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Unified Spectral Hamiltonian Results of Balanced Bipartite Graphs and Complementary Graphs

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Abstract

There have been researches on sufficient spectral conditions for Hamiltonian properties and path-coverable properties of graphs. Utilizing the Bondy–Chvátal closure, we provide a unified approach to study sufficient graph eigenvalue conditions for these properties and sharpen several former spectral Hamiltonian results on balanced bipartite graphs and complementary graphs.

Keywords Hamiltonian graphs · Traceable graphs · (Almost)balanced bipartite graphs · complementary graphs · (Signless Laplacian)spectral radius

Mathematics Subject Classification 05C50 · 15A18 · 15A36

1 Introduction

We study simple undirected graphs, with undefined terms and notation following [3]. As in [3], $\delta(G)$, $\kappa(G)$, $\kappa'(G)$ and $\overline{G}$ denote the minimum degree, the connectivity, the edge-connectivity and the complement of a graph G, respectively. For an integer k, a graph G is *k-connected* (resp. *k-edge-connected*) if $\kappa(G) \geq k$

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(resp. $\kappa'(G) \geq k$). Throughout this paper, for an integer $s \geq 1$, let sK_1 be the edgeless graph with s vertices. Let $S \subseteq V(G)$ be a subset. For any vertex $u \in V(G)$, define $N_S(u) = \{v \in S : uv \in E(G)\}$. If H is a subgraph of G, then we use $N_H(u)$ for $N_{V(H)}(u)$. In particular, $N_G(u) = \{v \in V(G) : uv \in E(G)\}$ and $d_G(u) = |N_G(u)|$. We often use $N(u)$ and $d(u)$ for $N_G(u)$ and $d_G(u)$, respectively, when G is understood from the context. A graph G is *nontrivial* if it has at least one edge. As in [3], G is *Hamiltonian* (resp., *traceable*) if G contains a spanning cycle (resp., spanning path), and is *Hamilton-connected* if any pair of distinct vertices are joined by a spanning path.

Definition 1.1 Let $q \geq 0$ be an integers and let G be a graph.

(i) G is *q-traceable* (resp. *q-Hamiltonian*, *q-Hamilton-connected*) if any removal of at most q vertices from G results in a traceable graph (resp., a Hamiltonian graph, a Hamilton-connected graph).

(ii) G is *q-edge-Hamiltonian* if any collection of vertex-disjoint paths with at most q edges altogether must belong to a Hamiltonian cycle in G.

(iii) G is *q-path-coverable* if $V(G)$ can be covered by no more than q vertex-disjoint paths.

Following [3], we use $G[X]$ to denote the subgraph of G induced by X. By Definition 1.1(i), a q-Hamiltonian graph is also a $(q+1)$-traceable graph. However, a $(q+1)$-traceable graph is not necessarily a q-Hamiltonian graph. For instance, the Petersen graph is 1-traceable, but not 0-Hamiltonian. Moreover, a traceable graph is a 0-traceable graph, and a Hamiltonian graph is both a 0-Hamiltonian and a 1-traceable graph. If G is Hamilton-connected, then for any pair of vertices $\{u, v\}$ of G, there is a Hamiltonian path connecting u and v. Thus, $G\left[V(G)\backslash\{u, v\}\right]$ contains a Hamiltonian path, and hence G is 2-traceable.

As in [3], the *join* $G \vee H$ of two disjoint graphs G and H is defined by $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \left\{xy : \ x \in V(G) \text{ and } y \in V(H)\right\}$. A *k-regular* graph is a graph with $d_G(u) = k$ for each vertex $u \in V(G)$. For two different nonnegative integers p and q, a *(p, q)-semi-regular bipartite* graph is a bipartite graph G with vertex bipartition (U, V) such that $d_G(u) = p$, $\forall u \in U$ and $d_G(v) = q$, $\forall v \in V$. As usual, let K_n, C_n and $K_{k,n-k}$ be the complete graph, cycle and complete bipartite graph with n vertices, respectively. Following [15], for nonnegative integers n, k and s satisfying $s \leq k \leq \frac{1}{2}(n+s-2)$, define the graph $M_n^{k,s}$ with n vertices and minimum degree k as follows:

$$M_n^{k,s} = K_s \vee \left(K_{n-k-1} \cup K_{k+1-s}\right).$$

In order to characterize the exceptional graphs in our main results, we introduce several graph families in the following.

Definition 1.2 Let n, k, p, q, r be five nonnegative integers, and s be an integer.

(i) Define $\mathbb{B}_{n,k,s,r} = \left\{\overline{G_1} \vee G_2 : G_1 = (U,V) \text{ is a connected } (k-s, n-k-1)\text{-semi-regular bipartite graph with } n-s-1+r \text{ vertices and } G_2 \text{ is a spanning subgraph of } K_{s+1-r}, \text{ where } 0 \le r \le s+1 \text{ and } r \ne 1\right\}$. In particular, $\mathbb{B}_{n,k,-1,0} = \{M_n^{k,0}\} = \left\{K_{n-k-1} \cup K_{k+1}\right\}$.

(ii) Define $\mathbb{C}_{n,s,r} = \left\{\overline{G_1} \vee G_2 : G_1 \text{ is a connected } (p, n-s-1-p)\text{-semi-regular bipartite graph with } n-s-1+r \text{ vertices and } G_2 \text{ is a spanning subgraph of } K_{s+1-r}, \text{ where } 0 \le r \le s+1,\ r \ne 1 \text{ and } 1 \le p \le \frac{n-s-1}{2}\right\}$. In particular, $\mathbb{C}_{n,-1,0} = \left\{K_p \cup K_{n-p} : \text{where } 1 \le p \le \frac{n}{2}\right\}$.

(iii) Suppose that $n = 2k+1-s$ and $s \le 1$. Define $\mathbb{H}_{n,k,s,r} = \left\{G_1 \vee G_2 : G_1 \text{ is a } r\text{-regular graph with } n-k+r \text{ vertices and } G_2 \text{ is a spanning subgraph of } K_{k-r}, \text{ where } 0 \le r \le k\right\}$. In particular, $\mathbb{H}_{n,k,s,k}$ is the set of all k-regular graphs with n vertices.

(iv) Let $\mathbb{D}_{n,s,r} = \left\{\left(\overline{G_1} \vee \overline{G_2}\right) \vee G_3 : G_1 \text{ and } G_2 \text{ are two connected } \frac{n-s-1}{2}\text{-regular graphs with } |V(G_1)| + |V(G_2)| = n-r \text{ and } G_3 \text{ is a spanning subgraph of } K_r \text{ with } \mu\left(\overline{G_3}\right) \le n-s-1, \text{ where } 0 \le r \le s-1\right\}$. In particular, $\mathbb{D}_{n,1,0} = \left\{K_{\frac{n}{2},\frac{n}{2}}\right\}$.

(v) Let $\mathbb{W}_{n,s,r} = \left\{\overline{G_1} \vee G_2\text{: } G_1 \text{ is a connected } \frac{n-s-1}{2}\text{-regular graph with } n-r \text{ vertices and } G_2 \text{ is a spanning subgraph of } K_r \text{ with } \mu\left(\overline{G_2}\right) \le n-s-1, \text{ where } 0 \le r \le \frac{n+s-1}{2}\right\}$. In particular, $\mathbb{W}_{n,-1,0}$ is the set of $\left(\frac{n}{2}-1\right)$-regular graphs.

