Discrete Mathematics 344 (2021) 112239

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

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On (s, t)-supereulerian graphs with linear degree bounds

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ARTICLE INFO

Article history:
Received 10 June 2020
Received in revised form 9 November 2020
Accepted 15 November 2020
Available online xxxx

Keywords:
Edge-connectivity
Closed trails
Supereulerian graphs
(s, t)-supereulerian
Collapsible graphs
Reduction

ABSTRACT

For integers $s \geq 0$ and $t \geq 0$, a graph G is (s, t)-supereulerian if for any disjoint edge sets $X, Y \subseteq E(G)$ with $|X| \leq s$ and $|Y| \leq t$, G has a spanning closed trail that contains X and avoids Y. Pulleyblank in [J. Graph Theory, 3 (1979) 309-310] showed that determining whether a graph is $(0, 0)$-supereulerian, even when restricted to planar graphs, is NP-complete. Settling an open problem of Bauer, Catlin in [J. Graph Theory, 12 (1988) 29–45] showed that every simple graph G on n vertices with $\delta(G) \geq \frac{n}{5} - 1$, when n is sufficiently large, is $(0, 0)$-supereulerian or is contractible to $K_{2,3}$. We prove the following for any nonnegative integers s and t.

(i) For any real numbers a and b with $0 < a < 1$, there exists a family of finitely many graphs $\mathcal{F}(a, b; s, t)$ such that if G is a simple graph on n vertices with $\kappa'(G) \geq t+2$ and $\delta(G) \geq an + b$, then either G is (s, t)-supereulerian, or G is contractible to a member in $\mathcal{F}(a, b; s, t)$.

(ii) Let ℓK_2 denote the connected loopless graph with two vertices and ℓ parallel edges. If G is a simple graph on n vertices with $\kappa'(G) \geq t + 2$ and $\delta(G) \geq \frac{n}{2} - 1$, then when n is sufficiently large, either G is (s, t)-supereulerian, or for some integer j with $t + 2 \leq j \leq s + t$, G is contractible to a jK_2.

1. Introduction

We consider finite loopless graphs that may have parallel edges and follow [4] for undefined terms and notation. For a vertex subset or an edge subset X of a graph G, $G[X]$ denotes the subgraph induced by X. As in [4], we use $\delta(G)$, $\kappa(G)$ and $\kappa'(G)$ to denote the minimum degree, connectivity and the edge-connectivity of a graph G, respectively.

We define a relation "$\sim$" on $E(G)$ such that $e_1 \sim e_2$ if $e_1 = e_2$, or if e_1 and e_2 form a cycle in G. It is routine to check that $\sim$ is an equivalence relation and edges in the same equivalence class are parallel edges with the same end vertices. We use $[uv]$ to denote the set of all edges between u and v in a graph, and shorten $|[uv]|$ to $|uv|$. If G is a graph, then $\mu(G) = \max\{|uv| : uv \in E(G)\}$ is the **multiplicity** of G. Let ℓK_2 denote the connected loopless graph with two vertices and ℓ parallel edges. Thus for each edge $e \in E(G)$, the edges parallel to e in G induces a subgraph isomorphic to $|e|K_2$.

A graph G is **supereulerian** if G has a spanning closed trail. The supereulerian problem, which aims to characterize supereulerian graphs, was first introduced by Boesch, Suffel and Tindell in [3]. Pulleyblank in [20] showed determining if a graph is supereulerian, even within planar graphs, is NP complete. Supereulerian graphs have been intensively studied, as can be seen in the survey of Catlin [6], as well as the additional updated surveys on the subject in [10,14].

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https://doi.org/10.1016/j.disc.2020.112239
0012-365X/

The notion of (s, t)-supereulerian was formally introduced in [16,17], as a generalization of supereulerian graphs. For integers $s \geq 0$ and $t \geq 0$, a graph G is (s, t)-supereulerian if for any disjoint edge sets $X, Y \subseteq E(G)$ with $|X| \leq s$ and $|Y| \leq t$, G has a spanning closed trail that contains X and avoids Y. Thus supereulerian graphs are precisely $(0, 0)$-supereulerian graphs. A number of research results on the (s, t)-supereulerian problem and similar topics have been obtained, as seen in [9,11–13,15–17,23], among others. Settling an open problem of Bauer posed in [1,2], Catlin [5] proved the following theorem.

Theorem 1.1 (*Catlin, Theorem 9 of [5]*). *Let G be a simple graph on n vertices with $\kappa'(G) \geq 2$. If $\delta(G) \geq \frac{n}{5} - 1$, then when n is sufficiently large, G is $(0, 0)$-supereulerian, or G can be contracted to a $K_{2,3}$.*

It is natural to consider whether Theorem 1.1 can be extended to (s, t)-supereulerian graphs for all possible values of s and t. By definition, if a graph G is (s, t)-supereulerian, then $\kappa'(G) \geq t + 2$. This motivates the current research. Our main results are the following.

Theorem 1.2. *For any nonnegative integers s and t, and any real numbers a and b with $0 < a < 1$, there exists a family of finitely many graphs $\mathcal{F}(a, b; s, t)$ such that if G is a simple graph on n vertices with $\kappa'(G) \geq t + 2$ and $\delta(G) \geq an + b$, then one of the following must hold.*
(i) G is (s, t)-supereulerian.
(ii) G is contractible to a member in $\mathcal{F}(a, b; s, t)$.

