u(e) : e \in M\}$ *as specified in (ii-1) is an independent set in G such that $d(u(e)) = k$ for any $e \in M$ and such that for any $e, e' \in M$, $u(e)v(e'), v(e)u(e') \in E(G)$.*

Proof Let $N(X)$ denote the set of vertices in G that is adjacent to a vertex in X. Since M is a maximum matching, by Theorem 2.1

$$G \text{ does not have an } M\text{-augmenting path.} \tag{1}$$

As $n \geq 2k+2$ and $|M| = k$, we have $|X| \geq 2$. Since M is a maximum matching, we have $N(X) \subseteq V(M)$, and it follows by $\delta(G) \geq k$ and by Lemma 2.1 that Lemma 2.2 (i) must hold. It remains to prove Lemma 2.2 (ii).

Case 1. $n = |V(G)| \geq 2k + 3$.

Let $m = |X|$. Then $m \geq 3$. By Lemma 2.2 (i),

$$|(X, V(M))_G| = mk. \tag{2}$$

It follows from (1) that

$$\text{for any } e \in M, \text{ the vertices of } e \text{ are incident with at most } m \text{ edges in } (X, V(M))_G. \tag{3}$$

Let $e = ab \in M$. If $a, b \in N(X)$, then by (1), there must be a unique $x \in X$ such that x is adjacent to both a and b; and for any $x' \in X - \{x\}$, x' is adjacent to neither a nor b. By Lemma 2.2 (i), x is adjacent to k vertices in $V(M)$. As $|(\{x\}, \{a, b\})_G| = 2$, it follows by (2) and $m \geq 3$ that

$$|(X, V(M) - \{a, b\})_G| = mk - 2 \geq mk - m + 1 = m(k - 1) + 1.$$

If $k > 1$, then there must be an edge $f \in M - \{e\}$ such that the vertices of f are incident with at least $m + 1$ edges in $(X, V(M))_G$, contrary to (3). Hence in this case we must have $k = 1$ and $M = \{e\}$. As $|X| \geq 3$, it follows from (2) with $m \geq 3$ that G must have an M-augmenting path, contrary to (1). Hence for any $e = ab \in M$, exactly one of a and b is adjacent to vertices in X.

Thus for each $e \in M$, let $v(e)$ denote the unique vertex of e that is adjacent to vertices in X and $u(e)$ the other vertex of e which is not adjacent to any vertex in X. Then as $|X| = m$, $|(v(e), X)_G| \leq m$. It follows from (2) that

$$km \geq \sum_{e \in M} |(v(e), X)_G| = |(V(M), X)_G| = mk.$$

This implies that for each $x \in X$ and for each $e \in M$, both $v(e)x$ is in $E(G)$. This proves (ii-1) for Case 1.

Let $Y = \{u(e) : e \in M\}$ and we shall show that Y is an independent set. In fact, if for some $e_1, e_2 \in M$, $u(e_1)u(e_2) \in E(G)$, then for any distinct $x_1, x_2 \in X$, $\{x_1 v(e_1), e_1, u(e_1)u(e_2), e_2, x_2 v(e_2)\}$ induces an M-augmenting path, contrary to (1). Thus each $u(e) \in Y$ can only be adjacent to vertices in $\{v(e) : e \in M\}$. As $|\{v(e) : e \in M\}| = k$ and as $\delta(G) \geq k$, we conclude that for each $e \in M$, $d(u(e)) = k$, and for any $e, e' \in M$, $u(e)v(e'), v(e)u(e') \in E(G)$. This proves (ii-2) for Case 1.

Case 2. G is connected and $|V(G)| = 2k + 2$.

Then $X = \{w, z\}$. Let $M = \{e_1, \cdots, e_k\}$. Let $M_w \subseteq M$ denote the edges in M each of which has a vertex adjacent to w. We define M_z similarly. By the assumption of Lemma 2.2, we have $\delta(G) \geq k$, and so $|M_w| \geq \frac{k}{2}$ and $|M_z| \geq \frac{k}{2}$.