(vi) Suppose that $n > k \ge 0$, $p \ge k+1$, and let (X, Y) be the vertex bipartition of $K_{n,p+q}$ with $|X| = n$ and $|Y| = p+q$. Let $X_1 \subset X$ be a subset with $|X_1| = n-k$, $Y_1 \subset Y$ be a subset with $|Y_1| = q \ge 1$ and $K = K_{n,p+q}[X_1 \cup Y_1]$ be the induced subgraph. Define $B_{k,n-k;p,q} = K_{n,p+q} - E(K)$. When k is understood from the context, we often write $B_{k,n-k;p,q}$ as $Z_{p,q}$ and define $Z^0_{p,q} = Z_{p,q} - e$, where $e = uv \in E(Z_{p,q})$ satisfying $d_{Z_{p,q}}(u) = n$ and $d_{Z_{p,q}}(v) = p$. To simplify the notation in the proofs, we define

$$F_{n,k,s} = Z_{n+s-k-1,k+1-s} \text{ and } F^0_{n,k,s} = Z^0_{n+s-k-1,k+1-s}. \tag{1.1}$$

As examples, let $L_1, L_2, \ldots, L_6$ be the six graphs depicted in Fig. 1. By setting $G_1 = K_{2,4}$ in Definition 1.2 (i)–(ii), we have $L_1 \in \mathbb{B}_{9,4,2,0} \cap \mathbb{C}_{9,2,0}$. By taking $G_1 = C_7$ in Definition 1.2 (iii), we have $L_2 \in \mathbb{H}_{9,4,0,2}$. Let $G_1 = G_2 = K_3$ and $G_3 = K_{1,2}$ in Definition 1.2 (iv), it follows that $L_3 \in \mathbb{D}_{9,4,3}$. As $\overline{L_4} = C_6 \cup K_3$, by Definition

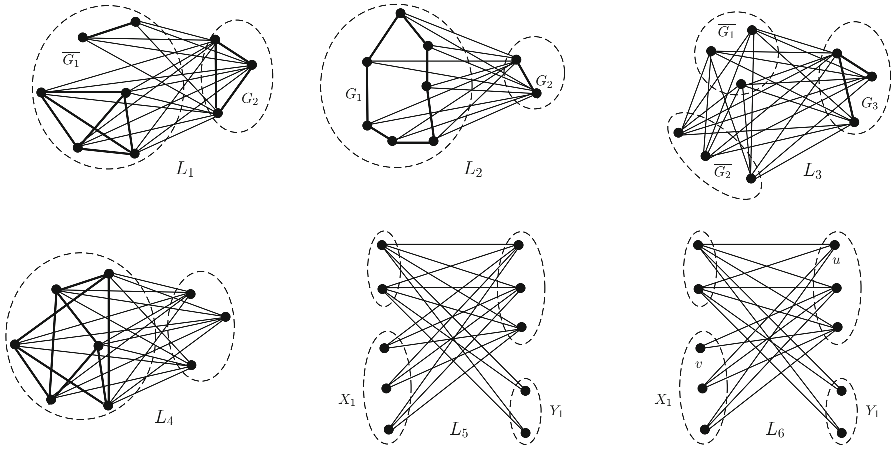

Fig. 1 The graphs $L_1, L_2, \ldots, L_6$

1.2(v), we have $L_4 \in \mathbb{W}_{9,4,3}$. Letting $n = 5$, $k = 2$ and $s = 1$ in Definition 1.2 (vi), we observe that $L_5 = B_{2,3;3,2} = Z_{3,2} = F_{5,2,1}$ and $Z^0_{3,2} = F^0_{5,2,1} = L_6$.

Following [3], we use $G = [U, V]$ to denote a bipartite graph with vertex bipartition (U, V); and G is *balanced* (respectively, *almost balanced*) if $|U| = |V|$ (respectively, if $|U| - |V| \in \{1, -1\}$). Let p and q be two nonnegative integers. A bipartite graph $G = [U, V]$ is *(p, q)-traceable* if for any subset $S \subset G$ satisfying $|S \cap U| = p$, $|S \cap V| = q$ and $|(|U| - p) - (|V| - q)| \leq 1$, $G - S$ is traceable; and $G = [U, V]$ is *(p, q)-Hamiltonian* if for any subset $S \subset G$ satisfying $|S \cap U| = p$, $|S \cap V| = q$ and $|U| - p = |V| - q$, $G - S$ is Hamiltonian.

For two graphs G and H, we write $H \subseteq G$ if H is a subgraph of G. For nonnegative integers n and k, let $\mathbb{G}_n$ be the class of graphs with n vertices, and define the *k-closure* of a graph G following [4], denoted by $\mathscr{C}_k(G)$, to be the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least k until no such pair remains nonadjacent. By definition, $G \subseteq \mathscr{C}_k(G)$. A graphical property P is *k-stable* if for any graph $G \in \mathbb{G}_n$, G has Property P if and only if $\mathscr{C}_k(G)$ has Property P. It is worth noting that this definition of k-stable is a slightly different from that in [2].

There is also a closure concept for bipartite graphs [2]. Let $k > 0$ be an integer and $G = [U, V]$ be a bipartite graph. The *bipartite closure graph* $\mathscr{B}_k(G)$ of G is the bipartite graph obtained from G by recursively joining pairs of nonadjacent vertices u, v with $u \in U$ and $v \in V$ whose degree sum is at least k until no such pair remains nonadjacent. By definition, $G \subseteq \mathscr{B}_k(G)$.

Let $A(G)$ and $D(G)$, respectively, be the adjacency matrix and the diagonal degree matrix of G. The *signless Laplacian matrix* of G is defined to be $Q(G) = D(G) + A(G)$. The *spectral radius* of G, denoted by $\rho(G)$, is the largest eigenvalue of $A(G)$, and the *signless Laplacian spectral radius* of G, denoted by $\mu(G)$, is the largest eigenvalue of $Q(G)$. Throughout this paper, let α be a nonnegative real number and define $\Theta(G,\alpha)$ be the largest eigenvalue of the matrix $A(G)+\alpha D(G)$. By definition, $\Theta(G,0)=\rho(G)$ and $\Theta(G,1)=\mu(G)$.

There have been lots of studies on graphical properties warranted by various kind of graph spectral conditions. Our current research is motivated by these studies, as revealed in the subsections in this section. We will have brief literature reviews on the relationship between graphical properties and the eigenvalues of the complement of a graph in Sect. 1.1, and those of balanced and almost balanced bipartite graphs in Sect. 1.2. As the properties involved are possessed by complete graphs or complete balanced bipartite graphs, and are stable under taking the corresponding Bondy–Chvátal closures, we in this paper investigate the relationship between different types of graph eigenvalues and the property when a related Bondy–Chvátal closure of the graph is a complete graph or a complete balanced bipartite graph. Our main results, as shown in Sects. 1.1 and 1.2, present unified conclusions that generalize several former results in a number of different problems.

1.1 Spectral Results of Complement Graphs on Hamiltonian Problem

There have been researches on describing the Hamiltonian properties of a graph G in terms of the eigenvalues of $\overline{G}$. The following are the related pioneer results.

Theorem 1.3 *Let G be a graph on n vertices.*

(i) (Fiedler and Nikiforov [8]) If $\rho(\overline{G})\le\sqrt{n-1}$, then G is traceable unless $G=M_n^{0,0}$.

(ii) (Fiedler and Nikiforov [8]) If $\rho(\overline{G})\le\sqrt{n-2}$, then G is Hamiltonian unless $G=M_n^{1,1}$.

(iii) (Yu and Fan [18]) If $n\ge 4$ and if $\rho(\overline{G})\le\sqrt{\frac{(n-2)^2}{n}}$, then G is Hamilton-connected.

(iv) (Li and Ning [12]) Suppose that $n\ge 2k+2$ and $\delta(G)\ge k\ge 0$. If $\rho(\overline{G})\le\rho\left(\overline{M_n^{k,0}}\right)$, then G is traceable unless $G\in\mathbb{B}_{n,k,-1,0}$ or $G\in\mathbb{H}_{n,k,-1,0}$.

(v) (Li and Ning [12]) Suppose that $n\ge 2k+1$ and $\delta(G)\ge k\ge 1$. If $\rho(\overline{G})\le\rho\left(\overline{M_n^{k,1}}\right)$, then G is Hamiltonian unless $G\in\mathbb{B}_{n,k,0,0}$ or $G\in\mathbb{H}_{n,k,0,0}$.

Extensions of some of the results stated in Theorem 1.3 have been obtained by several researchers, as seen in the theorem below.

Theorem 1.4 *Let G be a connected graph on n vertices.*

(i) (Yu et al. [20]) Suppose that $n\ge 2k+1$ and $\delta(G)\ge k\ge q+1\ge 1$. If $\rho(\overline{G})\le\sqrt{(k-q)(n-k-1)}$, then G is q-Hamiltonian and q-edge-Hamiltonian unless $G\in\mathbb{B}_{n,k,q,r}$ or $G\in\mathbb{H}_{n,k,q,r}$.

