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Polynomially determining spanning connectivity of locally connected line graphs

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ABSTRACT

For a graph G, an integer $s \geq 0$ and distinct vertices $u, v \in V(G)$, an $(s; u, v)$-path-system of G is a subgraph H consisting of s internally disjoint (u, v)-paths. The spanning connectivity $\kappa^*(G)$ is the largest integer s such that for any k with $0 \leq k \leq s$ and for any $u, v \in V(G)$ with $u \neq v$, G has a spanning $(k; u, v)$-path-system. It is known that $\kappa^*(G) \leq \kappa(G)$, and determining if $\kappa^*(G) > 0$ is an NP-complete problem. A graph G is maximally spanning connected if $\kappa^*(G) = \kappa(G)$. Let $msc(G)$ and $s_k(G)$ be the smallest integers m and m' such that $L^m(G)$ is maximally spanning connected and $\kappa^*(L^{m'}(G)) \geq k$, respectively. We show that every locally-connected line graph with connectivity at least 3 is maximally spanning connected, and that the spanning connectivity of a locally-connected line graph can be polynomially determined. As applications, we also determine best possible upper bounds for $msc(G)$ and $s_k(G)$, and characterize the extremal graphs reaching the upper bounds. Consequently, former results in Asratian (1996), Sheng et al. (1999) and Xiong et al. (2014) are extended.

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1. The problem

The research problem of this paper focuses on the spanning connectivity of finite graphs. As loops play no roles in our connectivity studies, we assume that graphs under considerations are loopless but with possible multiple edges. The definitions for notation and terms not specifically defined in this paper will follow those in Bondy and Murty [4]. Therefore we shall use $\delta(G)$, $\kappa(G)$ and $\kappa'(G)$ to denote the minimum degree, the connectivity and the edge connectivity of a graph G, respectively. If $x, y \in V(G)$ are vertices of a graph G, then a path (or a trail, respectively) of G with termini x and y is called an (x, y)-path (or an (x, y)-trail, respectively). For an integer $s \geq 0$ and for $u, v \in V(G)$ with $u \neq v$, a subgraph H of G consisting of s internally disjoint (u, v)-paths (or edge-disjoint (u, v)-trails, respectively) is called an $(s; u, v)$-**path-system** (or an $(s; u, v)$-**trail-system**, respectively) of G, and if $V(H) = V(G)$, then H is a spanning $(s; u, v)$-path-system (or a spanning $(s; u, v)$-**trail-system**, respectively).

By the well-known Menger's Theorems (Theorems 9.1 and 9.7 of [4]), we define a graph G to be k-**connected** (or k-edge-connected, respectively) if for any pair of distinct vertices u and v, G contains a $(k; u, v)$-path-system (or a $(k; u, v)$-trail-system, respectively). Therefore, the connectivity $\kappa(G)$ of a graph G (or the edge-connectivity $\kappa'(G)$ of G, respectively) equals the maximum number k such that for every pair of distinct vertices u and v, G has a $(k; u, v)$-path-system (or a

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$(k; u, v)$-trail-system, respectively). In [21], an $(s; u, v)$-path-system and a spanning $(s; u, v)$-path-system are also called a k-container and a k^*-container, respectively. The **spanning connectivity** $\kappa^*(G)$ of a graph G is the largest integer s such that for any integer k with $0 \le k \le s$ and for any $u, v \in V(G)$ with $u \ne v$, G has a spanning $(k; u, v)$-path-system. A graph G is s-**spanning connected** if $\kappa^*(G) \ge s$.

The concept of spanning connectivity of a network was initially formulated by Hsu in [20] to evaluate the performance of communication of interconnected networks. Many studies have investigated spanning connectivity and its applications in various types of communication effectiveness and fault-tolerant spanning laceability in interconnection networks as well as in the diffusion dynamics of multilayer networks. As of today, there have been lots of studies on spanning connectivity and an edge counterpart of it, as seen in [1,7,10,18,20,27–30,39–41], among others. As shown in [21], many former studies on spanning connectivity have been focused on results involving degree conditions to assure a simple graph to have spanning connectivity at least a given integer s; as well as investigations of spanning connectivity of certain family of graphs such as Harary graphs, hypercubes and hypercube-like graphs.

By definition, a graph G is hamiltonian if and only if for any distinct vertices $u, v \in V(G)$, G has a spanning $(2; u, v)$-path-system. Thus as remarked in [21], spanning connectivity of graphs can be viewed as a hybrid concept of Hamiltonicity and connectivity. Following [4], a graph G is **Hamilton-connected** if for any $u, v \in V(G)$ with $u \ne v$, G has a spanning (u, v)-path P. Thus $\kappa^*(G) \ge 1$ implies that G is Hamilton-connected. It is well known that every Hamilton-connected graph with at least 4 vertices must be 3-connected. Hence the following fact (1) is observed.

$$\text{If } G \text{ is a graph with } |V(G)| \ge 4 \text{ and } \kappa^*(G) > 0, \text{ then } \kappa(G) \ge 3. \tag{1}$$

As every Hamilton-connected graph must also be hamiltonian, we conclude that a graph G is Hamilton-connected if and only if $\kappa^*(G) > 0$. Thus determining if $\kappa^*(G) > 0$ in general is an NP-complete problem. One of the motivations of this research is to seek nontrivial common families of graphs in which spanning connectivity can be polynomially determined.

As it is known that the connectivity of a graph can be polynomially determined, (see, for example, [13,14]), the problem whether high connectivity could imply positive spanning connectivity was considered. While the complete bipartite graphs indicate that in general, high connectivity of a graph G does not warrant $\kappa^*(G) > 0$, researchers have been investigating graph families in which high connectivity of a graph G in these families would imply that $\kappa^*(G) > 0$. Thomassen in [38] first conjectured that every 4-connected line graph is hamiltonian. This most fascinating conjecture has attracted many researchers.

Let $L(G)$ denote the **line graph** of a graph G, which is a simple graph with vertex set $E(G)$, and with edge set $E(L(G)) = \{e'e'' : e', e'' \in E(G) \text{ and } e', e'' \text{ are adjacent in } G\}$. A graph that does not have an induced subgraph isomorphic to $K_{1,3}$ is a **claw-free** graph. Beineke [3] and Robertson (Page 74 of [19]) showed that line graphs are claw-free graphs. By several ingenious closure concepts developed by Ryjáček [33] and by Ryjáček and Vrána [34], Thomassen's above-mentioned conjecture is shown to be equivalent to each of the following.

Conjecture 1.1. *Let G be a graph and let Γ be a claw-free graph.*
(i) (Thomassen [38] and, Kučzel and Xiong [23]) Every 4-connected line graph has spanning connectivity at least 2.
(ii) (Matthews and Sumner [31], and Ryjáček and Vrána [34]) Every 4-connected claw-free graph has spanning connectivity at least 2.

There have been intensive studies towards Conjecture 1.1, as shown in the surveys [12,16,17]. By Menger's Theorem [32], see also Theorem 9.1 of [4], for any graph G, we always have $\kappa(G) \ge \kappa^*(G)$. Thus graphs G with $\kappa(G) = \kappa^*(G)$ are of particular interests. In view of (1), we define a connected graph G to be **maximally spanning connected** if both $\kappa(G) \ge 3$ and $\kappa(G) = \kappa^*(G)$. A similar concept of super spanning connected graph is formerly defined in [21], which implies that K_2 is super spanning connected. By the definition in this paper, K_2 is not maximally spanning connected. As examples, complete graphs of order at least 4 are maximally spanning connected, but complete bipartite graphs of any orders are not maximally spanning connected.