Let m, n, s, t be positive integers with $n = 2m \geq s + t$. Define G to be the graph from a disjoint union of two graphs G_1 and G_2, with $G_1 \cong G_2 \cong K_m$, and by adding a set W of $s + t - 1$ new edges linking vertices in G_1 to vertices in G_2. Then $\delta(G) = \frac{n}{2} - 1$. Choose a subset $X \subset W$ satisfying $1 < |X| \leq s$, $|W - X| \leq t$ and $|X| \equiv 1 \pmod 2$. As $|X| \equiv 1 \pmod 2$, $G - (W - X)$ cannot have a spanning closed trail containing X. This example indicates that the bound in the next result is best possible in some sense.

Theorem 1.3. *Let s and t be two nonnegative integers. If G is a simple graph on n vertices with $\kappa'(G) \geq t+2$ and $\delta(G) \geq \frac{n}{2} - 1$, then when n is sufficiently large, one of the following must hold.*
(i) G is (s, t)-supereulerian.
(ii) For some integer j with $t + 2 \leq j \leq s + t$, G is contractible to a jK_2.

In the next section, we summarize former results and needed tools in our arguments to prove the main results. The main results will be validated in the last section.

2. Mechanisms

Define $N_G[v] = N_G(v) \cup \{v\}$ for any vertex $v \in V(G)$. We write $H \subseteq G$ to mean that H is a subgraph of G. If X, Y are vertex subsets of $V(G)$, then define $E_G[X, Y] = \{xy \in E(G) : x \in X, y \in Y\}$ and $\partial_G(X) = E_G[X, V(G) - X]$. If $X = \{v\}$, then we often use $\partial_G(v)$ for $\partial_G(X)$. If $X \subseteq E(G)$, the **contraction** G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. We define $G/\emptyset = G$. If H is a subgraph of G, we write G/H for $G/E(H)$. If H is a connected subgraph of G and v_H is the vertex in G/H onto which H is contracted, then H is the **preimage** of v_H in G. A vertex v in the contraction G/X is **nontrivial** if its preimage in G has at least two vertices.

For an integer $i \geq 0$, let $D_i(G) = \{v \in V(G) : d_G(v) = i\}$ and $O(G)$ be the set of all odd degree vertices of G. A graph G is **collapsible** if for any subset R of $V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph H with $O(H) = R$. By definition, the singleton graph K_1 is collapsible. Collapsible graphs are introduced by Catlin in [5] (see also Proposition 1 of [14]) as a useful tool to study eulerian subgraphs. As when $R = \emptyset$, a spanning connected subgraph H with $O(H) = R$ is a spanning closed trail of G, collapsible graphs are supereulerian graphs. Let $H_1, H_2, \ldots, H_c$ denote the list of all maximal collapsible subgraphs. The graph $G' = G/(\cup_{i=1}^{c} H_i)$ is the **collapsible reduction** of G, or simply the **reduction** of G in short. A graph equaling its own reduction is a **collapsible reduced** graph, or simply a **reduced** graph in short. Theorem 2.1 below presents useful properties related to collapsible graphs.

Theorem 2.1. *Let G be a graph and let H be a collapsible subgraph of G. Let v_H denote the vertex onto which H is contracted in G/H. Each of the following holds.*
(i) (Catlin, Theorem 3 of [5]) G is collapsible (or supereulerian, respectively) if and only if G/H is collapsible (or supereulerian, respectively). In particular, G is collapsible if and only if the reduction of G is K_1.
(ii) (Catlin, Theorem 5 of [5]) A graph is reduced if and only if it does not have a nontrivial collapsible subgraph.
(iii) (Catlin [5]) Cycles of length at most 3 are collapsible.
(iv) (Catlin [5]) The contraction of a collapsible graph blue is collapsible.
(v) Let $X \subseteq E(G)$ be an edge subset of G. If $G - X$ is collapsible, then G has a spanning eulerian subgraph H with $X \subseteq E(H)$.

Proof. It remains to prove (v). Let $R = O(G[X])$. Then $R \subseteq V(G)$, and $|R| \equiv 0 \pmod 2$. Since $G - X$ is collapsible, $G - X$ has a spanning connected subgraph H_R with $O(H_R) = R$. It follows that $H = G[E(H_R \cup X)]$ is a spanning eulerian subgraph of G with $X \subseteq E(H)$. ■

For a graph G, let $\tau(G)$ be the maximum number of edge-disjoint spanning trees in G, and $F(G)$ be the minimum number of additional edges that must be added to G to result in a graph with two edge-disjoint spanning tree. Thus $\tau(G) \geq 2$ if and only if $F(G) = 0$. Theorem 2.2(iii) below can be obtained by applying Theorem 1.4 of [7] to maximal 2-connected subgraphs of G.

Theorem 2.2. *Let G be a connected graph. Each of the following holds.*
(i) (Catlin, Theorem 7 of [5]) If $F(G) \leq 1$, then G is collapsible if and only if $\kappa'(G) \geq 2$. In particular, every graph G with $\tau(G) \geq 2$ is collapsible.
(ii) (Catlin et al. Theorem 1.3 of [7]) If $F(G) \leq 2$, then either G is collapsible or its reduction is a member in $\{K_2, K_{2,t} : t \geq 1\}$.
(iii) (Catlin et al. Theorem 1.4 of [7]) If $F(G) \leq 2$ and $\kappa'(G) \geq 3$, then G is collapsible.
(iv) (Catlin et al. Lemma 2.3 of [7]) If G is a reduced graph with $|V(G)| \geq 2$, then $F(G) = 2|V(G)| - |E(G)| - 2$.