(ii) (Yu et al. [19], Chen and Zhang [5]) Suppose that $n\ge 2k$ and $\delta(G)\ge k\ge 2$. If $\rho(\overline{G})\le\sqrt{(k-1)(n-k-1)}$, then G is Hamilton-connected unless $G\in\mathbb{B}_{n,k,1,0}$ or $G\in\mathbb{H}_{n,k,1,r}$.

Analogous adjacency and signless Laplacian spectral conditions of the completeness of a graph to warrant similar or other properties have also been investigated. The following results come from Theorems 3.1, 3.4 and Corollary 3.2 of Yu et al. [20].

Theorem 1.5 (*Yu et al.* [20]) *Let G be a graph on n vertices.*

(i) Suppose that $n \geq 2k+1$, $\delta(G) \geq k \geq \max\{q-1, 1\}$ and $q \geq 1$. If $\rho(\overline{G}) \leq \sqrt{(k-q+2)(n-k-1)}$, then G is q-connected and q-edge-connected unless $G \in \mathbb{B}_{n,k,q-2,r}$ or $G \in \mathbb{H}_{n,k,q-2,r}$.

(ii) Suppose that $n \geq 2k+q+1$, $\delta(G) \geq k \geq 1$ and $q \geq 1$. If $\rho(\overline{G}) \leq \sqrt{(k+q)(n-k-1)}$, then G is q-path-coverable unless $G \in \mathbb{H}_{n,k,-q,r}$ or $G = K_{k+1} \cup K_{n-k-1}$ when $q = 1$.

Theorem 1.6 *Let G be a graph with n vertices.*

(i) (Zhou [21]) If $\mu(\overline{G}) \leq n$, then G is traceable unless $G \in \mathbb{C}_{n,-1,0}$ or $G \in \mathbb{W}_{n,-1,r}$.

(ii) (Zhou [21]) If $\mu(\overline{G}) \leq n-1$ and $n \geq 3$, then G is Hamiltonian unless $G \in \mathbb{C}_{n,0,0}$ or $G \in \mathbb{W}_{n,0,r}$, where $1 \leq r \leq \frac{n-1}{2}$.

(iii) (Yu and Fan [18]) If $\mu(\overline{G}) \leq n-2$ and $n \geq 6$, then G is Hamilton-connected unless $G \in \mathbb{C}_{n,1,0}$ or $G \in \mathbb{D}_{n,1,0}$ or $G \in \mathbb{W}_{n,1,r}$, where $1 \leq r \leq \frac{n}{2}$.

It is observed that in the theorems above, all the graphical properties warranted by the various spectral properties satisfy certain level of stability, as shown in the result of Bondy and Chvátal below.

Theorem 1.7 *Let n and q be two integers with $n \geq 3$ and $q \geq 0$. Each of the following holds for a graph on n vertices.*

(i) (Bondy and Chvátal [2]) The property that "G is q-connected" is $(n+q-2)$-stable.

(ii) (Bondy and Chvátal [2]) The property that "G is q-edge-connected" is $(n+q-2)$-stable.

(iii) (Bondy and Chvátal [2]) The property that "G is q-path-coverable" is $(n-q)$-stable.

(iv) (Bondy and Chvátal [2]) The property that "G is q-edge-Hamiltonian" is $(n+q)$-stable.

(v) (Bondy and Chvátal [2]) The property that "G is q-Hamiltonian connected" is $(n+q+1)$-stable.

(vi) (Bondy and Chvátal [2]) The property that "G is q-Hamiltonian" is $(n+q)$-stable.

(vii) (Liu et al. [15]) The property that "G is q-traceable" is $(n+q-1)$-stable.

These motivate our current study. The main result of this paper is the following.

Theorem 1.8 *Let n, k and s be three integers and let G be a graph on n vertices.*

(i) Suppose that $n \geq \max\{2k, 2k+1-s\}$ and $\delta(G) \geq k \geq \max\{s, 1\}$. If $\rho(\overline{G}) \leq \sqrt{(k-s)(n-k-1)}$, then either $\mathscr{C}_{n+s}(G) = K_n$ or $G \in \mathbb{B}_{n,k,s,r} \cup \mathbb{H}_{n,k,s,r}$, or both $s = k-1$ and $G = \overline{K_{1,k-1}} \vee \overline{K_{1,k-1}}$.

(ii) Suppose that $n \geq 3s+2$. If $\mu(\overline{G}) \leq n-s-1$, then either $\mathscr{C}_{n+s}(G) = K_n$, or $G \in \mathbb{C}_{n,s,r} \cup \mathbb{D}_{n,s,r} \cup \mathbb{W}_{n,s,r}$.

Since K_n is q-traceable for any $0 \leq q \leq n$, the corollary below follows immediately from Theorem 1.7(vii) and Theorem 1.8 with $s = q-1$.

Corollary 1.9 *Let n, k and q be three nonnegative integers and G be a graph with $|V(G)| = n$.*

(i) Suppose that $n \geq \max\{2k, 2k+2-q\}$ and $\delta(G) \geq k \geq \max\{q-1, 1\}$. If $\rho(\overline{G}) \leq \sqrt{(k+1-q)(n-k-1)}$, then G is q-traceable, unless $G \in \mathbb{B}_{n,k,q-1,r} \cup \mathbb{H}_{n,k,q-1,r}$ or both $G = \overline{K_{1,k-1}} \vee \overline{K_{1,k-1}}$ and $q = k$.

(ii) Suppose that $n \geq 3q-1$. If $\mu(\overline{G}) \leq n-q$, then G is q-traceable unless $G \in \mathbb{C}_{n,q-1,r} \cup \mathbb{D}_{n,q-1,r} \cup \mathbb{W}_{n,q-1,r}$.

As the complete graph has all the properties listed in Theorems 1.7 and 1.8 generalizes the corresponding results in Theorems 1.3, 1.4, 1.5 and 1.6, when s is taking different appropriate values. Motivated by Theorem 1.6, it is natural to consider whether the possibility that "$G \in \mathbb{W}_{n,q-1,r}$" can be removed from the statement of Corollary 1.9(ii). The following example suggests that the answer is negative.

Example 1.10 Let n and q be two integers such that $q \geq 2$, $n \geq 3q-5$ and $n+q$ is even. If G_1 is a $(q-2)$-regular graph with $\frac{n+q-2}{2}$ vertices, then $G = G_1 \vee \left(\frac{n-q+2}{2}\right)K_1$ is $\frac{n+q-2}{2}$-regular, and hence $G \in \mathbb{W}_{n,q-1,r}$. Note that any deletion of q vertices from G_1 to G results in a non-traceable graph. Thus, G is not q-traceable.

1.2 Spectral Results of Balanced Bipartite Graphs on Hamilton Problem

Researches on predicting traceable and Hamiltonian bipartite graphs by graph spectral conditions have been attracted many researchers, as seen in [9, 12, 14, 16, 17], among others. The following theorem displays some of the spectral results on Hamiltonian properties of balanced bipartite graphs.

Theorem 1.11 *Let G be a balanced bipartite graph on $2n$ vertices.*

(i) (Liu et al. [16]) If $n \geq 3$, $\delta(G) \geq 1$ and $\rho(G) \geq \sqrt{n^2-2n+3}$, then G is traceable.

(ii) (Li and Ning [13]) If $n \geq (k+2)^2$, $\delta(G) \geq k \geq 2$ and either $\rho(G) \geq \rho(F_{n,k,0})$ or $\mu(G) \geq \mu(F_{n,k,0})$, then G is traceable unless $G = F_{n,k,0}$.

(iii) (Jiang et al. [10]) If $n \geq \max\{(k+2)^2, \frac{k^2(k+1)}{2} + k + 3\}$, $\delta(G) \geq k \geq 2$ and $\rho(G) \geq \sqrt{n(n-k-1)}$, then G is traceable unless $G = F_{n,k,0}$.

(vi) (Liu et al. [16]) If $n \geq 4$, $\delta(G) \geq 2$ and $\rho(G) \geq \sqrt{n^2 - 2n + 4}$, then G is Hamiltonian unless $G = B_{2,n-2;n-2,2}$.