As of today, little is known on maximally spanning connected graph families other than the complete graphs and a few others. This motivates the current study. For a vertex $v \in V(G)$, define $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The vertex v is locally connected if the induced subgraph $G[N_G(v)]$ is connected. A graph G is **locally connected** if every vertex v of G is locally connected. Asratian [2] and Y. Sheng, F. Tian and B. Wei [37] studied connectivity conditions for a locally connected claw-free graph G to have spanning connectivity at least 2. As line graphs are claw-free, their result is also valid for line graphs. A class of maximally spanning connected line graphs has also been studied in [22] and [7].

Theorem 1.2. *Let G be a connected graph.*
(i) (Asratian [2] and Y. Sheng, F. Tian and B. Wei [37]) If G is an locally connected claw-free graph with $\kappa(G) \ge 3$, then $\kappa^*(G) \ge 2$.
(ii) (Huang and Hsu [22], and Chen et al. [7]) Let $k \ge 3$ be an integer. If a graph G has k-edge-disjoint spanning trees, then $L(G)$ is maximally spanning connected.

Theorem 1.3(i), one of our main results, has identified a new family of graphs whose line graphs are maximally spanning connected, which extends Theorem 1.2(i). As connectivity of a graph can be polynomially determined, Theorem 1.3(ii) follows from Theorem 1.3(i).

Theorem 1.3. *Each of the following holds.*
(i) Every 3-connected, locally connected line graph $L(G)$ is maximally spanning connected.
(ii) The spanning connectivity of a locally connected line graph can be polynomially determined.

For an integer $m > 0$, define $L^0(G) = G$, and the iterated line graph $L^m(G) = L(L^{m-1}(G))$. A path P of G is a **divalent path** of G if every internal vertex of P has degree 2 in G. Following [24,26,41,42], define

$$\ell(G) = \max\{m : G \text{ has a length } m \text{ divalent path that is not in a } K_3\}. \quad (2)$$

For discussional convenience, we in this paper denote $\mathcal{G}$ to be the family of all connected nontrivial graphs that are not isomorphic to a path, a cycle or a $K_{1,3}$. To study iterated line graphs, we only consider graphs in $\mathcal{G}$. The iterated line graph index problem is also an intensively studied topic in graph theory. Chartrand and Wall in [6] initiated the study of the smallest integer $k \geq 0$, called the hamiltonian index $h(G)$ of a graph G, such that the iterated line graph $L^k(G)$ becomes hamiltonian. Other hamiltonian like indices were defined and studied by Clark and Wormald in [11]. More generally, we have the following definition.

Definition 1.4 (*[25]*)**.** Let $\mathcal{P}$ denote a graphical property and G be a connected graph $G \in \mathcal{G}$. Then $\mathcal{P}(G)$, the $\mathcal{P}$-**index** of G, is defined by

$$\mathcal{P}(G) = \begin{cases} \min\{k : L^k(G) \text{ has property } \mathcal{P}\} & \text{if for some integer } j > 0,\ L^j(G) \text{ has property } \mathcal{P}, \\ \infty & \text{otherwise.} \end{cases}$$

Clark and Wormald in [11] studied the existence of the indices for the properties of being edge-hamiltonian, pancyclic, vertex-pancyclic, edge-pancyclic, hamiltonian-connected, respectively. Additional studies of these indices can also be found in [25]. In [35], Ryjáček, Woeginger and Xiong indicated that determining the value of $h(G)$ is a difficult problem. The index problem for graphical properties has been intensively studied, as seen in [6,8,9,11,15,24–26,35,36,41,42], among others.

In this research, we consider some indices related to spanning connectivity of graphs. For an integer $k \geq 2$, and a graph $G \in \mathcal{G}$, let $s_k(G)$ be the smallest integer m such that $\kappa^*(L^m(G)) \geq k$. When k is small, upper bounds for $s_k(G)$ have been investigated.

Theorem 1.5. *Let $G \in \mathcal{G}$ be a connected graph with maximum degree $\Delta(G)$.*
(i) (Chen et al. Theorem 22 of [9]) $s_2(G) \leq |V(G)| - \Delta(G) + 1$.
(ii) (Xiong et al. Theorem 1.3 of [41]) $s_3(G) \leq \ell(G) + 6$.

The results in Theorem 1.5 also motivate our current study. A divalent path P of G is a **bridge divalent path** if every edge of P is a cut edge of G; and is a **divalent** (s, t)-**path** if the two end vertices of P are of degree s and t, respectively. The next main result studies best possible bounds for $s_k(G)$. When $k = 2$, Theorem 1.6(iv) improves Theorem 1.5(i) and when $k = 3$, Theorem 1.6(iii) sharpens Theorem 1.5(ii).

Theorem 1.6. *Let $G \in \mathcal{G}$ be a graph and let $k \geq 3$ be an integer.*
(i) $s_2(G) \leq \ell(G) + 2$.
(ii) $s_k(G) \leq \ell(G) + k - 1$. Furthermore, $s_k(G) = \ell(G) + k - 1$ only if for some integer $t \geq 3$, G has a bridge divalent $(3, t)$-path of length $\ell(G)$.
(iii) $s_3(G) = \ell(G) + 2$ if and only if for some integer $t \geq 3$, G has a bridge divalent $(3, t)$-path of length $\ell(G)$.
(iv) $s_k(G) \leq |V(G)| - \Delta(G) + k - 2$.

For a graph $G \in \mathcal{G}$, define $msc(G)$ to be the smallest integer m such that $L^m(G)$ is maximally spanning connected. A best possible upper bound for $msc(G)$ is also obtained.

Theorem 1.7. *Let $G \in \mathcal{G}$ be a graph.*
(i) $msc(G) \leq \ell(G) + 2$, and for any integer $m \geq \ell(G) + 2$, $\kappa(L^m(G)) = \kappa^*(L^m(G))$. Moreover, $msc(G) = \ell(G) + 2$ if and only if for some integer $t \geq 3$, G has a bridge divalent $(3, t)$-path of length $\ell(G)$.
(ii) $msc(G) \leq |V(G)| - \Delta(G) + 2$, and for any integer $m \geq |V(G)| - \Delta(G) + 2$, $\kappa(L^m(G)) = \kappa^*(L^m(G))$.

The tools to assist our arguments to prove the main results are summarized and developed in the next section. In Section 3, we will prove the main results. Related open problems will be discussed in the last section.

2. Mechanisms

To facilitate our proofs of the main results, a number of tools will be displayed and developed in this section. Given a graph G and an integer $i \geq 0$, let $D_i(G)$ be the set of all vertices of degree i in G and $O(G) = \cup_{j\geq 0} D_{2j+1}(G)$ be the set of all odd degree vertices in G. By an n-**cycle** we mean a cycle C with $|V(C)| = n$; and C is a **short cycle** if $2 \leq |E(C)| \leq 3$. Extending the definition in [5], a graph G is **triangular** if every edge $e \in E(G)$ lies in a short cycle C_e of G.

By definition, a spanning $(2; u, v)$-path system is a Hamilton cycle and a spanning $(2; u, v)$-trail system is a spanning eulerian subgraph in a graph G. Harary and Nash-Williams proved a well-known relationship between Hamilton cycles in $L(G)$ and dominating eulerian subgraphs in G.

Theorem 2.1 (*Harary and Nash-Williams [19]*). *Let G a graph with $|E(G)| \geq 3$. The following are equivalent.*
(i) $L(G)$ has a Hamilton cycle.
(ii) G has an eulerian subgraph H such that $E(G - V(H)) = \emptyset$.