Subcase 1. $|M_w| = \frac{k}{2}$ or $|M_z| = \frac{k}{2}$.

Note that in this case, k must be even. We assume, without loss of generality, that $M_w = \{e_1, \cdots, e_{\frac{k}{2}}\}$. By (1), we must have $M_z = \{e_{\frac{k}{2}+1}, \cdots, e_k\}$. Again by (1), for each $x \in V(M_w)$ and $y \in V(M_z)$, we conclude that $xy \notin E(G)$. Thus

$$(V(M_w) \cup \{w\}, V(M_z) \cup \{z\})_G = \emptyset,$$

contrary to the assumption that G is connected. This shows that Subcase 1 cannot occur.

Subcase 2. $|M_w| > \frac{k}{2}$ and $|M_z| > \frac{k}{2}$.

Therefore, $M_w \cap M_z \neq \emptyset$. By (1), if an edge $e \in M$ whose vertices are adjacent to both w and z, then exactly one vertex of e can be adjacent to both w and z. Let $M' = M_w \cap M_z = \{e'_i = x_i y_i, (i = 1, \ldots, d; 1 \leq d \leq k)\} \subseteq M$. Without loss of generality, we assume that each e'_i has a unique vertex x_i with $x_i w, x_i z \in E(G)$. Let $M'' = M - M' = \{e''_j = r_j s_j, (j = 1, \ldots, k - d)\}$.

Claim 3. Each of the following holds.

(i) For each y_i, $y_i w, y_i z \notin E(G)$.
(ii) The set $\{y_1, y_2, \cdots, y_d\}$ is an independent set.
(iii) If Lemma 2.2 (ii-1) or (ii-2) does not hold, then $d \leq k - 1$.
(iv) For each $e = x_i y_i \in M'$, $|(\{x_i, y_i\}, \{w, z\})_G| = 2$.
(v) For each j with $1 \leq j \leq k - d$, there exists exactly one vertex in $\{w, z\}$ which is adjacent to both r_j and s_j.
(vi) For any $x_i y_i \in M'$ and for any $r_j s_j \in M''$, we have $y_i r_j, y_i s_j \notin E(G)$.

In fact, Claim 3 (i) and (ii) follow directly from (1), and Claim 3(iv) follows from Claim 3(i). For (iii), we assume that Lemma 2.2 (ii-1) or (ii-2) does not hold. and $d = k$. Then $M = M'$ and each x_i is adjacent to both w and z. By Claim 3(i), for each $e_i = x_i y_i \in M$, we have $y_i w, y_i z \notin E(G)$. Hence Lemma 2.2(ii-1) must hold. Also by Claim 3 (i) and (ii), each y_i can only be adjacent to $\{x_1, x_2, \cdots, x_d\}$. By the assumption of Lemma 2.2, we have $\delta(G) \geq k$ and so, for any i, we must have $d(y_i) = k$, and for any $1 \leq i, i' \leq k$, we must also have $x_i y_{i'} \in E(G)$. Hence Lemma 2.2(ii-2) holds as well. This contradiction implies Claim 3 (iii).

We argue by contradiction to prove Claim 3(v). By the definition of M'', for each j with $1 \leq j \leq k - d$, there exists at most one vertex in $\{w, z\}$ which is adjacent to both r_j and s_j. By contradiction, we assume that $r_1 s_1 \in M''$ with $|(\{w, z\}, \{r_1, s_1\})_G| \leq 1$. For any other $r_j s_j \in M''$ with $j \geq 2$, we have $|(\{w, z\}, \{r_j, s_j\})_G| \leq 2$. It follows from Lemma 2.2(i) and Claim 3(iv) that

$$\begin{aligned} 2k = & |(\{w, z\}, V(M))_G| \\ = & |(\{w, z\}, V(M'))_G| + |(\{w, z\}, \{r_1, s_1\})_G| \\ & + \sum_{j=2}^{k-d} |(\{w, z\}, \{r_j, s_j\})_G| \leq 2d + 1 + 2(k - d - 1) = 2k - 1 < 2k. \end{aligned}$$

This contradiction justifies Claim 3(v).