(v) (Li and Ning [12]) If $n \geq (k+1)^2$, $\delta(G) \geq k \geq 1$ and $\rho(G) \geq \rho(Z_{n-k,k})$, then G is Hamiltonian unless $G = Z_{n-k,k}$.

(vi) (Ge and Ning [9]) If $n \geq k^3 + 2k + 4$, $\delta(G) \geq k \geq 1$ and $\rho(G) \geq \sqrt{n(n-k)}$, then G is Hamiltonian unless $G = Z_{n-k,k}$.

(vii) (Jiang et al. [10]) If $n \geq \max\{(k+1)^2, \frac{k^3}{2} + k + 3\}$, $\delta(G) \geq k \geq 1$ and $\rho(G) \geq \sqrt{n(n-k)}$, then G is Hamiltonian unless $G = Z_{n-k,k}$.

(viii) (Li and Ning [12]) If $n \geq (k+1)^2$, $\delta(G) \geq k \geq 1$ and $\mu(G) \geq \mu(Z_{n-k,k})$, then G is Hamiltonian unless $G = Z_{n-k,k}$.

Our current research is also motivated by the results in Theorem 1.11. The following is a useful tool.

Theorem 1.12 (*Bondy and Chvátal* [2]) *A balanced bipartite graph G with $2n$ vertices is Hamiltonian if and only if $\mathscr{B}_{n+1}(G)$ is Hamiltonian.*

To extend results in Theorem 1.11, we need a more general form of Theorem 1.12 in our arguments, as stated in the following Proposition 1.13.

Proposition 1.13 *Let G be a balanced bipartite graph with $2n$ vertices and $q \geq 0$ be an integer,*

(i) G is (q, q)-Hamiltonian if and only if $\mathscr{B}_{n+q+1}(G)$ is (q, q)-Hamiltonian.

(ii) G is (q, q)-traceable if and only if $\mathscr{B}_{n+q+1}(G)$ is (q, q)-traceable.

The main result in this subsection is to find a unified approach as a generalization of all the former results stated in Theorem 1.11, as shown in the following theorem.

Theorem 1.14 *Let k and s be two nonnegative integers and let G be a balanced bipartite graph with $|V(G)| = 2n \geq 8k(k+1)$ and $\delta(G) \geq k \geq \max\{s, 1\}$. If either $\rho(G) \geq \rho(F^0_{n,k,s})$ or $\mu(G) \geq \mu(F^0_{n,k,s})$, then $\mathscr{B}_{n+s}(G) = K_{n,n}$ unless $G \in \left\{F_{n,k,s}, F^0_{n,k,s}\right\}$.*

Since $K_{n,n}$ is (q, q)-Hamiltonian for $0 \leq q \leq n-2$, from Proposition 1.13(i) and Theorem 1.14 we deduce the following result.

Corollary 1.15 *Let q and k be two nonnegative integers and let G be a balanced bipartite graph with $|V(G)| = 2n \geq 8k(k+1)$. If $\delta(G) \geq k \geq q+1$ and if either $\rho(G) \geq \rho(F^0_{n,k,q+1})$ or $\mu(G) \geq \mu(F^0_{n,k,q+1})$, then G is (q, q)-Hamiltonian unless $G \in \left\{F_{n,k,q+1}, F^0_{n,k,q+1}\right\}$.*

Theorem 1.16 *Let q and k be two nonnegative integers and let G be a balanced bipartite graph with $|V(G)| = 2n \geq \max\{6k(k+1), 4(k+2)^2\}$ and $\delta(G) \geq k \geq 2$. If either $\rho(G) \geq \rho(F^0_{n,k,0})$ or $\mu(G) \geq \mu(F^0_{n,k,0})$, then G is traceable unless $G \in \left\{F_{n,k,0}, F^0_{n,k,0}\right\}$.*

With Proposition 1.17 below, Corollary 1.15 and Theorem 1.16 extend Theorem 1.11 for sufficiently large n.

Proposition 1.17 *Let n, k and s be three nonnegative integers. If $n \geq \max\{\frac{1}{2}(k^2+4)(k+1), (k+1)(k-s+2)+2\}$ and $k \geq \max\{s, 1\}$, then $\rho\left(F^0_{n,k,s}\right) < \sqrt{n(n+s-k-1)}$.*

In [15], studies have been done on the relationship between Hamiltonian properties of a graph G and the value of $\Theta(G, \alpha)$, the largest eigenvalue of the matrix $A(G) + \alpha D(G)$ for a real number α. To further the studies in [15], we in this paper will show a lower bound to $\Theta(G, \alpha)$ that assures a balanced bipartite graph G to be (q, q)-traceable as well as to be (q, q)-Hamiltonian. Towards this aim, for integers k, n, s and a real number α, we define

$$\begin{aligned} \varepsilon_0(s) =& n(n+s-k-2) + (k+1)(k+2-s), \text{ and} \\ \Theta_0(s) =& \alpha\left(\frac{\varepsilon_0(s)}{n} + n\right) + (1-\alpha)\sqrt{\varepsilon_0(s)}. \end{aligned} \tag{1.2}$$

Theorem 1.18 *Let k and $s \geq -2$ be two integers and let G be a balanced bipartite graph with $|V(G)| = 2n \geq 6k+8$ and $\delta(G) \geq k \geq \max\left\{|s|, 1\right\}$. If $\Theta(G, \alpha) > \Theta_0(s)$ and $0 \leq \alpha \leq 1$, then either $\mathscr{B}_{n+s}(G) = K_{n,n}$ or $G \subseteq F_{n,k,s}$.*

As $K_{n,n}$ is (q, q)-Hamiltonian for $0 \leq q \leq n-2$, Corollary 1.19 follows immediately from Proposition 1.13(i) and Theorem 1.18.

Corollary 1.19 *Let q and k be two nonnegative integers and let G be a balanced bipartite graph with $|V(G)| = 2n \geq 6k+8$. If $\delta(G) \geq k \geq q+1$ and if both $\Theta(G, \alpha) > \Theta_0(q+1)$ and $0 \leq \alpha \leq 1$, then G is (q, q)-Hamiltonian unless $G \subseteq F_{n,k,q+1}$.*

The following theorem summarizes some of the former results using the spectral radius $\rho(G)$ or the signless Laplacian spectral radius $\mu(G)$ to study the traceability of an almost balanced bipartite graph G.

Theorem 1.20 *Let $G[U, V]$ be an almost balanced bipartite graph with $|V(G)| = 2n-1$.*

(i) (Liu et al. [16]) Suppose that $n \geq 4$, $\delta(G) \geq 1$, and for any $v \in V$, $d_G(v) \geq 2$. If $\rho(G) \geq \sqrt{n^2 - 3n + 4}$, then G is traceable unless $G \in \left\{B_{2,n-2;n-3,2}, B_{2,n-2;n-2,1}\right\}$.

(ii) (Yu et al. [19]) Suppose that $n \geq \max\left\{\frac{1}{2}(k^3+2k+4),\ (k+1)^2\right\}$ and $\delta(G) \geq k \geq 1$. If $\rho(G) > \sqrt{n(n-k-1)}$, then G is traceable unless $G = Z_{n-k-1,k}$.

(iii) (Yu et al. [19]) Suppose that $n \geq (k+1)^2$ and $\delta(G) \geq k \geq 1$. If $\mu(G) > 2n-k-2+\frac{(k+1)^2}{n}$, then G is traceable unless $G \subseteq Z_{n-k-1,k}$.

This also motivates our research along the same line. For a real number α, define

$$\begin{aligned}\Omega(\alpha) =&\alpha\left(2n+q-k-2+\frac{(k+1)(k+1-q)}{n}\right)\\ &+(1-\alpha)\sqrt{n(n+q-k-2)+(k+1)(k+1-q)}.\end{aligned} \tag{1.3}$$

Our main result in this direction is Theorem 1.21, which generalizes Theorem 1.20 when n is sufficiently large.

Theorem 1.21 *Let q and k be two nonnegative integers and let G be an almost balanced bipartite graph with $|V(G)| = 2n-1$ and $\delta(G) \geq k \geq q+1$.*

(i) If $n \geq 3k+4$, $0 \leq \alpha \leq 1$, and $\Theta(G,\alpha) > \Omega(\alpha)$, then G is (q, q)-traceable unless $G \subseteq Z_{n+q-k-1,k-q}$.