Chen et al. in [7] extended Theorem 2.1 by displaying a relationship between spanning connectivity in $L(G)$ and certain type of dominating trail systems in G. This will be a key tool in our arguments. As in [4], a trail in a graph G can be expressed as a sequence

$$T = v_0, e_1, v_1, e_2, \ldots, e_k, v_k \tag{3}$$

such that for each i with $1 \leq i \leq k$, the edge e_i is incident with the two vertices v_{i-1} and v_i, and such that if $1 \leq i < j \leq k$, then $e_i \neq e_j$. A trial T (with the notation in (3)) is **open** (or **closed**, respectively) if $v_0 \neq v_k$ (or $v_0 = v_k$, respectively). We define the **internal vertices** of the trail in (3) to be the set $\{v_1, v_2, \ldots, v_{k-1}\}$, if T is open, and to be $V(T)$ if T is closed. As in an open trail, vertices may occur more than once, it is also possible for the end vertices v_0 or v_k in (3) to be internal. A trail T of G is **dominating** if every edge of G is incident with an internal vertex of T, and is **spanning** if it is dominating with $V(T) = V(G)$.

Let $e', e'' \in E(G)$ be two edges of G. A trail T of G is an (e', e'')-**trail** of G if the two end edges of T are e' and e'', respectively. As an example, the trail in (3) is an (e_1, e_k)-trail. Two (e', e'')-trails T_1 and T_2 are **internally edge-disjoint** if $E(T_1) \cap E(T_2) = \{e', e''\}$. For a given integer $s \geq 0$, an $(s; e', e'')$-trail system in G is a subgraph J consisting of s internally edge-disjoint (e', e'')-trails $(T_1, T_2, \ldots, T_s)$. A vertex v is an **internal vertex of** J if for some i with $1 \leq i \leq s$, v is an internal vertex of T_i. For an $(s; e', e'')$-trail system J, define

$$\partial_G(J) = \{e \in E(G) - E(J) : \text{e is incident with an internal vertex of } J\}.$$

An $(s; e', e'')$-trail system J is **dominating** if $E(G) - E(J) = \partial_G(J)$, and is **spanning** if it is dominating with $V(G) = V(J)$.

Theorem 2.2 (*Chen et al. Theorem 2.1 of [7]*). *Let G be a graph with $|E(G)| \geq 3$ and let $s \geq 3$ be an integer. Then $\kappa^*(L(G)) \geq s$ if and only if for any edge $e', e'' \in E(G)$, and for each integer k with $1 \leq k \leq s$, G has a dominating $(k; e', e'')$-trail-system.*

Recall that $\ell(G)$ is defined in (2), the connectivity of iterated line graphs have been investigated. The following former results will be useful in our arguments.

Theorem 2.3. *Let $k > 0$ be an integer and $G \in \mathcal{G}$ be a graph.*
(i) (Zhang et al. Lemma 3.2 [42]) If $G \in \mathcal{G}$, then $L^{\ell(G)}(G)$ is triangular.
(ii) (Zhang et al. Proposition 2.3 [43]) If G is a connected triangular simple graph, then $L(G)$ is triangular. If, in addition, G is k-connected, then $L(G)$ is $(k+1)$-connected.
(iii) For any integer $m \geq \ell(G) + k - 1$, $\kappa(L^m(G)) \geq m - \ell(G) + 1 \geq k$.

Proof. It suffices to justify (iii). Let $\ell = \ell(G)$. By Theorem 2.3(i), $L^{\ell}(G)$ is 1-connected and triangular. By repeated application of Theorem 2.3 (ii), $\kappa(L^m(G)) \geq m - \ell(G) + 1 \geq k$. ■

In the following of this section, we always assume that $G \in \mathcal{G}$ is a connected graph. We shall show certain relationship between the subgraphs of a graph G and the subgraphs of its line graph $L(G)$. Let $\mathcal{H}(G)$ denote the collection of all edge-induced subgraphs of G and let $\mathcal{L}(G)$ denote the collection of all induced subgraphs of $L(G)$. Thus for any subgraph $H \in \mathcal{H}(G)$, we have $L(H) = L(G[E(H)]) \in \mathcal{L}(G)$. If $J \in \mathcal{L}(G)$ then $V(J) \subseteq E(G)$ and so the edge-induced subgraph $G[V(J)] \in \mathcal{H}(G)$ satisfying $L(G[V(J)]) = J$. Thus we may view $L : \mathcal{H}(G) \to \mathcal{L}(G)$ as a bijective mapping and let L^{-1} denote the inverse mapping of L. By the definition of iterated line graphs, if $s \geq 1$ is an integer, then we denote L^s to be the mapping that maps subgraphs in $\mathcal{H}(G)$ into subgraphs in $L^s(G)$, and we use L^{-s} to denote the pull back mapping that sends the induced subgraphs in $L^s(G)$ back to the subgraphs in $\mathcal{H}(G)$. For notational convenience, If j and k are nonnegative integers, then we also use L^j to denote the corresponding mapping from $\mathcal{H}(L^k(G))$ to $\mathcal{L}(L^{k+j}(G))$, and L^{-j} its corresponding pull back mapping. Using the notation thus defined, we summarize some observations from the definition of line graphs in the following proposition.

Proposition 2.4. *Let $G \in \mathcal{G}$ be a connected graph and let $L : \mathcal{H}(G) \to \mathcal{L}(G)$ denote the bijection mapping defined above. For each edge $e \in E(G)$ (also viewed as the subgraph induced by the single edge e), define $v_e = L(e)$. Each of the following holds.*
(i) For each edge $e \in E(G)$, the vertex v_e is a cut vertex of $L(G)$ if and only if $\{e\}$ is an essential edge-cut of G.
(ii) Let $e_1, e_2 \in E(G)$. Then if $v_{e_1}v_{e_2}$ is an edge in $E(L(G))$ not lying in a complete graph of order at least 3 in $L(G)$, then $G[\{e_1, e_2\}]$ is a divalent path of G.
(iii) Let P be a divalent path in G with $|E(P)| = h > 0$. For any integer k with $0 \leq k < h$, $L^k(P)$ is a divalent path in $L^k(G)$ with $|E(L^k(P))| = h - k$, and $L^h(P)$ is a vertex of $L^h(G)$. Furthermore, if P is a bridge divalent path of G, then $L^k(P)$ is also a bridge divalent path in $L^k(G)$, and $L^h(P)$ is a cut vertex of $L^h(G)$.
(iv) Let s and t be integers with $s \geq t \geq 2$. If v is a cut vertex of $L^s(G)$, then, viewing v as a subgraph induced by the single vertex v, $L^{-t}(v)$ is a bridge divalent path of length t in $L^{s-t}(G)$ in which every edge is an essential cut edge; likewise, if e is an edge which is not in a complete subgraph of order at least 3 in $L^s(G)$, then $L^{-t}(e)$ is a divalent path of length $t+1$ in $L^{s-t}(G)$.
(v) Let $e', e'' \in E(G)$ be distinct edges and let $s \geq 1$ be an integer. If $L(G)$ has an $(s; e', e'')$-path-system, then G has an $(s; e', e'')$-trail-system.