As $F(G) = 0$ amounts to $\tau(G) \geq 2$, utilizing the spanning tree packing theorem of Nash-Williams [19] and Tutte [21], the following is obtained.

Theorem 2.3 (*Catlin et al. Theorems 1.1 and 1.3 of [8]*)**.** *Let G be a graph, $\epsilon \in \{0, 1\}$ and $\ell \geq 1$ be integers. The following are equivalent:*
(i) G is $(2\ell + \epsilon)$-edge-connected;
(ii) For any $X \subseteq E(G)$ with $|X| \leq \ell + \epsilon$, $\tau(G - X) \geq \ell$.

Theorem 2.3 has a seemingly more general corollary, as stated below.

Corollary 2.4 (*Xiong et al. [22]*)**.** *Let G be a connected graph, and ϵ, k, ℓ be integers with $\epsilon \in \{0, 1\}$, $\ell \geq 2$ and $2 \leq k \leq \ell$. The following are equivalent.*
(i) $\kappa'(G) \geqslant 2\ell + \epsilon$.
(ii) For any $X \subseteq E(G)$ with $|X| \leq 2\ell - k + \epsilon$, $\tau(G - X) \geq k$.

An **elementary subdivision** of an $e = uv \in E(G)$ is the operation to form a new graph $G(e)$ from $G - e$ by adding a path uv_ev with v_e being a new vertex in $G(e)$. If $X \subseteq E(G)$ is an edge subset, then $G(X)$ denotes the resulting graph formed by elementarily subdividing each edge in X. Observation 2.5 follows immediately from the definition.

Observation 2.5. *For an edge subset $X \subseteq E(G)$, let $V_X = \{v_e : e \in X\}$, $E_X = \{uv_e, v_ev : e = uv \in X\}$ and $E'_X = \{v_ev : e = uv \in X\}$. Each of the following holds.*
(i) $V_X = V(G(X)) - V(G)$ and $E_X = E(G(X)) - E(G)$.
(ii) There exists a bijection between X and $\{v_eu : e \in X\}$ and so $G(X)/E'_X \cong G$.
(iii) For any 2-edge-connected subgraph H' of $G(X)$, and for any $e = uv \in X$, if $v_e \in V(H')$, then both $v_eu, v_ev \in E(H')$; and if $\{uv_e, vv_e\} \cap E(H') \neq \emptyset$, then $\{uv_e, vv_e\} \subset E(H')$. Thus in view of Observation 2.5(ii), $H = H'/(E'_X \cap E(H'))$ is a subgraph of G, called the **restoration** *of H' in G.*
(iv) G has a spanning eulerian subgraph H with $X \subseteq E(H)$ and $Y \cap E(H) = \emptyset$ if and only if $(G - Y)(X)$ is supereulerian.

Chen, Chen and Luo (Theorem 4.1 of [9]) prove that if $\kappa'(G) \geq 4$, $t \leq \frac{\kappa'(G)}{2}$ and $s + t + 1 \leq \kappa'(G)$, then G is (s, t)-supereulerian. Proposition 2.6(ii) below extends this result when $\kappa'(G) \geq 5$.

Proposition 2.6. *Let s, t be nonnegative integers and let G be a graph. Each of the following holds.*
(i) If G is (s, t)-supereulerian, then any contraction of G is also (s, t)-supereulerian.
(ii) Suppose that H is a graph with $\kappa'(H) \geq \max\{s + t + 1, t + 2, 5\}$. Then H is (s, t)-supereulerian.
(iii) If $H = \ell K_2$ with $\ell \geq \max\{s + t + 1, t + 2, 4\}$, then G is (s, t)-supereulerian if and only if G/H is (s, t)-supereulerian.

Proof. Suppose that G is (s, t)-supereulerian and $e_0 \in E(G)$. Let $\Gamma = G/e_0$. To prove (i), it suffices to show that Γ is also (s, t)-supereulerian. Let $X, Y \subseteq E(\Gamma)$ be arbitrary edge subsets with $X \cap Y = \emptyset$, $|X| \leq s$ and $|Y| \leq t$. As $E(\Gamma) \subseteq E(G)$, and since G is (s, t)-supereulerian, it follows from Observation 2.5(iv) that $(G - Y)(X)$ has a spanning eulerian subgraph J. As $e_0 \in E(G - (X \cup Y))$, let $J + e_0$ denote the subgraph of $(G - Y)(X)$ induced by $E(J) \cup \{e_0\}$. Since J is eulerian, it follows that $J' = (J + e_0)/e_0$ is also a connected graph without a vertex of odd degree, and so J' is a spanning eulerian subgraph of Γ. Hence (i) holds.