(ii) If $n \geq 4k(k+1)$, and if either $\rho(G) \geq \rho\left(Z^0_{n+q-k-1,k-q}\right)$ or $\mu(G) \geq \mu\left(Z^0_{n+q-k-1,k-q}\right)$, then G is (q, q)-traceable unless $G \in \left\{Z_{n+q-k-1,k-q},\ Z^0_{n+q-k-1,k-q}\right\}$.

The organization of this paper is as follows. In Sect. 2, we present the proof to Theorem 1.8. Proposition 1.13 will be justified in Sect. 3. Section 4 is denoted to the verification of Theorem 1.18. Utilizing Theorem 1.18, we then present the proof of Theorem 1.14 in Sect. 5. In Sect. 6, we prove Theorem 1.16 and Proposition 1.17, and then we complete the proof of Theorem 1.21 in Sect. 7.

2 The proof of Theorem 1.8

We start with a few additional lemmas, which are needed in our arguments.

Lemma 2.1 *Let G be a graph with $|E(G)| > 0$. Each of the following holds*:

(i) (Cvetković et al. [7]) $\mu(G) \geq \min\{d(u)+d(v) : uv \in E(G)\}$. Moreover, if G is connected, then equality holds if and only if G is regular or semi-regular bipartite.

(ii) (Li and Ning [12]) $\rho(G) \geq \min\left\{\sqrt{d(u)d(v)} : uv \in E(G)\right\}$. Moreover, if G is connected, then equality holds if and only if G is regular or semi-regular bipartite.

Lemma 2.2 *Let p and q be integers with $p \geq q \geq 1$. If G is a (p, q)-semi-regular bipartite graph, then $|V(G)| \neq q + p + 1$.*

Proof By contradiction, let $G = [U, V]$ be a (p, q)-semi-regular bipartite graph with $|V(G)| = q + p + 1$. By the definition of a (p, q)-semi-regular bipartite graph, we have

$$|U| + |V| = p + q + 1,\ p|U| = q|V|,\ |U| \geq q \geq 1, \text{ and } |V| \geq p \geq 1.$$

Since the order of a (p, q)-semi-regular bipartite graph $[U, V]$ is $p + q + 1$, either $(|U|, |V|) = (q, p + 1)$ or $(|U|, |V|) = (q + 1, p)$, but, as $p > 0$ and $q > 0$, neither of the two possibilities is consistent with $p|U| = q|V|$. □

Proof of Theorem 1.8 For the sake of notational simplicity, throughout the proof, we let $H = \mathscr{C}_{n+s}(G)$. Our argument is to assume that $H \neq K_n$ to prove that in Theorem 1.8 (i), $G \in \mathbb{B}_{n,k,s,r} \cup \mathbb{H}_{n,k,s,r}$ unless $G = \overline{K_{1,k-1}} \vee \overline{K_{1,k-1}}$ and $s = k - 1 \geq 2$, and in Theorem 1.8 (ii), $G \in \mathbb{C}_{n,s,r} \cup \mathbb{D}_{n,s,r} \cup \mathbb{W}_{n,s,r}$.

Since $H \neq K_n$, $\overline{H}$ contains at least one non-trivial component. We shall let F denote a non-trivial component of $\overline{H}$. For any $u, v \in V(H)$ with $uv \notin E(H)$ and $d_H(u) \geq d_H(v)$, as $H = \mathscr{C}_{n+s}(G)$, we conclude that $d_H(u) + d_H(v) \leq n + s - 1$, and so for any edge $uv \in E(\overline{H})$,

$$d_{\overline{H}}(u) + d_{\overline{H}}(v) \geq 2(n - 1) - (n + s - 1) = n - s - 1. \tag{2.1}$$

□

Proof of Theorem 1.8 (i) Our proof of Theorem 1.8 (i) takes an approach similar to those in the justifications of Theorem 1.6(ii) in [12] and of Theorem 3.1 in [20]. Here, for the completeness of the proof, we present it in detail. By (2.1), we have

$$d_{\overline{H}}(u) d_{\overline{H}}(v) \geq d_{\overline{H}}(u)\left(n - s - 1 - d_{\overline{H}}(u)\right). \tag{2.2}$$

Since $\delta(H) \geq \delta(G) \geq k$, we have $d_{\overline{H}}(v) \leq n - k - 1$. This, together with (2.1), implies that $d_{\overline{H}}(v) \geq d_{\overline{H}}(u) \geq n - s - 1 - d_{\overline{H}}(v) \geq k - s$. Hence for each $uv \in E(\overline{H})$, we have

$$k - s \leq d_{\overline{H}}(u) \leq d_{\overline{H}}(v) \leq n - k - 1.$$

Let $\Phi(x) = x(n - s - 1 - x)$ with $k - s \leq x \leq n - k - 1$. The concavity of quadratic functions implies that

$$\Phi(x) \geq \min\left\{\Phi(k - s),\ \Phi(n - k - 1)\right\} = (k - s)(n - k - 1). \tag{2.3}$$

By Lemma 2.1(ii), and by (2.2) and (2.3), we have

$$\begin{aligned}\sqrt{(k - s)(n - k - 1)} &\leq \min\left\{\sqrt{d_{\overline{H}}(u) d_{\overline{H}}(v)}\ :\ uv \in E(F)\right\}\\ &\leq \rho\left(\overline{H}\right) \leq \rho\left(\overline{G}\right) \leq \sqrt{(k - s)(n - k - 1)}.\end{aligned}$$

Claim 1 below follows from Lemma 2.1(ii). □

Claim 1 For any nontrivial component F of $\overline{H}$, each of the following holds.

(i) F is either regular or semi-regular bipartite.
(ii) For any edge $uv \in E(F)$, we have $d_{\overline{H}}(u) = k - s \leq n - k - 1 = d_{\overline{H}}(v)$.
(iii) $n - k \leq |V(F)| \leq n$.

We shall complete the proof of Theorem 1.8(i) by examining the following two cases.

Case 1. $\overline{H}$ contains at least two non-trivial components.

Let F_1 and F_2 be two non-trivial components of $\overline{H}$. By Claim 1, each of F_1 and F_2 is either regular or semi-regular bipartite, and for any edge $uv \in E(\overline{H})$, $d_{\overline{H}}(u) = k - s \leq n - k - 1 = d_{\overline{H}}(v)$. By Claim 1 (iii), we have $2(n-k) \leq n$, and so $2k \leq \max\{2k + 1 - s, 2k\} \leq n \leq 2k$. Thus $n = 2k$, $s \geq 1$ and $\overline{H}$ must have exactly two non-trivial components F_1 and F_2 with $|V(F_1)| = |V(F_2)| = k$. Pick an $F_i \in \{F_1, F_2\}$.

If F_i is regular, then $k - s = n - k - 1 = k - 1$, and so $s = 1$. As $|V(F_1)| = |V(F_2)| = k$ and by Claim 1(ii), $F_1 = F_2 = K_k$. Since $\rho(\overline{G}) = \rho(\overline{H}) = k - 1$ and $\overline{H} \subseteq \overline{G}$, we have $\overline{G} = K_k \cup K_k$, and so $G \in \mathbb{H}_{n,k,1,0}$.

If F_i is semi-regular bipartite, then $d_{\overline{H}}(v) = n - k - 1 = k - 1 = |V(F_i)| - 1$, and so $F_i = K_{1,k-1}$ and $1 = d_{\overline{H}}(u) = k - s$. As $s = k - 1$ and by Claim 1(ii), it follows that $F_1 = F_2 = K_{1,k-1}$. Since $\rho(\overline{G}) = \rho(\overline{H}) = \sqrt{k-1}$ and $\overline{H} \subseteq \overline{G}$, we have $\overline{G} = K_{1,k-1} \cup K_{1,k-1}$, and so $G = \overline{K_{1,k-1}} \vee \overline{K_{1,k-1}}$, as desired.

Case 2. $\overline{H}$ contains only one non-trivial component.