Proof. Proposition 2.4(i), (ii) and (iii) follow from the definitions of line graphs, of divalent paths and of bridge divalent paths. To prove (iv), let $v \in V(L^s(G))$ be a vertex. Then there must be an edge $e \in E(L^{s-1}(G))$ such that $v = v_e := L(e)$, or $e = L^{-1}(v)$. Since $G \in \mathcal{G}$, it follows by the definition of line graphs that $L^{s-1}(G) \in \mathcal{G}$, and so $L^s(G) - v$ has at least two components, which implies that e is a bridge divalent path of length 1 in $L^{s-1}(G)$ and $\{e\}$ is an essential edge-cut of $L^{s-1}(G)$; and $L^{-2}(v)$ is a bridge divalent path of length 2 in $L^{s-2}(G)$ in which every edge is an essential cut edge of $L^{s-2}(G)$. Inductively, for $t \geq 2$, $L^{-t}(v)$ is a bridge divalent path of length t in $L^{s-t}(G)$ in which every edge is an essential cut edge of $L^{s-t}(G)$. The proof for the edge part is similar and so it is omitted.

We are to prove (v). Let H be an $(s; e', e'')$-path-system consisting of s internally disjoint (e', e'')-paths $P_1, P_2, \ldots, P_s$. Choose such an $(s; e', e'')$-path-system H. For each i with $1 \leq i \leq s$, as $V(P_i) \subseteq E(G)$, $G[V(P_i)]$ is an edge-induced connected subgraph in G containing both edges e' and e'', and so $G[V(P_i)]$ contains an (e', e'')-trail T_i. Since $P_1, P_2, \ldots, P_s$ are internally disjoint in $L(G)$, we conclude that $P_1, P_2, \ldots, P_s$ are internally edge-disjoint in G, and so G has an $(s; e', e'')$-trail-system. ■

3. The main results and their proofs

The **symmetric difference** of two sets X and Y, is

$$X \triangle Y = X \cup Y - (X \cap Y).$$

Let G be a connected graph and $k > 0$ be an integer. An edge-cut X of G is an **essential k-edge-cut** of G if $|X| = k$ and each side of $G - X$ has an edge. The **essential edge-connectivity** of a connected graph G, denoted by $ess'(G)$, is the smallest integer k such that G has an essential k-edge-cut, if G has at least one essential edge cut; or $ess'(G) = |E(G)| - 1$, if G does not have an essential edge cut. We say that G is essentially k-edge-connected if $ess'(G) \geq k$. By the definition of a line graph, we observe that

$$\kappa(L(G)) \geq k \text{ if and only if } ess'(G) \geq k. \tag{4}$$

3.1. Maximally spanning connectedness in locally connected line graphs

We start with some preliminary results to understand the impact of local connectedness of $L(G)$ on the graph G. For a vertex $v \in V(G)$, define $E_G(v) = \{e \in E(G) : e \text{ is incident with } v \text{ in } G\}$.

Lemma 3.1. *Let G be a connected graph with $|E(G)| \geq 3$. The following are equivalent.*
(i) $L(G)$ is locally connected.
(ii) Every edge $e = uv \in E(G)$ with $\min\{d_G(u), d_G(v)\} \geq 2$ lies in a short cycle C_e of G.

Proof. Assume (i). Let $e = uv \in E(G)$ be an edge with $\min\{d_G(u), d_G(v)\} \geq 2$, which is not lying in a cycle of length 2. By the definition of a line graph, $N_{L(G)}(e) = (E_G(u) \cup E_G(v)) - \{e\}$. Since $\min\{d_G(u), d_G(v)\} \geq 2$, each of $E_G(u)$ and $E_G(v)$ is not empty. Since $L(G)[N_{L(G)}(e)]$ is connected, there must be an edge $e_u \in E_G(u)$ and $e_v \in E_G(v)$ such that $e_u e_v \in E(L(G))$. It follows that e_u and e_v would share a common vertex in G, and so $C_e = G[\{e, e_u, e_v\}]$ is a 3-cycle in G that contains e. Thus (ii) must hold.

Conversely, we assume that (ii) holds. Let $e \in V(L(G))$ be given. We shall show that e is a locally connected vertex in $L(G)$. By symmetry, we assume that $e = uv \in E(G)$ with $|N_G(u)| \geq |N_G(v)|$. If $|N_G(v)| = 1$, then $N_{L(G)}(e) = E_G(u) - \{e\}$, and so $L(G)[N_{L(G)}(e)]$ is a complete graph. Assume that $|N_G(v)| \geq 2$. By definition, $N_{L(G)}(e) = E_G(u) \cup E(G(v) - \{e\})$, and so $N_{L(G)}(e)$ is spanned by two complete subgraphs $L(G)[E_G(u) - \{e\}]$ and $L(G)[E_G(v) - \{e\}]$. By (ii), e lies in a short cycle C_e of G. If $E(C_e) = \{e, e_1\}$, then $e_1 \in (E_G(u) \cap E(G(v)) - \{e\})$, and so $L(G)[N_{L(G)}(e)]$ is connected. Now assume that $E(C_e - e) = \{e_1, e_2\}$. We may assume that $e_1 \in E_G(u)$ and $e_2 \in E_G(v)$. Since C_e is a 3-cycle, e_1 and e_2 are incident with a common vertex in G, and so in $L(G)$, $e_1 e_2 \in E(L(G))$. This implies that in any case, (i) must hold. ■

In view of Lemma 3.1, we define a graph G to be **almost triangular** if every edge $e = uv \in E(G)$ with $\min\{d_G(u), d_G(v)\} \geq 2$ lies in a short cycle in G. A subgraph H is **near spanning** in G if $V(G) - D_1(G) = V(H)$. The next lemma is useful.

Lemma 3.2. *Let $s \geq 1$ be an integer and G be a connected almost triangular graph with $ess'(G) \geq 3$. For any $e', e'' \in E(G)$, if G has an $(s; e', e'')$-trail system, then G has a near spanning and dominating $(s; e', e'')$-trail system.*

Proof. Suppose that G has an $(s; e', e'')$-trail system. Choose an $(s; e', e'')$-trail system J of G such that

$$|V(J)| + |E(J)| \text{ is maximized, among all } (s; e', e'')\text{-trail systems of } G, \tag{5}$$

and subject to (5),

$$|\partial_G(J)| \text{ is as large as possible.} \tag{6}$$

Let $J = (T_1, T_2, \ldots, T_s)$, where each T_i is an (e', e'')-trail, $1 \leq i \leq s$.

Claim 1. $V(J) = V(G) - D_1(G)$.

By contradiction, and as G is connected, we assume that there must be a vertex $v \in V(G) - (V(J) \cup D_1(G))$ such that for some $w_v \in V(J)$, $vw_v \in E(G)$. Assume further that there exists a vertex $v \in V(G) - V(J)$ such that for some i with $1 \le i \le s$, w_v is an internal vertex T_i. As $v \notin D_1(G)$ and w_v is an internal vertex of J, we have $\min\{d_G(v), d_G(w_v)\} \ge 2$. Since G is almost triangular, there must be a short cycle C_{vw} with $vw_v \in E(C_{vw})$. Since $v \notin V(J)$, both edges incident with v in C_{vw} are not in J. If $|E(C_{vw})| = 2$, then T_i can be extended to $G[E(T_i) \cup E(C_{vw})]$, which is also an (e', e'')-trail, internally edge-disjoint from the other (e', e'')-trail in J, contrary to (5). Hence we assume that $|E(C_{vw})| = 3$.