Assume that $\kappa'(H) \geq \max\{s+t+1, t+2, 5\}$. Let X, Y be disjoint edge subsets of H with $|X| \leq s$ and $|Y| \leq t$. By adding edges to X if needed, we assume that $|X| = s$. If $s+t \leq \kappa'(H) - 2$, then by Corollary 2.4 (with $k = 2$), $H - (X \cup Y)$ has two edge-disjoint spanning trees, and so by Theorem 2.1(i), $H - (X \cup Y)$ is collapsible. It follows from Theorem 2.1(iii) that $H - Y$ has a spanning eulerian subgraph containing X. Hence we assume that $s + t = \kappa'(H) - 1$, and so $s = \kappa'(H) - t - 1 \geq 1$. Let $W \subseteq X \cup Y$ with $|W| = 2$ and $|W \cap X| > 0$ such that if $s \geq 2$, then $W \subseteq X$; and let $Z = (X \cup Y) - W$. Hence $|Z| \leq s + t - 2$, and so $\kappa'(H - Z) \geq 3$. By Corollary 2.4, $\tau(H - Z) \geq 2$. It follows that $F((H - Z)(W)) \leq 2$. As $\kappa'(H - Z) \geq 3$, then only edge cuts of size 2 in $(H - Z)(W)$ are those of the form $\partial_{(H-Z)(W)}(v_e)$ for some $e \in W$. By Theorem 2.2(ii), either $(H - Z)(W)$ is collapsible or the reduction of $(H - Z)(W)$ is a $K_{2,|W|} = K_{2,2}$. As the latter case contradicts to the fact that $\kappa'(H - Z) \geq 3$, we conclude that $(H-Z)(W)$ is collapsible. By Theorem 2.1(v), $(H-Y)(W)$ has a spanning eulerian subgraph that contains $X-W$, and so $H-Y$ has a spanning eulerian subgraph that contains X. This proves (ii).

By (i), to prove (iii), it remains to assume that G/H is (s,t)-supereulerian to show that G is (s,t)-supereulerian. Let $G_H = G/H$ and let v_H denote the vertex in G_H onto which H is contracted. By (ii), we may assume that H is not a spanning subgraph of G, and so G_H is nontrivial. Let X, Y be disjoint edge subsets of G with $|X| \le s$ and $|Y| \le t$. Define $X' = X - E(H)$, $X'' = X \cap E(H)$, $Y' = Y - E(H)$, and $Y'' = Y \cap E(H)$. Then $|X'| \le s$ and $|Y'| \le t$. Since G_H is a nontrivial (s,t)-supereulerian graph, it follows by Observation 2.5(iv) that $(G_H - Y')(X')$ contains a spanning eulerian subgraph L'.

We need to extend L' to a spanning eulerian subgraph of $(G-Y)(X)$. Let $G'' = (G-Y)(X)$ and $H'' = (H-Y'')(X'')$. Then as $E(L') \cap Y'' = \emptyset$, by their definitions, both $E(L') \subseteq E((G_H - Y')(X')) \subseteq E(G'')$ and H'' is a subgraph of G''. It follows that

$$(G_H - Y')(X') = (G/H - Y')(X') = (G-Y')(X')/H = (G-Y)(X)/[(H-Y'')(X'')] = G''/H''.$$

Since $H = \ell K_2$ with $\ell \ge \max\{s+t+1, t+2, 4\}$, and since $|X''| \le s$ and $|Y''| \le t$, H'' is a graph in which every edge lies in a cycle of length at most 3, and so by Theorem 2.1(i) and (iii), H'' is collapsible. Let $R = O(G''[E(L')])$. Then $|R| \equiv 0 \pmod 2$. As L' is an eulerian subgraph of $(G_H - Y')(X') = (G-Y)(X)/H = G''/H''$, we have $R \subseteq V(H'')$. Since H'' is collapsible, H'' has a spanning connected subgraph L'' with $O(L'') = R$. It follows that $G''[E(L') \cup E(L'')]$ is a spanning eulerian subgraph of $G'' = (G-Y)(X)$. By definition, G is (s,t)-supereulerian. ■

For given non negative integers s and t, let $\mathcal{L}_{s,t}$ denote the family of all (s,t)-supereulerian graphs. By definition, $K_1 \in \mathcal{L}_{s,t}$. A graph H is a **contractible configuration** of $\mathcal{L}_{s,t}$ (or (s,t)**-contractible**, in short), if for any graph G containing H as a subgraph, the following always holds:

$$G \in \mathcal{L}_{s,t} \text{ if and only if } G/H \in \mathcal{L}_{s,t}.$$

Proposition 2.6 indicates that $\mathcal{L}_{s,t}$ is closed under taking contraction, and, if $\ell \ge \max\{s+t+1, t+2, 4\}$, then ℓK_2 is a contractible configuration of $\mathcal{L}_{s,t}$. A a graph Γ is (s,t)**-reduced** if Γ does not contain any nontrivial subgraph that is a contractible configuration of $\mathcal{L}_{s,t}$. For a graph G, the (s,t)**-reduction** of G, is the graph Γ formed from G by contracting all maximal (s,t)-contractible subgraphs of G. By definition, if Γ is the (s,t)-reduction of G, then

$$G \in \mathcal{L}_{s,t} \text{ if and only if } \Gamma \in \mathcal{L}_{s,t}. \tag{1}$$

For a graph G, the (collapsible) reduction of G and the (s,t)-reduction of G may not be the same. To describe the relationship between the two, we need a few more terms.

Definition 2.7. Let s and t be nonnegative integers, G be a graph, X and Y be disjoint edge subsets of G with $|X| \le s$ and $|Y| \le t$, and let $J = (G-Y)(X)$ and J' be the reduction of J. For any vertex $z \in V(J')$, let H'_z denote the preimages of z in J, and let H_z be the restorations of H'_z in $G-Y$. Define

$$\begin{aligned} M &= G[\bigcup_{z \in V(J')} E(H_z)], \\ M' &= J[\bigcup_{z \in V(J')} E(H'_z)], \\ X' &= X \cap E(M') \text{ and } J'' = (G-Y)(X')/M'. \end{aligned}$$

Define $Y' = \{uv \in Y : \text{there exists a graph } L \in \{H_z : z \in V(J')\} \text{ such that } u, v \in V(L)\}$, and $Y'' = Y - Y'$.