Let F denote this only nontrivial component of $\overline{H}$. By Claim 1(i), F is a regular or semi-regular bipartite graph. Assume first that F is a semi-regular bipartite graph. By Claim 1(ii), F is a connected $(n-k-1, k-s)$-semi-regular bipartite graph, and so for some integer r with $0 \leq r \leq k$, $|V(F)| = n - s - 1 + r$. It follows that $\overline{H} = F \cup (s + 1 - r)K_1$. Since $\rho(\overline{H}) = \rho(\overline{G})$ and $\overline{H}$ is a spanning subgraph of $\overline{G}$, we have $F \cup (s + 1 - r)K_1 \subseteq \overline{G} \subseteq F \cup K_{s+1-r}$, and so $\overline{F} \vee (s + 1 - r)K_1 \subseteq G \subseteq \overline{F} \vee K_{s+1-r}$. By Lemma 2.2, this implies that $G \in \mathbb{B}_{n,k,s,r}$.

Hence we may assume that F is regular. By Claim 1 (ii), $k - s = n - k - 1$ and so $2k + 1 - s = n \geq 2k$, implying $s \leq 1$. By Claim 1 (iii), we conclude that $|V(F)| = n - k + r$, for some integer r with $0 \leq r \leq k$. It follows that $\overline{H} = F \cup (k - r)K_1$. As $\rho(\overline{H}) = \rho(\overline{G})$ and $\overline{H}$ is a spanning subgraph of $\overline{G}$, we have $F \cup (k - r)K_1 \subseteq \overline{G} \subseteq F \cup K_{k-r}$, and so $\overline{F} \vee (k - r)K_1 \subseteq G \subseteq \overline{F} \vee K_{k-r}$. Since $\overline{F}$ is a r-regular graph with $|V(\overline{F})| = n + r - k$, by Definition 1.2(ii), $G \in \mathbb{H}_{n,k,s,r}$.

This completes the proof of Theorem 1.8 (i). □

Proof of Theorem 1.8 (ii) By (2.1) and Lemma 2.1(i), we conclude that, for each nontrivial component F of $\overline{H}$, $n - s - 1 \leq \mu(F) \leq \mu(\overline{H}) \leq \mu(\overline{G}) \leq n - s - 1$, F is either a regular or a semi-regular bipartite graph, and for any $uv \in E(\overline{H})$,

$$\mu(F) = \mu(\overline{H}) = \mu(\overline{G}) = d_{\overline{H}}(u) + d_{\overline{H}}(v) = n - s - 1. \tag{2.4}$$

Similar to the proof of Theorem 1.8 (i), we justify Theorem 1.8 (ii) by a case analysis. □

Claim 2 If $\overline{H}$ has a semi-regular bipartite component, then $\overline{H}$ has exactly one nontrivial component.

Assume that F is a semi-regular bipartite component of $\overline{H}$. By (2.4), $|V(F)| \geq n - s - 1$. If $\overline{H} - V(F)$ contains a nontrivial component F', then by (2.4), $|V(F')| \geq \frac{1}{2}(n - s + 1)$. It follows from $n - s - 1 + \frac{n-s+1}{2} \leq n$ that $n \leq 3s + 1$, contrary to the assumption that $n \geq 3s + 2$. Hence F is the unique non-trivial component of $\overline{H}$. This validates the claim.

Case 1. $\overline{H}$ has a semi-regular bipartite component.

We assume that F is a semi-regular bipartite component of $\overline{H}$. By Claim 2, F is the only nontrivial component of $\overline{H}$. We may assume that F is a connected (p, q)-semi-regular graph with $1 \leq p \leq q$, and for some integer r with $0 \leq r \leq s + 1$, $|V(F)| = n - s - 1 + r$. Thus $s \geq -1$, $\overline{H} = F \cup (s + 1 - r)K_1$, and $1 \leq p \leq \frac{1}{2}(n - s - 1)$ by (2.4).

Since $\mu(\overline{G}) = \mu(\overline{H})$ and $\overline{H}$ is a spanning subgraph of $\overline{G}$, we have $F \cup (s + 1 - r)K_1 \subseteq \overline{G} \subseteq F \cup K_{s+1-r}$, and so $\overline{F} \vee (s + 1 - r)K_1 \subseteq G \subseteq \overline{F} \vee K_{s+1-r}$. By Lemma 2.2, we conclude that $G \in \mathbb{C}_{n,s,r}$. This proves Theorem 1.8 (ii) if Case 1 occurs.

Case 2. $\overline{H}$ does not have a semi-regular bipartite component.

By Lemma 2.1(i) and the assumption of Case 2, every non-trivial component of $\overline{H}$ is regular. Let F denote a component of $\overline{H}$. Then for any vertex $u \in V(F)$, by (2.4), $d_{\overline{H}}(u) = \frac{1}{2}(n - s - 1)$, and so $|V(F)| \geq \frac{1}{2}(n - s + 1)$.

If $\overline{H}$ contains at least three non-trivial components, then $\frac{3(n-s+1)}{2} \leq n$, implying $n \leq 3(s - 1)$, contrary to the assumption that $n \geq 3s + 2$. Hence $\overline{H}$ contains at most two nontrivial components. Let F' denote the possible nontrivial component of $\overline{H} - V(F)$, if it exists.

We first suppose that H is regular, and so $\overline{H}$ is $\frac{1}{2}(n - s - 1)$-regular. In this case, either $\overline{H} = F$ or $\overline{H} = F \cup F'$, where F and F' are both connected $\frac{1}{2}(n - s - 1)$-regular. Since $\overline{H} \subseteq \overline{G}$ and $\mu(\overline{G}) = \mu(\overline{H})$, it follows that $\overline{H} = \overline{G}$, and so $G = H \in \mathbb{W}_{n,s,0}$ for $\overline{H} = F$ or $G = H \in \mathbb{D}_{n,s,0}$ for $\overline{H} = F \cup F'$.

Hence we may assume that H is not regular, and so $\overline{H} \neq F$. Assume first that F and F' are two nontrivial components of $\overline{H}$ containing $\frac{1}{2}(n - s + 1) + r_1$ and $\frac{1}{2}(n - s + 1) + r_2$ vertices, respectively. Thus $\overline{H} = F \cup F' \cup (s - 1 - r_1 - r_2)K_1$. Since $\mu(\overline{G}) = \mu(\overline{H})$ and since $\overline{H}$ is a spanning subgraph of $\overline{G}$, we conclude that $\overline{F} \vee \left(\overline{F'} \vee (s - 1 - r_1 - r_2)K_1\right) \subseteq G \subseteq \overline{F} \vee \left(\overline{F'} \vee K_{s-1-r_1-r_2}\right)$, and so $G \in \mathbb{D}_{n,s,r}$, where $1 \leq r \leq s - 1$.

Therefore, we may assume that F is the only non-trivial component of $\overline{H}$, and so $\overline{H} = F \cup rK_1$, where $r = |V(G) \backslash V(F)|$. Since F is $\frac{1}{2}(n - s - 1)$-regular, we have $\frac{1}{2}(n - s + 1) \leq |V(F)| \leq n - 1$. Since $\mu(\overline{G}) = \mu(\overline{H})$ and since $\overline{H}$ is a spanning subgraph of $\overline{G}$, it follows that $F \cup rK_1 \subseteq \overline{G} \subseteq F \cup K_r$, and so $\overline{F} \vee rK_1 \subseteq G \subseteq \overline{F} \vee K_r$. This implies that $G \in \mathbb{W}_{n,s,r}$, where $1 \le r \le \frac{1}{2}(n+s-1)$. □

3 The Proof of Proposition 1.13

The following result initiated the study of the Bondy–Chvátal closure concept for balanced bipartite graphs.

Lemma 3.1 (*Lemma* 7.3.5 *of* [1]) *Let $G = [U, V]$ be a balanced bipartite graph with $2n$ vertices. Let $u \in U$ and $v \in V$ be two non-adjacent vertices with $d_G(u) + d_G(v) \ge n + 1$. Then G is Hamiltonian if and only if $G + uv$ is Hamiltonian.*

To prove Proposition 1.13, it suffices to prove the following two lemmas.