We again argue by contradiction to prove Claim 3(vi). Assume that $y_i r_j \in E(G)$. By Claim 3(v), we may assume that $r_j s_j \in M_w$, and so $\{w s_j, r_j s_j, r_j y_i, x_i y_i, z x_i\}$ will induce an M-augmenting path, contrary to (1). When we have $y_i s_j \in E(G)$, the same argument would also lead the a contradiction. This justifies Claim 3(vi), and completes the proof for Claim 3.

We argue by contradiction to prove Lemma 2.2(ii). As $M' = M_w \cap M_z \neq \emptyset$, we have $d \geq 1$ and so y_1 exists. By Claim 3(i), (ii) and (vi), the neighbors of y_1 can only be among $\{x_1, x_2, \cdots, x_d\}$. Hence $d(y_1) \leq |\{x_1, x_2, \cdots, x_d\}| = d$. As $\delta(G) \geq k$ and by Claim 3(iii), we must have $k \leq d(y_1) \leq d \leq (k-1)$. This contradiction implies that we must have $d = k$, and so by Claim 3(iii), Lemma 2.2(ii) must hold in Case 2. □

Theorem 2.2 *Let G be a simple graph with $n = |V(G)| \geq 2$ and $k = \alpha'(G)$ such that G is connected when $n = 2k+2$. If $\delta(G) \geq k$, then $\kappa'(G) \geq k$. Furthermore, if, in addition, $n \geq 2k+2$, then $\kappa'(G) = k$.*

Proof Let M be a matching of maximum size of G. Assume first that $2k \leq n \leq 2k+1$.

Arbitrarily pick a nonempty proper subset $X \subset V(G)$, and let $Y = V(G) - X$. As $|X| + |Y| = n \leq 2k+1$, we have either $1 \leq |X| \leq k$ or $1 \leq |Y| \leq k$. By symmetry, we assume that $1 \leq |X| = m \leq k$. Since $\kappa'(G) \geq k$, for each $x \in X$, we have $|(\{x\}, Y)_G| \geq k - (m-1)$. Thus $|\partial_G(X)| \geq m(k-(m-1)) = -m^2 + m(k+1)$. As this is a quadratic function with $1 \leq m \leq k$, it follows that $|\partial_G(X)| \geq -m^2 + m(k+1) \geq k$. Thus $\kappa'(G) \geq k$, and so the theorem holds if $2k \leq n \leq 2k+1$.

Hence we assume that $n \geq 2k+2$. By Corollary 2.1, it suffices to show that $\kappa'(G) \geq k$. Let X be an arbitrary nonempty proper vertex subset with satisfying $\emptyset \neq X \subset V(G)$. We will prove $\kappa'(G) \geq k$ by showing that $|\partial_G(X)| \geq k$.

Let $Z = V(G) - V(M)$. By Lemma 2.2 (ii-1), for any $e = uv \in M$, there exists a unique $v(e) \in \{u, v\}$ such that for any $z \in Z$, $v(e)z \in E(G)$. Let $M_v = \{v(e) : e \in M\}$, and $M_u = V(M) - M_v$. Let $m \geq 2$ be the integer satisfying $n = 2k + m$. By Lemma 2.2, for any $v \in M_v$, and any $u \in Z \cup M_u$, we have $vu \in E(G)$.