The following lemma describes a relationship between the (collapsible) reduction of G and the (s,t)-reduction of G, and will be needed in our arguments.

Lemma 2.8. *We adopt the notation in Definition 2.7 and let $X'' = X - X'$. Each of the following holds.*

(i) $X'' \subseteq E(J'')$ and $J'' = (G-Y)(X')/M' = (G-Y'')(X')/M'$.

(ii) $J' = J''(X'') = ((G-Y'')/M)(X'')$.

(iii) If J is not supereulerian, then G can be contracted to an (s,t)-reduced and non (s,t)-supereulerian graph with order at most $|V(J')|$.

Proof. Let G, J and J' be graphs defined as in Definition 2.7, for given edge subsets $X, Y \subseteq E(G)$ with $X \cap Y = \emptyset$, $|X| \le s$ and $|Y| \le t$.

Since J' is the reduction of $J = (G-Y)(X)$, for any vertex $z \in V(J')$, let H'_z denote the preimage of z in J, and let H_z be the restoration of H'_z in $G-Y$. Thus $V(G) = V(G-Y) = \cup_{z \in V(J')} V(H_z)$.

By Definition 2.7, $J'' = (G-Y)(X')/M' = (G-Y'')(X')/M'$. As $X' = X \cap E(M')$, we have $X'' \subseteq E(J'')$, and so (i) follows.

Fix an arbitrary vertex $z \in V(J')$. Since H'_z is collapsible, $\kappa'(H_z) \ge 2$, and so for any vertex $v \in V(H_z) \cap V_X$, both edges incident with v in J must also be in $E(H'_z)$. It follows from Theorem 2.1(iv) that H_z is a collapsible subgraph of G. By definition, $J' = J/M'$. Then by their definitions, the edges in Y' will become loops and be deleted in the process of contracting M'. It follows that $J' = J/M' = [(G-Y)(X)]/M' = [(G-Y'')(X)]/M' = [(G-Y'')(X')]/M'(X'') = J''(X'')$. By Definition 2.7, $J'' = (G-Y)(X')/M' = (G-Y'')/M$, and so $J' = J''(X'') = ((G-Y'')/M)(X'')$. This justifies (ii).

Since J is not supereulerian, it follows by Theorem 2.1(i) that J' is not supereulerian. By Lemma 2.8(i) and (ii), the graph

$$[(G/M) - Y''](X'') = ((G - Y'')/M)(X'') = [(G - Y'')(X')]/M'(X'') = J''(X'') = J' \tag{2}$$

is not supereulerian. Since $|X''| \le |X| \le s$ and $|Y''| \le |Y| \le t$, G/M is not (s,t)-supereulerian. Let Γ be the (s,t)-reduction of G/M. It follows by (1) that Γ is not (s,t)-supereulerian. By (2), the restoration of J' is $G/M - Y''$ and so $|V(\Gamma)| \le |V(G/M - Y'')| = |V(G/M)| \le |V(J')|$. This completes the proof of the lemma. ■

In [22], an edge-connectivity necessary condition for (s,t)-supereulerian graph has been found.

Proposition 2.9 (*Xiong et al. [22]*). *Let s, t be nonnegative integers. Define*

$$j_0(s,t) = \begin{cases} s + t + \frac{1-(-1)^s}{2} & \text{if } s \ge 1 \text{ and } s + t \ge 3, \\ t + 2 & \text{otherwise.} \end{cases} \tag{3}$$

If a graph G is (s,t)-supereulerian, then $\kappa'(G) \ge j_0(s,t)$.

The next lemma is also useful.

Lemma 2.10 (*Liu et al. Lemma 3.1 of [18]*). *Let G be a simple graph with $\delta = \delta(G)$, and $X \subseteq V(G)$ be a subset. If $|\partial_G(X)| < \delta$, then $|X| \ge \delta + 1$.*

3. Proof of Theorem 1.2

Let a, b, s, t be given as in the statement of Theorem 1.2, $\ell = \max\{s + t + 1, t + 2, 5\}$, and

$$c = \max\left\{\frac{10a}{1+a} + 1, 4\right\}. \tag{4}$$

Define $N = N(a, b, s, t)$ by

$$N = \max\left\{\frac{1}{a} + s + 3, \frac{4-b}{a}, \frac{|b+1| - a(b+1)}{a^2}, \frac{c+t-b+1}{a}, \frac{(1+a)(c+1) - 10a}{a(c-3)}\right\}, \tag{5}$$

and define $\mathcal{F} = \mathcal{F}(a, b; s, t)$ to be the family of all (s,t)-reduced non (s,t)-supereulerian graphs of order at most N. By Proposition 2.6(iii), every graph G in $\mathcal{F}$ has multiplicity at most $\ell - 1$. Thus $\mathcal{F}$ is a family of finitely many graphs. In particular, by Proposition 2.9,

$$\{jK_2 : 1 \le j \le j_0 - 1\} \subset \mathcal{F}. \tag{6}$$

To prove Theorem 1.2, we argue by contradiction, and assume that there exists a counterexample graph G with $n = |V(G)|$ minimized among all counterexample to the theorem. We have the following observations, stated as Claim 1.

Claim 1. *The graph G satisfies the hypotheses of Theorem 1.2, as well as each of the following.*
(i) G cannot be contracted to a member in $\mathcal{F}$, and so $n \ge N + 1$.
(ii) There exist disjoint edge subsets $X, Y \subseteq E(G)$ with $|X| = s$ and $|Y| = t$ such that $G - Y$ does not have a spanning closed trail that contains all edges in X.