Let e_v denote the edge in C_{vw} that is not incident with v. We assume that if $e_v \in E(J)$, (including the case when $e_v \in \{e', e''\}$), then $e_v \in E(T_i)$. Define

$$T_i' = \begin{cases} G[E(T_i) \triangle E(C_{vw})] & \text{if } e_v \notin \{e', e''\} \\ G[E(T_i) \cup E(C_{vw})] & \text{if } e_v \in \{e', e''\} \end{cases}.$$

Then T_i' is also an (e', e'')-trail. As $v \notin V(J)$, T_i' is also internally edge-disjoint from T_j, where $1 \le j \le s$ and $j \ne i$. It follows that $J' = (T_1, \dots, T_{i-1}, T_i', T_{i+1}, \dots, T_s)$ is an $(s; e', e'')$-trail system with $|V(J)| + |E(J)| < |V(J')| + |E(J')|$, contrary (5). Hence we assume that no vertex in $V(G) - (D_1(G) \cup V(J))$ is incident with an internal vertex of any T_i. This implies that any path connecting a vertex in $V(J)$ and a vertex in $V(G) - V(J)$ must use at least one of the edges $\{e', e''\}$. This implies that $\{e', e''\}$ is an essential edge cut of G, contrary to the assumption that $ess'(G) \ge 3$. This justifies Claim 1.

By Claim 1, J is near spanning. If J is also dominating, then done. Hence we assume that

$$\text{there exists an edge } e_0 \in E(G) - E(J) \text{ not incident with any internal vertex of } J. \tag{7}$$

Suppose that $e_0 = u_0v_0$ is incident with $v_0 \in D_1(G)$. By $ess'(G) \ge 3$, $d_G(u_0) \ge 4$. By (7), u_0 cannot be an internal vertex of J. Hence we may assume that $e' = u'v'$ with $u_0 = u'$ not being an internal vertex of J. This implies that v' must be an internal vertex of J. It follows that we have $\min\{d_G(u_0), d_G(v')\} \ge 2$. Since G is almost triangular, G has a short cycle C_0 containing $e' = u_0v$. If $E(C_0) - E_G(u_0)$ has an edge in $E(J)$, then we may assume that this edge is in T_1. Define $T_1'' = G[(E(T_1) \triangle E(C_0)) \cup \{e'\}]$. Thus $J'' = (T_1'', T_2, \dots, T_s)$ is also an $(s; e', e'')$-trail system of G with $|V(J)| + |E(J)| \le |V(J'')| + |E(J'')|$ and $\partial_G(J) \cup \{e_0\} \subseteq \partial_G(J'')$, contrary to (6).

Hence e_0 is not incident with a vertex of degree 1 in G. By (7), we may assume that $e' = u'v'$, $e'' = u''v''$, $e_0 = u'u''$ and for any i with $1 \le i \le s$, T_i is an (u', u'')-trial with the first edge being e' and the last edge being e''. Since G is triangular, $e' = u'v'$ lies in a short cycle $C_{e'}$ of G. Let e_1' denote the edge in $C_{e'} - \{e'\}$ that is incident with u'. If $e_1' \in E(J)$, then u' is an internal vertex of J, contrary to (7). Hence $e_1' \notin E(J)$. By definition, $|E(C_{e'})| \in \{2, 3\}$. When $E(C_{e'}) = \{e', e_1', e_2'\}$, we assume by symmetry that, if $e_2' \in E(J)$, then $e_2' \in E(T_1)$. With this assumption, define $T_1''' = G[(E(T_1) \triangle E(C_{e'})) \cup \{e'\}]$. Thus $J''' = (T_1''', T_2, \dots, T_s)$ is also an $(s; e', e'')$-trail system of G with $|V(J)| + |E(J)| \le |V(J''')| + |E(J''')|$ and $\partial_G(J) \cup \{e_0\} \subseteq \partial_G(J'')$, contrary to (6). Thus every possibility of the assumption (7) always leads to a contradiction, and so J must be dominating. This completes the proof of the lemma. ■

Lemma 3.3. *Let $k \ge 1$ be an integer and G be a graph.*
(i) Let $e', e'' \in E(G)$. If $L(G)$ has a $(k; e', e'')$-trial system, then G has a $(k; e', e'')$-trail system.
(ii) Suppose that G is a connected almost triangular graph with $ess'(G) \ge 3$. Then $L(G)$ is maximally spanning connected.

Proof. For any $e', e'' \in E(G)$, assume that $L(G)$ has a $(k; e', e'')$-trial system H consisting of internally edge-disjoint (e', e'')-trails $P_1, P_2, \dots, P_k$ with $|E(H)|$ minimized. Then by the minimality of $|E(H)|$, each P_i in H must be a path. By Proposition 2.4(v), G has a $(k; e', e'')$-trail system. This proves (i).

Let $\kappa(L(G)) = k$. As $ess'(G) \ge 3$, by (4), $k \ge 3$. Thus for every integer s with $1 \le s \le k$, $L(G)$ is s-connected. By Menger theorem, for any $e', e'' \in E(G)$, $L(G)$ has an $(s; e', e'')$-path system. By Lemma 3.3(i), G has an $(s; e', e'')$-trail system.

Fix an integer s with $1 \le s \le k$, and let e', e'' be arbitrarily chosen edges in G. Since G is a connected almost triangular graph with $ess'(G) \ge 3$, and since G has an $(s; e', e'')$-trail system, it follows by Lemma 3.2 that G must also have a dominating $(s; e', e'')$-trail system. By Theorem 2.2, $L(G)$ has a spanning $(s; e', e'')$-path system. As e', e'' are arbitrarily chosen, it follows from the definition of spanning connectivity that $k = \kappa(L(G)) \ge \kappa^*(L(G)) \ge k$, completing the proof for (ii). ■

Proof of Theorem 1.3(i). Since $L(G)$ is locally connected, by Lemma 3.1, G is almost triangular. Since $\kappa(L(G)) \ge 3$, it follows from (4) that G must be essentially 3-edge-connected. By Lemma 3.3(ii), $L(G)$ is maximally connected. This proves Theorem 1.3(i). ■

3.2. The spanning connected indices of graphs

The main purpose of this subsection is to prove Theorems 1.6 and 1.7. Before proving these theorems, we present the following examples, which are useful to illustrate the process determining the graphs in Theorems 1.6(i) and 1.7(i) that reach the upper bounds. (See Fig. 1 for an illustration of the iterated line graphs of a bridge divalent $(3, t)$-path with $\ell = 2$ in Example 3.4.)

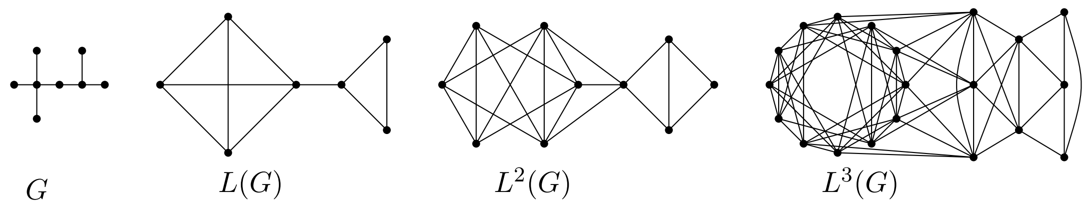

Fig. 1. Illustration for Example 3.4.

Example 3.4. Let d', d'' and ℓ be positive integers with $d'' \neq 2$, $d' \geq 3$ and $\ell \geq 2$; and $G \in \mathcal{G}$ be a graph with $\ell(G) = \ell$ that contains a bridge divalent (d', d'')-path $P = v_0e_1v_1 \cdots v_{\ell-1}e_\ell v_\ell$ with $d_G(v_0) = d'$ and $d_G(v_\ell) = d''$. If $d' = 3$, then let $e'_1, e'_2, e_1 \in E(G)$ be the three edges incident with v_0 in G. It is routine to apply Proposition 2.4 to verify the following.
(i) For any j with $0 \leq j \leq \ell - 1$, $L^j(G)$ has a bridge divalent (d', d'')-path.
(ii) If $d' = 3$ and $d'' \geq 3$, then $L^\ell(G)$ has a cut vertex which is incident with an essential edge cut of size 2.
(iii) If $d' = 3$ and $d'' \geq 3$, then $L^{\ell+1}(G)$ is triangular and has a vertex 2-cut.
(iv) If $d' = 3$ and $d'' \geq 3$, then $msc(G) \geq \ell + 2$.
(v) If G does not have a bridge divalent $(3, t)$-path for some integer $t \geq 3$, then $L^\ell(G)$ is essentially 3-edge-connected.