Lemma 3.2 *Let $G = [U, V]$ be a balanced bipartite graph with $2n$ vertices and q be a nonnegative integer. Let $w_1 \in U$ and $w_2 \in V$ be two vertices satisfying $w_1w_2 \notin E(G)$ and $d_G(w_1) + d_G(w_2) \ge n + q + 1$. Then the following are equivalent.*

(i) G is (q, q)-Hamiltonian.
(ii) $G' = G + w_1w_2$ is (q, q)-Hamiltonian.

Proof As (i) implies (ii) by definition, it remains to show that (ii) implies (i). Let $S \subset V(G)$ satisfying $|S \cap U| = |S \cap V| = q$ and $G_1 = G\big[V(G) \backslash S\big]$. We are to show that G_1 has a Hamilton cycle.

Since G' is (q, q)-Hamiltonian, $G'\big[V(G) \backslash S\big]$ contains a Hamilton cycle C. If C is not a Hamilton cycle of G_1, then $w_1w_2 \in E(C)$, and so this Hamilton cycle C can be expressed as $C = w_1w_2 \ldots w_{2n-2q}w_1$. Since $w_1 \in U$ and $w_2 \in V$, we observe that $|N_S(w_1)| + |N_S(w_2)| \le |S| = 2q$. As $d_G(w_1) + d_G(w_2) \ge n + q + 1$, we have $d_{G_1}(w_1) + d_{G_1}(w_2) \ge n - q + 1$.

Note that G_1 is a balanced bipartite graph with $2(n - q)$ vertices. By Lemma 3.1, G_1 is Hamiltonian if and only if $G_1 + w_1w_2$ is Hamiltonian. □

Lemma 3.3 *Let $G = [U, V]$ be a balanced bipartite graph with $2n$ vertices and q be a nonnegative integer. Let $w_1 \in U$ and $w_2 \in V$ be two vertices satisfying $w_1w_2 \notin E(G)$ and $d_G(w_1) + d_G(w_2) \ge n + q + 1$. Then the following are equivalent.*

(i) G is (q, q)-traceable.
(ii) $G' = G + w_1w_2$ is (q, q)-traceable.

Proof By definition, we observe that (i) implies (ii), and so it suffices to show that (ii) implies (i). Let $S \subset V(G)$ satisfying $|S \cap U| = |S \cap V| = q$ and $G_1 = G\big[V(G) \backslash S\big]$. We are to show that G_1 has a Hamilton path. Since G' is (q, q)-traceable, $G_1 + w_1w_2$ contains a Hamilton path P.

If P is not a Hamilton path of G_1, then $w_1w_2 \in E(P)$. We suppose that $P = u_1u_2 \cdots u_{2n-2q}$, where $w_1 = u_i$ and $w_2 = u_{i+1}$. As $G'[U, V]$ is bipartite with

$u_i \in U$ and $u_{i+1} \in V$, we observe that $|N_S(u_i)| + |N_S(u_{i+1})| \leq |S| = 2q$. By the assumption that $d_G(u_i) + d_G(u_{i+1}) \geq n + q + 1$, we conclude that

$$d_{G_1}(u_i) + d_{G_1}(u_{i+1}) \geq n - q + 1. \tag{3.1}$$

Case 1. $u_1 \in U$.

Then, as $u_i \in U$, i is odd. If $u_i u_{2n-2q} \in E(G)$, then $P - \{u_i u_{i+1}\} + \{u_i u_{2n-2q}\}$ is a Hamilton path of G_1. Hence we may assume that $u_i u_{i+1}, u_i u_{2n-2q} \notin E(G)$. Similarly, we have $u_{i+1} u_1 \notin E(G)$.

Claim 1. There is an index j with either $i + 3 \leq j \leq 2n - 2q - 2$ or $2 \leq j \leq i - 3$, such that $u_i u_j$, $u_{i+1} u_{j+1} \in E(G_1)$.

Since G is bipartite, we may suppose that $N_{G_1}(u_i) = \{u_{s_1}, u_{s_2}, \ldots, u_{s_p}\}$, where $2n - 2q \notin \{s_1, s_2, \ldots, s_p\} = \emptyset$ and for $t \in \{1, 2, \ldots, p\}$, s_t is even. If Claim 1 fails, then $N_{G_1}(u_{i+1}) \subseteq \{u_1, u_3, u_5, \ldots, u_{2n-2q-1}\} \backslash \{u_{s_1+1}, u_{s_2+1}, \ldots, u_{s_p+1}\}$. Thus, $d_{G_1}(u_{i+1}) \leq n - q - p$ and so by (3.1), $n - q + 1 \leq d_{G_1}(u_i) + d_{G_1}(u_{i+1}) \leq p + (n - q - p) = n - q$, a contradiction. This completes the proof of Claim 1.

By Claim 1, either for some j with $2 \leq j \leq i - 3$, both $u_i u_j \in E(G_1)$ and $u_{i+1} u_{j+1} \in E(G_1)$, whence $u_1 u_2 \ldots u_j u_i u_{i-1} \ldots u_{j+1} u_{i+1} u_{i+2} \cdots u_{2n-2q}$ is a Hamiltonian path of $G[V(G) \backslash S]$; or for some j with $i + 3 \leq j \leq 2n - 2q - 2$, both $u_i u_j \in E(G_1)$ and $u_{i+1} u_{j+1} \in E(G_1)$, whence $u_1 u_2 \ldots u_i u_j u_{j-1} \ldots u_{i+1} u_{j+1} u_{j+2} \ldots u_{2n-2q}$ is a Hamiltonian path of $G[V(G) \backslash S]$. This proves that Lemma 3.3 (ii) implies Lemma3.3 (i) in this case.

Case 2. $u_1 \in V$.

As $u_i \in U$, i is even. We first justify the following claim.

Claim 2. There is an index j with either $i + 3 \leq j \leq 2n - 2q - 1$ or $1 \leq j \leq i - 3$, such that $u_i u_j$ and $u_{i+1} u_{j+1} \in E(G_1)$.

Since G is bipartite, we may suppose that $N_{G_1}(u_i) = \{u_{s_1}, u_{s_2}, \ldots, u_{s_p}\}$, where s_t is odd for $t \in \{1, 2, \ldots, p\}$. If Claim 1 fails, then $N_{G_1}(u_{i+1}) \subseteq \{u_2, u_4, u_6, \ldots, u_{2n-2q}\} \backslash \{u_{s_1+1}, u_{s_2+1}, \ldots, u_{s_p+1}\}$. Thus, by (3.1), $n - q + 1 \leq d_{G_1}(u_i) + d_{G_1}(u_{i+1}) \leq p + (n - q - p) = n - q$, a contradiction. This completes the proof of Claim 2.

By Claim 2, either for some j with $1 \leq j \leq i - 3$, both $u_i u_j \in E(G_1)$ and $u_{i+1} u_{j+1} \in E(G_1)$, whence $u_1 u_2 \ldots u_j u_i u_{i-1} \ldots u_{j+1} u_{i+1} u_{i+2} \cdots u_{2n-2q}$ is a Hamiltonian path of $G[V(G) \backslash S]$; or for some j with $i + 3 \leq j \leq 2n - 2q - 1$, both $u_i u_j \in E(G_1)$ and $u_{i+1} u_{j+1} \in E(G_1)$, whence $u_1 u_2 \ldots u_i u_j u_{j-1} \ldots u_{i+1} u_{j+1} u_{j+2} \ldots u_{2n-2q}$ is a Hamiltonian path of $G[V(G) \backslash S]$. Thus in any case, Lemma3.3 holds always. □

4 The Proof of Theorem 1.18

Following the notation in [3], if A, B are disjoint subsets of $V(G)$, then define $E_G[A, B] = \{xy \in E(G) : x \in A \text{ and } y \in B\}$ and $e_G(A, B) = |E_G[A, B]|$. The functions $\varepsilon_0(s)$ and $\Theta_0(s)$, defined in (1.2), will be used in the arguments in this section.