Let X and Y be the edge subsets assured by Claim 1(ii), and define $J = (G - Y)(X)$. We adopt the notation in Observation 2.5 for the definition of V_X and E_X. As $\kappa'(G) \ge t + 2$ and by Observation 2.5(iv),

$$\kappa'(J) \ge 2 \text{ and } J \text{ is not supereulerian.} \tag{7}$$

Let J' denote the reduction of J, and define $h = |D_2(J')|$ and $h_1 = |D_2(J') \cap V_X|$. We have the following claim.

Claim 2. $F(J') \ge 3$.

Suppose that $F(J') \le 2$. By Theorem 2.2(ii), either J' is supereulerian, whence by Theorem 2.1(i), J is supereulerian; or $J' = K_{2,h}$ with $h \equiv 1 \pmod 2$ and $h \ge 3$. By (7), we must have $J' = K_{2,h}$. Let $D_h(J') = \{u_1, u_2\}$, and let H'_1, H'_2 be the preimages of u_1 and u_2 in J, respectively; and let H_1 and H_2 be the restorations of H'_1 and H'_2 in $G - Y$, respectively. Thus $V(G) = V(G - Y) = V(H_1) \cup V(H_2)$.

If $h = h_1$, then $h \le |X| \le s \le \max\{s + t, 1\}$, and so by (6), $G/(H_1 \cup H_2) = hK_2$ is a member in $\mathcal{F}$, contrary to Claim 1(i). Thus we must have $h > h_1$. Then for each vertex $z \in D_2(J') - V_X$, let H'_z denote the preimage of z in J, and H_z be the restoration of H'_z in $G - Y$. Since H'_z is collapsible, we have $\kappa'(H_z) \ge 2$. Pick a vertex $v \in V(H_z)$. As $z \in D_2(J') - V_X$ and by $n > N \ge \frac{4-b}{a}$, we have $|V(H_z)| \ge |N_G[v]| - 2 \ge an + b - 1 \ge 3$, It follows that there must be a vertex $v' \in V(H_z)$ such that $N_G[z'] \subseteq V(H_z)$. Thus for each $z \in D_2(J') - V_X$, $|V(H_z)| \ge an + b + 1$. This implies, by $n > N \ge \frac{|b+1| - a(b+1)}{a^2}$ in (5), that

$$h - h_1 \le \frac{n}{an + b + 1} = \frac{an + b + 1 - b - 1}{a(an + b + 1)} = \frac{1}{a} - \frac{b+1}{a(an + b + 1)} < \frac{1}{a} + 1.$$

It follows by $h_1 \le s$ and (5) that $|V(J')| = 2 + h = 2 + h_1 + (h - h_1) < \frac{1}{a} + s + 3 \le N$. By Lemma 2.8 and by (7), G can be contracted to an (s, t)-reduced graph with at most N vertices, which is in $\mathcal{F}$, contrary to Claim 1(i). This proves Claim 2.

For each integer i, let $d_i = |D_i(J')|$. By Claim 2, $F(J') \ge 3$ and so by Theorem 2.2(iv), we have $4|V(J')| - 2|E(J')| \ge 10$. As $|V(J')| = \sum_{i\ge 2} d_i$ and $2|E(J')| \ge \sum_{i\ge 2} id_i$, we have

$$2d_2 + d_3 \ge 10 + \sum_{i\ge 5}(i-4)d_i. \tag{8}$$

For each vertex $v \in V(J') - V_X$, let H'_v be the maximal collapsible subgraph in J which is the contraction preimage of v, and let H_v be the restoration of H'_v. Thus H_v is a subgraph of G.

Claim 3. *Let $n' = |V(J')|$, and define $Z_c = \{v \in V(J') : d_{J'}(v) \le c\}$. Each of the following holds.*
(i) For any $z \in Z_c$, $|V(H_z)| \ge an + b + 1$.
(ii) $|Z_c| \le \frac{1}{a} + 1$.
(iii) $n' \le N$.

Fix a vertex $z \in Z_c$. Then by (5), for any $v \in V(H_z)$, as $n > N \ge \frac{c+t-b+1}{a}$, we have $|\partial_G(V(H_z))| \le c + t < an + b$. It follows by Lemma 2.10 that $|V(H_z)| \ge an + b + 1$. Thus (i) holds. By (i), we have

$$n = |V(G)| \ge \sum_{z\in Z_c} |V(H_z)| \ge |Z_c|(an + b + 1), \text{ and so } |Z_c| \le \frac{n}{an + b + 1}.$$

By (5), $n \ge N \ge \frac{|b+1|-a(b+1)}{a^2}$, implying that $|Z_c| \le \frac{1}{a} + 1$, and so (ii) follows as well.

To prove (iii), we observe that for any vertex $v \in V(J') - Z_c$, $d_{J'}(v) \ge c + 1$, and so by $F(J') \ge 3$,

$$(c+1)|V(J') - Z_c| \le \sum_{v\in V(J')} d_{J'}(v) = 2|E(J')| \le 4n' - 10.$$

It follows that $|V(J') - Z_c| \le \frac{4n'-10}{c+1}$, and so by Claim 3(ii),

$$\frac{1}{a} + 1 \ge |Z_c| = n' - |V(J') - Z_c| \ge n' - \frac{4n' - 10}{c+1} = n'\left(1 - \frac{4}{c+1}\right) + \frac{10}{c+1}. \tag{9}$$

By algebraic manipulations and by (9), (4) and (5), we have

$$n' \le \frac{(1+a)(c+1) - 10a}{a(c-3)} \le N.$$

Thus (iii) holds, and so the claim is justified.