Proof. By the definition of line graphs, an edge e incident with a vertex of degree d lies in a maximal clique of order d in the line graph. Hence the edge incident with the vertex of degree s in a bridge divalent (s, t)-path of length at least 2 becomes a vertex in a bridge divalent path of degree s. By Proposition 2.4(iii), $L^j(G)$ has a bridge divalent (d', d'')-path. This justifies (i).

Assume that $d' = 3$ and $d'' \geq 3$. By (i), $L^{\ell-1}(G)$ has a bridge divalent $(3, d'')$-path of length 1, which is a cut edge $f_0 = w_1w_2$ in $L^{\ell-1}(G)$. Assume that the edges incident with w_1 in $L^{\ell-1}(G)$ are f_0, f_1, f_2. Since $d'' \geq 3$, f_0 is an essential cut edge. By Proposition 2.4(ii), in $L^\ell(G)$, f_0 is a cut vertex, and so $\{f_0f_1, f_0f_2\}$ is an essential edge cut in $L^\ell(G)$. This proves (ii).

As (iii) implies that $L^{\ell+1}(G)$ is not 3-connected, (iv) follows from (iii), and so it suffices to justify (iii). By Theorem 2.3(i) and (ii), $L^{\ell+1}(G)$ is triangular. By (ii), the essential edge cut of size 2 in $L^\ell(G)$ becomes a vertex 2-cut in $L^{\ell+1}(G)$. Hence (iii) must hold.

Now assume that G does not have a bridge divalent $(3, t)$-path for some integer $t \geq 3$. Let X be an essential edge cut of $L^\ell(G)$. By Theorem 2.3(i), $L^\ell(G)$ is triangular, and so $|X| \geq 2$. By contradiction, we assume that $X = \{f_1, f_2\}$ is an edge cut of $L^\ell(G)$. As $L^\ell(G)$ is triangular, f_1, f_2 must be incident with a common vertex w_0 in $L^\ell(G)$. Since $\{f_1, f_2\}$ is an essential edge cut of $L^\ell(G)$, w_0 must be a cut vertex of $L^\ell(G)$. By Proposition 2.4, for some integer $t \geq 3$, $L^{-\ell}(w_0)$ is a bridge divalent $(3, t)$-path of length ℓ in G, contrary to the assumption of (v). ■

Lemma 3.5. *Let $G \in \mathcal{G}$ with $\ell = \ell(G)$. Then*

$$L^{\ell(G)+1}(G) \text{ is triangular and } \kappa'(L^{\ell(G)+1}(G)) \geq \kappa(L^{\ell(G)+2}(G)) \geq 3. \tag{8}$$

Proof. By Theorem 2.3(i), $L^{\ell(G)}(G)$ is triangular. By definition, a connected triangular graph must also be 2-edge-connected. It follows by Theorem 2.3(ii) that $L^{\ell(G)+1}(G)$ is triangular and $\kappa'(L^{\ell(G)+1}(G)) \geq \kappa(L^{\ell(G)+1}(G)) \geq 3$. ■

For graphs that does not have bridge divalent $(3, t)$-path of length ℓ, a slightly stronger assertion can be stated.

Lemma 3.6. *Let $G \in \mathcal{G}$ be a connected graph and let $\ell = \ell(G) \geq 2$. Suppose that for any integer $t \geq 3$, G does not have a bridge divalent $(3, t)$-path of length ℓ. Each of the following holds.*
(i) $L^\ell(G)$ is triangular with $ess'(L^\ell(G)) \geq 3$.
(ii) For any integer $j \geq 1$, $L^{\ell+j}(G)$ is triangular with $\kappa^*(L^{\ell+j}(G)) = \kappa(L^{\ell+j}(G)) \geq j + 2$.

Proof. (i) follows from Theorem 2.3(i) and Example 3.4(v). By Lemma 3.6(i), (4), and by Lemma 3.3(ii), Lemma 3.6(ii) holds when $j = 1$. Inductively, assume that (ii) holds for smaller values of j and $j \geq 2$. Then by induction, $L^{\ell+j-1}(G)$ is triangular with $\kappa^*(L^{\ell+j-1}(G)) = \kappa(L^{\ell+j-1}(G)) \geq (j-1)+2$. By Theorem 2.3(ii) and Lemma 3.3(ii), we conclude that $L^{\ell+j}(G)$ is triangular with $\kappa^*(L^{\ell+j}(G)) = \kappa(L^{\ell+j}(G)) \geq j + 2$, and so (ii) follows by induction. This proves the lemma. ■

The iterated line graphs of graphs with special structures will reach the triangular state and higher spanning connectivity somewhat earlier, as seen in the example below. The justification of the conclusions in Example 3.7 is similar to those in Lemma 3.6.

Example 3.7. Let $\ell \geq 4$ and $d \geq 3$ be integers, and let $C_\ell = v_1v_2...v_\ell v_1$ denote a cycle on ℓ vertices and let $\{u_1, u_2, \ldots, u_{d-2}, w_1, w_2\}$ be a set of vertices disjoint from $V(C_\ell)$. Let $V = V(C_\ell) \cup \{u_1, u_2, \ldots, u_d\}$. Define graphs $Q_1(\ell, d)$, $Q_2(\ell, d)$, $Q_3(\ell, d)$, $Q_4(\ell, d)$ and, when $d \geq 4$, $Q_5(\ell, d)$ as follows.

$$V(Q_1(\ell, d)) = V \text{ and } E(Q_1(\ell, d)) = E(C_\ell) \cup \{v_1u_i : 1 \leq i \leq d-2\}.$$

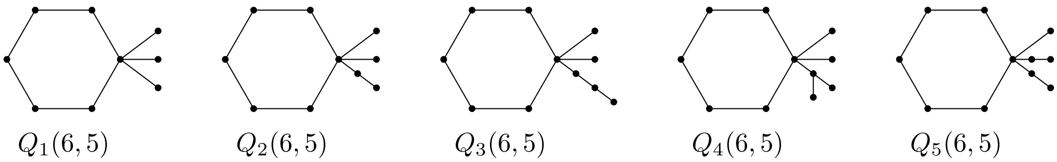

Fig. 2. Some graphs in Example 3.7.

$$V(Q_2(\ell,d)) = V \cup \{w_1\} \text{ and } E(Q_2(\ell,d)) = E(C_\ell) \cup \{v_1u_i : 1 \le i \le d-2\} \cup \{u_1w_1\}.$$
$$V(Q_3(\ell,d)) = V \cup \{w_1, w_2\} \text{ and } E(Q_3(\ell,d)) = E(C_\ell) \cup \{v_1u_i : 1 \le i \le d-2\} \cup \{u_1w_1, w_1w_2\}.$$
$$V(Q_4(\ell,d)) = V \cup \{w_1, w_2\} \text{ and } E(Q_4(\ell,d)) = E(C_\ell) \cup \{v_1u_i : 1 \le i \le d-2\} \cup \{u_1w_1, u_1w_2\}.$$
$$V(Q_5(\ell,d)) = V \cup \{w_1, w_2\} \text{ and } E(Q_4(\ell,d)) = E(C_\ell) \cup \{v_1u_i : 1 \le i \le d-2\} \cup \{u_1w_1, u_2w_2\}.$$