Subcase 2.1. $(M_u \cup Z) \subseteq X$ (or $(M_u \cup Z) \cap X = \emptyset$). We assume that $M_u \cup Z \subseteq X$ as by symmetry, the proof for $(M_u \cup Z) \cap X = \emptyset$ is similar. As $V(G) = M_u \cup M_v \cup Z$, there exists a $y \in M_v - X \subseteq V(G) - X$. By Lemma 2.2, $|\partial_G(X)| \geq |(M_u \cup Z, \{y\})_G| = |Z| + |M_u| = k + m > k$.

Subcase 2.2. $M_v \subseteq X$ (or $M_v \subseteq V(G) - X$). We assume that $M_v \subseteq X$, as by symmetry, the proof for $M_v \subseteq V(G) - X$ is similar. Then $M_u \cup Z - X \neq \emptyset$. Pick $y \in M_u \cup Z - X$. Then by Lemma 2.2, $|\partial_G(X)| \geq |(M_v, \{y\})_G| = |M_v| = k$.

Subcase 2.3. Both $M_u \cup Z - X \neq \emptyset$ and $X \cap (M_u \cup Z) \neq \emptyset$, and both $M_v - X \neq \emptyset$ and $X \cap M_v \neq \emptyset$. In this subcase, we pick an $x \in X \cap (M_u \cup Z)$ and a $y \in (M_u \cup Z) - X$. It follows by Lemma 2.2 that $|\partial_G(X)| \geq |(\{x\}, M_v - X)_G| + |(M_v \cap X, \{y\})_G| = |M_v - X| + |M_v \cap X| = |M_v| = k$. It follows that we always have $|\partial_G(X)| \geq k$, and so $\kappa'(G) \geq k$. By Corollary 2.1, $\kappa'(G) = k$. This completes the proof of the theorem. □

3 The Main Result

Our main result, Theorem 1.4, will be proved in this section. Theorem 1.3 in Sect. 1 will be utilized here.

Suppose that G is a connected simple graph with $|V(G)| \geq 2$ and $\alpha'(G) = 1$. For each vertex $w \in V(G)$, let $E_G(w)$ be the set of edges in G incident with w. Let $e = uv \in E(G)$ be such that $\{e\}$ is a maximum matching of G. We assume that $d_G(u) \geq d_G(v)$. Since $\{e\}$ is a maximum matching of G, we have $E_G(u) = E(G)$ and $E_G(v) = \{e\}$, unless $G = K_3$. We state this observation as the lemma below.

Lemma 3.1 *Let G be a connected simple graph with $n = |V(G)| \geq 2$ and $\alpha'(G) = 1$. Then either $G = K_3$ or $G = K_{1,n-1}$.*

We will prove a slightly stronger version of Theorem 1.4 in this section, as stated below.

Theorem 3.1 *Let G be a simple graph with $n = |V(G)| \geq 2$ and $k = \alpha'(G)$ such that G is connected when $n = 2k + 2$. Suppose that $\delta(G) \geq k$. Then G is supereulerian if and only if $G \neq K_{1,n-1}$ if n is even or $G \neq K_{2,n-2}$ if n is odd.*

Proof It is routine to show that for any integer $n > 0$, the graphs $K_{1,n-1}$ and $K_{2,n-2}$ (when n is odd) are not supereulerian. It suffices to show the sufficiency. By Theorem (2.2), we have $\kappa'(G) \geq k$.

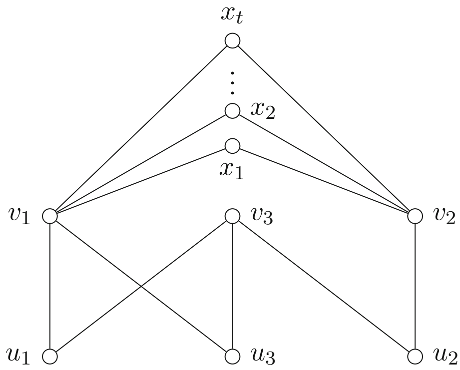
If $k = 1$, then Theorem 3.1 follows from Lemma 3.1. Hence we assume that $k \geq 2$. If $k \geq 4$, then by Theorem 1.3, G must be supereulerian. If $k = 3$, then by Theorem 1.2(iv), G must also be supereulerian as the Petersen graph has a matching of size 5. Hence we may assume that $\kappa'(G) \geq \alpha'(G) = 2$.