By Claim 3(iii), and by Lemma 2.8, G can be contracted to a member in $\mathcal{F}$, contrary to Claim 1(i). This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.3

Let G be a graph satisfying the hypothesis of Theorem 1.3, and set

$$N = \max\{2t + 9, 2(2s + t + 2)\}. \tag{10}$$

We shall assume that $n \ge N$ and that Theorem 1.3(i) fails to show that Theorem 1.3(ii) must hold. As Theorem 1.3(i) fails, by Observation 2.5(iv), there exist edge disjoint subsets $X, Y \subseteq E(G)$ such that $|X| \le s$, $|Y| \le t$ and

$$(G - Y)(X) \text{ is not supereulerian.} \tag{11}$$

Let $J = (G - Y)(X)$ and J' be the reduction of J. Since $\kappa'(G) \ge t + 2$, we have $\kappa'(J') \ge 2$. If $F(J') \le 1$, then by Theorem 2.2(i), J' is collapsible, and so by Theorem 2.1(i), J is supereulerian, contrary to (11). Hence we must have $F(J') \ge 2$. For each integer i, we again let $d_i = |D_i(J')|$. By Theorem 2.2(iv), $2|V(J')| - |E(J')| - 2 = F(J') \ge 2$, and so $4\sum_{i\ge 2} d_i \ge 8 + \sum_{i\ge 2} id_i$. It follows that

$$2d_2 + d_3 \ge 8 + \sum_{i\ge 5}(i-4)d_i. \tag{12}$$

We will validate the following claim.

Claim 4. *Each of the following holds.*
(i) $\Delta(J') \le 2s$.
(ii) Every vertex in $(\bigcup_{i=3}^{2s} D_i(J')) \cup (D_2(J') - V_X)$ is nontrivial.
(iii) Let m be the number of nontrivial vertices in J'. Then $m \le 2$.
(iv) Let $h = |D_2(J')|$. Then $h \equiv 1 \pmod 2$, $h \ge 3$, $J' \cong K_{2,h}$ and $D_2(J') \subseteq V_X$.

By contradiction, we assume that $\Delta(J') \geq 2s+1$. Then for some $j \geq 2s+1$, $d_j > 1$, and so by (12), $2(d_2+d_3) \geq 8+(2s+1-4) = 2s+5$. As both sides of the inequality are integers, we have $d_2+d_3 \geq s+3$. Since $|V_X \cap D_2(J')| \leq |V_X| = s$, there must be at least three vertices $z_1, z_2, z_3 \in D_2(J') \cup D_3(J') - V_X$. For each $i \in \{1, 2, 3\}$, let H'_{z_i} denote the contraction preimage of z_i in J, and let H_{z_i} denote the restoration of H'_{z_i} in $G-Y$. By (10), $n \geq N \geq 2t+9$, and so $\delta(G) \geq \frac{n}{2} - 1 > 3+t \geq |\partial_G(H_{z_i})|$. By Lemma 2.10, $|V(H_{z_i})| \geq \frac{n}{2}$. It follows that $n = |V(G)| \geq \sum_{i=1}^{3} |V(H_{z_i})| \geq \frac{3n}{2}$, contrary to the fact $n > 0$. This proves (i).

Let $z \in (\bigcup_{i=3}^{2s} D_i(J')) \cup (D_2(J') - V_X)$, H'_z be the contraction preimage of z in J, and H_z denote the restoration of H'_z in $G-Y$. By (10), $n \geq N \geq 2(2s+t+2) \geq 4$, and so $\delta(G) \geq \frac{n}{2} - 1 > 2s+t \geq |\partial_G(H_z)|$. By Lemma 2.10, $|V(H_z)| \geq \frac{n}{2} \geq 2$, and so (ii) follows.

By contradiction, we assume that J' has at least three nontrivial vertices, say w_1, w_2, w_3. For each $i \in \{1, 2, 3\}$, let H'_{w_i} denote the contraction preimage of w_i in J, and let H_{w_i} denote the restoration of H'_{w_i} in $G-Y$. By (10), $n \geq N \geq 2(2s+t+2)$, and so by Claim 4(i) that $\delta(G) \geq \frac{n}{2} - 1 > 2s+t \geq |\partial_G(H_{w_i})|$. By Lemma 2.10, $|V(H_{w_i})| \geq \frac{n}{2}$. It follows that $n = |V(G)| \geq \sum_{i=1}^{3} |V(H_{w_i})| \geq \frac{3n}{2}$, contrary to the fact $n > 0$. This proves (iii).

By Claim 4(i), $d_j = 0$ for any $j \geq 2s+1$, and so by Claim 4(ii), $|V(J')| - |D_2(J') \cap V_X| = \sum_{i \geq 2} d_i - |D_2(J') \cap V_X| \leq 2$. Thus $|V(J')| \leq |D_2(J')| + 2$. By Claim 4(iii), $m \leq 2$. Let $u_1, \ldots, u_m$ denote the nontrivial vertices of J'. If at least one of the w_i's is of even degree in J', then since the number of odd degree vertices of a graph must be even, it follows by $m \leq 2$ that J' is an eulerian graph, and so supereulerian. By Theorem 2.1(i), J is supereulerian, contrary to (11). Hence we must have $m = 2$ and both u_1 and u_2 are of odd degree in J'. Since J' is reduced, J' contains no cycles of length at most 3, and so we must have $N_{J'}(u_1) = N_{J'}(u_2) = D_2(J')$. By (11), J' cannot be eulerian, and so $h \equiv 1 \pmod 2$. Since $\kappa'(J') \geq 2$, we must have $h \geq 3$. Finally, since both u_1 and u_2 are not in $D_2(J')$, it follows by Claim 4(ii) and (iii) that $D_2(J') \subseteq V_X$. This proves (iv), as well as Claim 4.