For each i with $1 \le i \le 5$, let $\mathcal{Q}_i$ denote the family of graphs such that $G \in \mathcal{Q}_i$ if and only is G is spanned by a $Q_i(\ell, d)$ with $V(C_\ell - v_1) \subseteq D_2(G)$ and $\Delta(G) = d$. For any $Q_i \in \mathcal{Q}_i$, let $G \in \{Q_1, Q_2, Q_3, Q_4, Q_5\}$, $n = |V(G)|$ and $\Delta = \Delta(G)$. We have the following observations (see Fig. 2 for illustrative examples).
(i) $\ell(Q_i(\ell,d)) = n - \Delta + 3 - i$ if $i \in \{1,2\}$ and $\ell(Q_3(\ell,d)) = \ell(Q_4(\ell,d)) = \ell(Q_5(\ell,d)) = n - \Delta$.
(ii) $L^{n-\Delta}(G)$ is triangular with $ess'(L^{n-\Delta}(G)) \ge 3$.
(iii) For any integer $j \ge 1$, $L^{n-\Delta+j}(G)$ is triangular with $\kappa^*(L^{n-\Delta+j}(G)) = \kappa(L^{n-\Delta+j}(G)) \ge j+2$.

Lemma 3.8. *Let $G \in \mathcal{G} - (\mathcal{Q}_1 \cup \mathcal{Q}_2)$ with $\ell = \ell(G) \ge 2$. Each of the following holds.*
(i) $\ell \le |V(G)| - \Delta$.
(ii) If G has a bridge divalent path P of length ℓ, then $\ell = |V(G)| - \Delta$ only if G has a unique bridge divalent $(\Delta, 1)$-path of length ℓ.
(iii) If for some integers $d', d'' \ge 3$, G has a bridge divalent (d', d'')-path P of length ℓ, then $\ell(G) \le |V(G)| - \Delta - 2$.

Proof. Let $G \in \mathcal{G} - (\mathcal{Q}_1 \cup \mathcal{Q}_2)$ be a graph with $\Delta = \Delta(G)$ and $\ell = \ell(G)$. Since $G \in \mathcal{G}$, we have $\Delta > 2$. Pick a vertex $w_0 \in V(G)$ with $d_G(w_0) = \Delta$ with $N_G(w_0) = \{u_1, u_2, \dots, u_\Delta\}$. Let P be a longest divalent path in G. Thus $|E(P)| = \ell(G)$. Since P is a divalent path, $|E(P) \cap \{u_1w_0, u_2w_0, \dots, u_\Delta w_0\}| \le 2$, where equality holds if and only if P is a cycle C_ℓ with ℓ vertices and contains w_0. In this case, if $|V(G) - (V(C_\ell) \cup N_G(w_0))| \le 1$, then $G \in \mathcal{Q}_1 \cup \mathcal{Q}_2$, contrary to the assumption. Thus either $|V(G) - (V(C_\ell) \cup N_G(w_0))| \ge 2$, or $|E(P) \cap \{u_1w_0, u_2w_0, \dots, u_\Delta w_0\}| \le 1$. In any case, at least $\Delta + 1$ vertices in G cannot be the internal vertices of P, which implies that $\ell = |E(P)| \le |V(G)| - (\Delta+1) + 1 = |V(G)| - \Delta$. This justifies Lemma 3.8(i).

To prove (ii) and (iii), we assume that

$$P = v_0v_1 \dots v_{\ell-1}v_\ell \tag{9}$$

is a bridge divalent path of length ℓ in G. By symmetry, we assume that $d' = d_G(v_0) \ge d_G(v_\ell) = d''$. (We allow that in the proof for (ii), $d'' = 1$.) By definition of $\ell(G)$, $v_0, v_\ell \notin D_2(G)$. Assume that $d_G(v_0) < \Delta$ and w_0 is a vertex in $V(G)$ with $d_G(w_0) = \Delta$. Then since P is a divalent path, we observe that $w_0 \notin V(P)$, and so $N_G(w_0) \cap V(P) = \emptyset$. Hence $V(P) \subseteq V(G) - (N_G(w_0) \cup \{w_0\})$. It follows that

$$\begin{aligned} \ell &= |E(P)| \le |V(G) - (N_G(w_0) \cup \{w_0\})| - 1 \\ &= |V(G)| - (\Delta+1) - 1 = |V(G)| - \Delta - 2. \end{aligned} \tag{10}$$

To complete the proof for (ii), we assume that $\ell = |V(G)| - \Delta$. By (10) we may assume that $d_G(v_0) = \Delta$. If $d_G(v_\ell) \ne 1$, then $d_G(v_\ell) \ge 3$. Since P is a bridge divalent path and $\ell \ge 2$, $|N_G(v_0) \cap N_G(v_\ell)| \le 1$. Hence the vertices in $(N_G(v_0) \cup N_G(v_\ell)) - (N_G(v_0) \cap N_G(v_\ell))$ cannot be internal vertices of P. It follows that

$$\ell = |E(P)| \le |V(G)| - |(N_G(v_0) \cup N_G(v_\ell)) - (N_G(v_0) \cap N_G(v_\ell))| - 1 \tag{11}$$
$$\le |V(G)| - (\Delta + 3 - 1) - 1 = |V(G)| - \Delta - 3, \tag{12}$$

a contradiction. This forces that $d_G(v_\ell) = 1$, and so P must be a divalent $(\Delta, 1)$-path.

To show that uniqueness in (ii), we assume that G has two bridge divalent $(\Delta, 1)$-paths P and P', each of length ℓ. As shown above, $|V(P) \cap V(P')| \le 1$ and there is at most one vertex of degree Δ in $V(P) \cap V(P')$. Thus there are at most two internal vertices of P and P' incident with a vertex of degree Δ in G, and so there are at least $\Delta - 1$ vertices in $N_G(w_0) \cup \{w_0\}$ that cannot be internal vertices of P or P'. It follows that the total number of internal vertices of P and P' is at most $|V(G)| - (\Delta - 1) - 2 = |V(G)| - \Delta - 1$. This implies that $2\ell = |E(P)| + |E(P')| \le |V(G)| - \Delta - 1 + 2$. As $\ell = |V(G)| - \Delta$, this forces that $|V(G)| = \Delta + 1$, implying that G is spanned by a $K_{1,|V(G)|-1}$, contrary to the assumption that $\ell \ge 2$. Thus (ii) must hold.

To prove (iii), we assume that the path P in (9) is a bridge divalent (d', d'')-path P of length ℓ with $d \geq 3$. By contradiction assume that $\ell(G) > |V(G)| - \Delta - 2$. By (10), we must have $d_G(v_0) = \Delta$, and so $V(G) = N_G(v_0) \cup V(P) \cup N_G(v_\ell)$. It follows by $d'' \geq 3$ that $\ell = |E(P)| = |V(G) - (N_G(v_0) \cup N_G(v_\ell))| + 1 = |V(G)| - \Delta - 2$, contrary to the assumption that $\ell(G) > |V(G)| - \Delta - 2$. ■

We are now ready to complete the proofs of the main results. For some technical reason, we first prove Theorem 1.7.