By Theorem 1.2(iii), G is supereulerian if and only if G does not have a connected subgraph H and an odd integer $t \geq 3$ such that $G/H = K_{2,t}$. If $H = K_1$, then $G = K_{2,t}$ with $n - 2 = t$ being odd. Hence the sufficiency of Theorem 3.1 holds. Therefore we assume that H is a connected nontrivial simple graph. Let v_H be the vertex in G/H onto which H is contracted, u, u' be the two vertices of degree t in this $K_{2,t}$ and let $v_1, v_2, \cdots, v_t$ be the vertices of degree 2 in this $K_{2,t}$. Let $e_i = uv_i$ and $e'_i = u'v_i$, for each i with $1 \leq i \leq t$. Note that $\{e_i, e'_i : 1 \leq i \leq t\} \subseteq E(G)$, by the definition of contraction.

If v_H has degree 2 in G/H, then we may assume that $v_H = v_1$. Since H is a connected nontrivial simple graph, $E(H)$ contains an edge e_H. It follows that $\{e_H, e_2, e'_3\}$ is a matching of G, contrary to the assumption that $\alpha'(G) = 2$. Hence v_H has degree t in G/H, and so we may assume that $v_H = u$.

Denote $e_i = u_i v_i$ with $u_i \in V(H)$, for $1 \leq i \leq t$. If for some $1 \leq i' < i'' \leq t$, we have $u_{i'} \neq u_{i''}$, then as $t \geq 3$, there exist an $i''' \notin \{i', i''\}$ with $1 \leq i''' \leq t$. Thus $\{e_{i'}, e_{i''}, e'_{i'''}\}$ is a matching of size 3 in G. Hence we may assume that $u_1 = u_2 = \ldots = u_t \in V(H)$. As H is nontrivial, u_1 is a cut vertex of G, and so $\kappa'(H) \geq \kappa'(G) \geq 2$. Since H is simple with $\kappa'(H) \geq 2$, H contains an edge e'_H not incident with u_1. It follows that $\{e'_H, e_1, e'_2\}$ is a matching of size 3 in G, contrary to that $\alpha'(G) = 2$. These contradictions force that $H = K_1$ and so $G = K_{1,n-2}$. This completes the proof of Theorem 3.1. □

Fig. 1 The graph G_t



4 Sharpness Discussion

Theorem 1.4 is sharp in the sense that there exist infinitely many connected nonsupereulerian graphs G with $\delta(G) < \alpha'(G)$, such that G is isomorphic to neither a $K_{1,n}$ nor to a $K_{2,n-2}$.

Example 4.1 For each integer $t \geq 1$,, let G_t be a graph with

$$V(G_t) = \{v_1, v_2, v_3, u_1, u_2, u_3, x_1, x_2, ..., x_t\}$$

and

$$E(G_t) = \{v_1u_1, u_1v_3, v_3u_3, u_3v_1, v_2u_2, u_2v_3\} \cup (\cup_{j=1}^{t}\{v_1x_j, x_jv_2\}),$$

as depicted in Fig. 1. Then $\{G_t\}_{t\geq 1}$ is an infinite family of connected graphs with $\delta(G) = 2$ and $\alpha'(G) = 3$. To see that each G_t is not supereulerian, we assume otherwise that G_t has a spanning closed trail S. Then as S is spanning, we must have $\{u_1v_3, u_2v_3, u_3v_3\} \subset E(S)$. However, as $d_{G_t}(v_3) = 3$, it implies that $d_S(v_3) = 3$, contrary to the assumption that S is an eulerian graph.