By Claim 4(iv), $J' \cong K_{2,h}$ for some odd integer $h \geq 3$. We continue using u_1, u_2 to denote the two vertices of degree h in J', and define H'_{u_i} to be the preimage of u_i in J, and H_{u_i} the restoration of H'_{u_i} in $G-Y$. By Claim 4(iv), $D_2(J') \subseteq V_X$. Let $X'' = \{e \in X : v_e \in D_2(J')\}$. Since $J' \cong K_{2,h}$, we have $V(G) = V(H_{u_1}) \cup V(H_{u_2})$ and $X'' \subseteq E_G[V(H_{u_1}), V(H_{u_2})] \subseteq X'' \cup Y$. Let $j = |E_G[V(H_{u_1}), V(H_{u_2})]|$. Then by $\kappa'(G) \geq t+2$, we have $t+2 \leq j \leq |X''| + |Y| \leq s+t$ and $G/(H_{u_1} \cup H_{u_2}) = jK_2$. Thus Theorem 1.3(ii) must hold. This completes the proof of Theorem 1.3.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

This research is partially supported by General Project of Natural Science Foundation of Chongqing, China (No. cstc2019jcyj-msxmX0579) and by National Science Foundation of China grants (Nos. 11861066, 11961067, 12001465, 11771039, 11771443).

References

[1] D. Bauer, On Hamiltonian cycles in line graphs, Stevens Research Report PAM No. 8501, Stevens Institute of Technology, Hoboken, NJ.
[2] D. Bauer, A note on degree conditions for Hamiltonian cycles in line graphs, Congres. Numer. 49 (1985) 11–18.
[3] F.T. Boesch, C. Suffel, R. Tindell, The spanning subgraphs of Eulerian graphs, J. Graph Theory 1 (1977) 79–84.
[4] J.A. Bondy, U.S.R Murty, Graph Theory, Springer, New York, 2008.
[5] P.A. Catlin, A reduction method to find spanning Eulerian subgraphs, J. Graph Theory 12 (1988) 29–45.
[6] P.A. Catlin, Supereulerian graphs: A survey, J. Graph Theory 16 (1992) 177–196.
[7] P.A. Catlin, Z. Han, H.-J. Lai, Graphs without spanning closed trails, Discrete Math. 160 (1996) 81–91.
[8] P.A. Catlin, H.-J. Lai, Y. Shao, Edge-connectivity and edge-disjoint spanning trees, Discrete Math. 309 (2009) 1033–1040.
[9] W.-G. Chen, Z.H. Chen, W.-Q. Luo, Edge connectivities for spanning trails with prescribed edges, Ars Combin. 115 (2014) 175–186.
[10] Z.-H. Chen, H.-J. Lai, Reduction techniques for super-Eulerian graphs and related topics-a survey, in: Combinatorics and Graph Theory '95, Vol. 1 (Hefei), World Sci. Publishing, River Edge, NJ, 1995, pp. 53–69.
[11] Z.H. Chen, W.-Q. Luo, W.-G. Chen, Spanning trails containing given edges, Discrete Math. 306 (2006) 87–98.
[12] R. Gu, H.-J. Lai, Y. Liang, Z. Miao, M. Zhang, Collapsible subgraphs of a 4-edge-connected graph, Discrete Appl. Math. 260 (2019) 272–277.
[13] H.-J. Lai, Eulerian subgraph containing given edges, Discrete Math. 230 (2001) 61–69.
[14] H.-J. Lai, Y. Shao, H. Yan, An update on supereulerian graphs, WSEAS Trans. Math. 12 (2013) 926–940.
[15] L. Lei, X. Li, S. Song, Y. Xie, On (st)-supereulerian generalized prisms, 2020, submitted for publication.
[16] L. Lei, X.M. Li, B. Wang, On (s,t)-supereulerian locally connected graphs, ICCS 2007, in: Proceedings, Lecture Notes in Computer Sciences, vol. 4489, 2007, pp. 384–388.
[17] L. Lei, X. Li, B. Wang, H.-J. Lai, On (s,t)-supereulerian graphs in locally highly connected graphs, Discrete Math. 310 (2010) 929–934.
[18] Q. Liu, Y. Hong, X. Gu, H.-J. Lai, Note on edge-disjoint spanning trees and eigenvalues, Linear Algebra Appl. 458 (2014) 128–133.
[19] C.St.J.A. Nash-williams, Edge-disjoint spanning trees of finite graphs, J. Lond. Math. Soc. 36 (1961) 445–450.
[20] W.R. Pulleyblank, A note on graphs spanned by Eulerian graphs, J. Graph Theory 3 (1979) 309–310.
[21] W.T. Tutte, On the problem of decomposing a graph into n factors, J. Lond. Math. Soc. 36 (1961) 221–230.
[22] Wei Xiong, S. Song, H.-J. Lai, Polynomially determine if a graph is $(s,3)$-supereulerian, 2020, submitted for publication.
[23] J. Xu, Z.-H. Chen, H.-J. Lai, M. Zhang, Spanning trails in essentially 4-edge-connected graphs, Discrete Appl. Math. 162 (2014) 306–313.