Proof of Theorem 1.7. By (8) and by Lemmas 3.3 and 3.2, we must have $\kappa(L^{\ell(G)+2}(G)) = \kappa^*((L^{\ell(G)+2}(G)))$. Thus $L^{\ell(G)+2}(G)$ is triangular and maximally spanning connected. For any integer $m > \ell(G) + 2$, assuming that $L^{m-1}(G)$ is 3-connected, triangular and maximally spanning connected. By Theorem 2.3(ii), $L^m(G)$ is also 3-connected and triangular; and by Lemma 3.2, $L^m(G)$ is maximally spanning connected. It follows by induction that for any $m \geq \ell(G) + 2$, $L^m(G)$ is also maximally spanning connected.

By Example 3.4 (iv), if for some integer $t \geq 3$, G has a bridge divalent $(3, t)$-path, then $msc(G) \geq \ell(G)+2$. This, together with the conclusions above, forces that $msc(G) = \ell(G) + 2$. Conversely, we assume that $G \in \mathcal{G}$ satisfies $msc(G) = \ell(G) + 2$. If G does not have a bridge divalent $(3, t)$-path for some integer $t \geq 3$, then by Example 3.4(v) and Theorem 2.3(i), $L^\ell(G)$ is essentially 3-edge-connected and triangular. Hence by Lemma 3.2, $L^{\ell+1}(G)$ is maximally spanning connected, contrary to the assumption of $msc(G) = \ell(G) + 2$. This completes the proof of Theorem 1.7(i).

If $\ell(G) = 1$, then $|V(G)| - \Delta(G) + 2 \geq 3$. By Theorem 1.7(i), for any $m \geq 3$, $L^m(G)$ is maximally spanning connected. Assume that $\ell(G) \geq 2$. By Example 3.7 (if $G \in \mathcal{Q}_1 \cup \mathcal{Q}_2$) or by Lemma 3.8(i), $\ell(G) \leq |V(G)| - \Delta(G) + 1$, and so by Theorem 1.7(i), we have

$$msc(G) \leq \ell(G) + 2 \leq |V(G)| - \Delta + 3. \tag{13}$$

If $G \in \mathcal{G}$ satisfying $msc(G) = |V(G)| - \Delta + 3$, then by (13), we must have $msc(G) = \ell(G)+2$. It follows by Theorem 1.7(i) that G must have bridge divalent path of length ℓ. By Lemma 3.8(ii), G has a unique bridge divalent $(\Delta, 1)$-path of length ℓ. By Theorem 2.3(i), $L^\ell(G)$ is triangular. By Example 3.4(v), $L^\ell(G)$ is essentially 3-edge-connected. It follows from Lemma 3.2 and Theorem 2.2 that $L^{\ell+1}(G)$ is maximally spanning connected. This contradicts to the assumption of $msc(G) = |V(G)| - \Delta + 3$. Hence, for any $G \in \mathcal{G}$, we must have $msc(G) \leq |V(G)| - \Delta + 2$. By (8), Lemmas 3.3 and 3.2, it is routine to show that for any $m \geq |V(G)| - \Delta + 2$, $L^m(G)$ is maximally spanning connected. This completes the proof of Theorem 1.7. ■

Proof of Theorem 1.6. Let $\ell = \ell(G)$. Assume first that $k \in \{2, 3\}$. By Theorem 1.7(i), $L^{\ell+2}(G)$ is maximally spanning connected. By Lemma 3.5, $\kappa^*(L^{\ell+2}(G)) \geq 3$. Thus $s_2(G) \leq s_3(G) \leq \ell + 2$. This proves Theorem 1.6(i).

Let $k \geq 3$ be an integer and let $m(k) = \ell + k - 1 \geq \ell + 2$. By Theorem 1.7(i), $L^{m(k)}(G)$ is maximally spanning connected. This, together with Theorem 2.3(iii), implies $\kappa^*(L^{m(k)}(G)) = \kappa(L^{m(k)}(G)) \geq k$. This shows that $s_k(G) \leq m(k) = \ell + k - 1$. Suppose that for any integer $t \geq 3$, G does not have a bridge divalent $(3, t)$-path of length ℓ. By Lemma 3.6(ii) with $j = k - 2$, we conclude that if for any integer $t \geq 3$, G does not have a bridge divalent $(3, t)$-path of length ℓ, then $s_k(G) \leq \ell + k - 2$. This completes the proof of Theorem 1.6(ii).

To prove (iii), by Theorem 1.6(ii) with $k = 3$, we assume that for some integer $t \geq 3$, G has a bridge divalent $(3, t)$-path. By Example 3.4(iii), $\kappa(L^{\ell+1}(G)) < 3$, and so $s_3(G) \geq \ell(G) + 2$. This implies that in this case, we must have $s_3(G) = \ell(G) + 2$. This completes the proof of Theorem 1.6(iii).

Let $k \geq 3$ be an integer. If G has a bridge divalent $(3, t)$-path P of length ℓ for some integer $t \geq 3$, then by Lemma 3.8(iii), $\ell(G) \leq |V(G)| - \Delta - 2$. By Theorem 1.6(ii), $s_k(G) \leq |V(G)| - \Delta + k - 3$. Theorem 1.6(iv) follows in this case.

Suppose that for any integer $t \geq 3$, G does not have a bridge divalent $(3, t)$-path. If every bridge divalent (d', d'')-path P of length ℓ satisfies $\min\{d', d''\} = 1$, then as the degree 1 vertex cannot be an internal vertex of P, there are at least $\Delta + 2$ vertices in G that are not internal vertices of P. It follows that $\ell \leq |V(G)| - \Delta - 1$. If every divalent (d', d'')-path P of length ℓ satisfies $\min\{d', d''\} \geq 4$, then either P is a bridge divalent path, whence $\ell = |E(P)| \leq |V(G)| - \Delta$; or P is not a bridge divalent path, whence $\ell \leq |E(P)| < |V(G)| - \Delta$. By Lemma Lemma 3.6(ii) with $j = k - 2$, we conclude that $s_k(G) \leq \ell + k - 2 \leq |V(G)| - \Delta - 1 + k - 2 = |V(G)| - \Delta + k - 3$. Thus Theorem 1.6(iv) follows in this case also.

Hence we assume that G has a bridge divalent path of length ℓ, and every bridge divalent path P of length ℓ is a (d', d'')-path with $\min\{d', d''\} \geq 4$. By Example 3.4(iv) and Theorem 2.3(i), $L^\ell(G)$ is triangular with $ess'(G) \geq 3$. ■

4. Concluding remarks

The research has found a new family of maximally spanning connected line graphs and the tools developed in this research have also improved some of the former results. The existence of other natural and commonly studied graph families that are also maximally spanning connected would be of interests. Motivated by Conjecture 1.1, we present the following problems for future researches.

Problem 4.1. Let G be a connected graph and $s \geq 2$ be an integer.
(i) Determine the existence of, and if it exists, the smallest value of an integer $f(s)$, such that every $f(s)$-connected line graph is s-spanning connected.
(i) Determine the existence of, and if it exists, the smallest value of an integer $h(s)$, such that every $h(s)$-connected claw-free graph is s-spanning connected.

As every line graph is a claw-free graph, we have $h(s) \geq f(s)$ if they exist. As stated in Conjecture 1.1, Thomassen [38] and, Kučzel and Xiong [23] conjecture that $f(2) = 4$, and Matthews and Sumner [31], and Ryjáček and Vrána [34] conjectured that $h(2) = 4$ also. Furthermore, Ryjáček and Vrána [34] proved that $f(2) = 4$ is equivalent to $h(2) = 4$. We conjecture that these values $f(s)$ and $h(s)$ exist for all $s \geq 2$, and Theorem 1.3 supports the conjecture that $f(s)$ exists.

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