Example 4.2 For any odd integers $t \geq 3$ and $k \geq \frac{5(t+1)}{2}$, there exists a 2-connected nonsupereulerian graph G with $\delta(G) = t$ and $\alpha'(G) \geq k$.

Let $\{v_1, v_2, v_3, v_4, v_5\}$ denote the vertex set of a $K_{2,3}$. Obtain a 2-connected graph G by blowing up each of v_1, v_2, v_3 and v_4 into a K_{t+1}, and by blowing up v_5 into a $K_{2k'}$, where $k' = k - 2(t+1)$. Then each of these complete subgraphs of G isomorphic to K_{t+1} contains a perfect matching of size $\frac{t+1}{2}$, and $K_{2k'}$ contains a perfect matching of size k'. It follows that $\alpha'(G) \geq 4 \cdot \frac{t+1}{2} + k' = k$, and $\delta(G) = \delta(K_{t+1}) = t$. Since G is contractible to a nonsupereulerian graph $K_{2,3}$, G must also be nonsupereulerian as well.

Acknowledgements The research of second author is supported in part by NNSFC (Nos. 11771039 and 11771443).

References

1. An, M., Xiong, L.: Supereulerian graphs, collapsible graphs and matchings, Acta Math. Appl. Sin. **39**, 871–877 (2016). (Chinese)
2. Bang-Jensen, J., Maddaloni, A.: Sufficient conditions for a digraph to be supereulerian. J. Graph Theory **79**(1), 8–20 (2015)
3. Berge, C.: Two theorems in graph theory. Proc. Nat. Acad. Sci. USA **43**, 842–844 (1957)
4. Boesch, F.T., Suffel, C., Tindell, R.: The spanning subgraphs of eulerian graphs. J. Graph Theory **1**, 79–84 (1977)
5. Bondy, J.A., Murty, U.S.R.: Graph Theory. Springer, New York (2008)
6. Catlin, P.A.: Reduction method to find spanning Eulerian subgraphs. J. Graph Theory **12**, 29–44 (1988)
7. Catlin, P.A.: Supereulerian graphs: a survey. J.Graph Theory **16**, 177–196 (1992)
8. Chen, Z.H., Lai, H.-J.: Reduction Techniques for Super-Eulerian Graphs and Related Topics-A Survey, Combinatorics and Graph Theory, vol. 95, pp. 53–69. World Science Publishing, River Edge (1995)
9. Chvátal, V., Erdös, P.: A note on Hamiltonian circuits. Discret. Math. **2**, 111–113 (1972)
10. Han, L., Lai, H.-J., Xiong, L., Yan, H.: The Chvátal-Erdös condition for supereulerian graphs and the Hamiltonian index. Discret. Math **310**, 2082–2090 (2010)
11. Harary, F., Nash-Williams, CStJA: On eulerian and hamiltonian graphs and line graphs. Can. Math. Bull. **8**, 701–709 (1965)
12. Jaeger, F.: A note on subeulerian graphs. J. Graph Theory **3**, 91–93 (1979)
13. Lai, H.-J., Yan, H.Y.: Supereulerian graphs and matchings. Appl. Math. Lett. **24**, 1867–1869 (2011)
14. Lai, H.-J., Shao, Y., Yan, H.: An Update on Supereulerian Graphs. WSEAS Trans. Math. **12**, 926–940 (2013)
15. Pulleyblank, W.R.: A note on graphs spanned by Eulerian graphs. J. Graph Theory **3**, 309–310 (1979)
16. Tian, R., Xiong, L.: The Chvátal-Erdös condition for a graph to have a spanning trail. Graphs Comb. **31**, 1739–1754 (2015)
17. Xu, J., Li, P., Miao, Z., Wang, K., Lai, H.-J.: Supereulerian graphs with small matching number and 2-connected hamiltonian claw-free graphs. Int. J. Comput. Math. **91**, 1662–1672 (2014)