Throughout this section, let k, n and s be integers, and unless otherwise stated, we always assume that $G=[U,V]$ is a balanced bipartite graph with $|V(G)|=2n$ and $H=\mathscr{B}_{n+s}(G)$. By definition, we have

$$\begin{aligned}&\delta(H)\geq\delta(G),\ |E(H)|\geq|E(G)|,\\ &\quad\text{and }\forall u\in U,\ v\in V\text{ with }uv\notin E(H),\ d_H(u)+d_H(v)\leq n+s-1.\end{aligned}\tag{4.1}$$

Lemma 4.1 *If* $n\geq 3k+4,\ s\geq -2,\ \delta(G)\geq k\geq\max\{|s|,1\}$, *and* $|E(G)|>\varepsilon_0(s)$, *then* $\mathscr{B}_{n+s}(G)=K_{n,n}$ *unless* $K_{n,n+s-k-1}\subseteq\mathscr{B}_{n+s}(G)$.

Proof We assume that $H\neq K_{n,n}$ to prove that $K_{n,n+s-k-1}$ must be a subgraph of H. Define

$$\begin{aligned}U_0=&\Big\{w\in U: d_G(w)\geq\frac{1}{2}(n+s)\Big\}, n_U=|U_0|,\\ V_0=&\Big\{w\in V: d_G(w)\geq\frac{1}{2}(n+s)\Big\}\text{ and }n_V=|V_0|.\end{aligned}$$

Claim 1. $n_U\geq k+s+3$ and $n_V\geq k+s+3$.

By symmetry, it suffices to prove $n_U\geq k+s+3$. Direct counting yields that

$$\begin{aligned}|E(G)|\leq|E(H)|=&\sum_{v\in U}d_H(v)=\sum_{v\in U_0}d_H(v)\\ &+\sum_{v\in U-U_0}d_H(v)\leq nn_U+\frac{1}{2}(n+s-1)(n-n_U).\end{aligned}\tag{4.2}$$

It follows by (4.2) and by $|E(G)|>\varepsilon_0(s)$ that

$$\begin{aligned}n_U\geq&\frac{2|E(G)|}{n+1-s}-\frac{n(n+s-1)}{n+1-s}\\ >&\frac{2n(n+s-k-2)+2(k+1)(k+2-s)-n(n+s-1)}{n+1-s}\\ =&\frac{n^2-(2k+3-s)n+2(k+1)(k-s+2)}{n+1-s}\\ =&k+s+2+\frac{\Phi(n)}{n+1-s},\end{aligned}\tag{4.3}$$

where $\Phi(n)=n^2-(2k+3-s)n+2(k+1)(k-s+2)-(n+1-s)(k+s+2)=n^2-(3k+5)n+2k^2-ks+5k+s^2-s+2$.

Since $n\geq 3k+4$, we have $\Phi'(n)=2n-(3k+5)>0$, and so $\Phi(n)\geq\Phi(3k+4)=k(2k-s)+2(k-1)+s(s-1)>0$. It follows by (4.3) that Claim 1 holds.

Let p_0 and q_0 be two positive integers such that $p_0\geq q_0$ and $p_0+q_0=\max\{p+q,\ \text{where}\ K_{p,q}\subseteq H\}$. By Claim 1, we may assume that $p_0\geq q_0\geq k+s+3$. Let $U'\subseteq U$ and $V'\subseteq V$ such that $H[U'\cup V']=K_{p_0,q_0}$ with $|U'|=p_0$ and $|V'|=q_0$. For any $v\in V\backslash V'$, if v will be adjacent with every vertex of U', then a violation to the maximality of p_0+q_0 occurs. Hence v is not adjacent to at least one vertex in U'. By the definition of the $(n+s)$-closure of G and by symmetry, we have

$$\forall v \in V \backslash V',\ d_H(v) \le n+s-q_0-1 \text{ and } \forall u \in U \backslash U',\ d_H(u) \le n+s-p_0-1. \tag{4.4}$$

Claim 2. $q_0 \ge n+s-k-2$.

Assume that Claim 2 does not hold. Then $k+s+3 \le q_0 \le n+s-k-3$. Define $\Phi_1(x) = x^2-(n+s-1)x+n(n+s-1)$. Since H is bipartite, and by (4.4), we have

$$\begin{aligned}|E(H)| &= \sum_{v\in V} d_H(v) = \sum_{v\in V'} d_H(v) + \sum_{v\in V-V'} d_H(v)\\ &\le nq_0 + \left(n-q_0\right)\left(n+s-q_0-1\right)\\ &= q_0^2-(n+s-1)q_0+n(n+s-1) = \Phi_1(q_0).\end{aligned}$$

As $k+s+3 \le q_0 \le n+s-k-3$, we have

$$\Phi_1(q_0) \le \max\left\{\Phi_1\left(k+s+3\right),\ \Phi_1\left(n+s-k-3\right)\right\}.$$

Since $n \ge 3k+4$, we have both $\varepsilon_0(s)-\Phi_1\left(n+s-k-3\right) = n-(2k-s+4) \ge k+s \ge 0$, and $\varepsilon_0(s)-\Phi_1\left(k+s+3\right) = (s+2)(n-2k-5) \ge 0$. Thus, $\varepsilon_0(s) < |E(G)| \le |E(H)| \le \Phi_1(q_0) \le \varepsilon_0(s)$, a contradiction. This completes the proof of Claim 2.

Claim 3. $p_0+q_0 \ge 2n+s-k-1$.

Assume that Claim 3 fails, and so $p_0+q_0 \le 2n+s-k-2$. By Claim 2, we have $p_0 \ge q_0 \ge n+s-k-2$. By (4.4),

$$\begin{aligned}|E(H)| &\le e_H\left(U',\ V'\right)+e_H\left(U\backslash U',\ V\right)+e_H\left(U,\ V\backslash V'\right)\\ &\le p_0q_0+(n+s-1-p_0)(n-p_0)+(n+s-1-q_0)(n-q_0).\end{aligned} \tag{4.5}$$

If $p_0 \ge n+s-k$, then as $\delta(G) \ge k$, it follows from the definition of the $(n+s)$-closure of G that each vertex of V' must be adjacent to every vertex of U, and so $p_0 = n$ and $q_0 = n+s-k-2$. It follows by (4.5) that $\varepsilon_0(s) < |E(G)| \le |E(H)| \le n(n+s-k-2)+(k+1)(k+2-s) = \varepsilon_0(s)$, a contradiction. Hence we may assume that $n+s-k-2 \le q_0 \le p_0 \le n+s-k-1$.

If $p_0 = q_0 = n+s-k-1$, then by (4.5) we have $|E(H)| \le (n+s-k-1)^2+2k(k+1-s)$. As $n \ge 3k+4$, this leads to

$$\begin{aligned}&\varepsilon_0(s)-|E(H)|\\ &\quad\ge \varepsilon_0(s)-\left((n+s-k-1)^2+2k(k+1-s)\right)\\ &\quad= (k-s)(n+s-2k-1)+1 \ge (k-s)((3k+4)+s-2k-1)+1\\ &\quad= (k-s)(k+s+3)+1 > 0.\end{aligned}$$

Hence $|E(G)| \le |E(H)| < \varepsilon_0(s)$, contrary to the assumption of the lemma.

If $p_0 = n+s-k-1$ and $q_0 = n+s-k-2$, then by (4.5) and $n \ge 3k+4$ we have $|E(G)| \le (n+s-k-1)(n+s-k-2) + k(k+1-s) + (k+1)(k+2-s) < \varepsilon_0(s)$, again a contradiction.

If $p_0 = n+s-k-2 = q_0$, then by (4.5) and $n \ge 3k+4$ we have $|E(G)| \le (n+s-k-2)^2 + 2(k+1)(k+2-s) < \varepsilon_0(s)$, contrary to the assumption of the lemma, and so Claim 3 is justified.

If $p_0 = n$, then the lemma follows from Claim 3. Assume that $p_0 \le n-1$, and so $q_0 \ge n+s-k$ by Claim 3. As $\delta(G) \ge k$, we conclude that every vertex of U' must be adjacent to all vertices of V, implying that $p_0 \ge q_0 = n$, contrary to the assumption that $H \ne K_{n,n}$. □

Theorem 4.2 *If $n \ge 3k+4$, $s \ge -2$, $\delta(G) \ge k \ge \max\{|s|, 1\}$ and $|E(G)| > \varepsilon_0(s)$, then $\mathscr{B}_{n+s}(G)$ is isomorphic to a member in $\{K_{n,n}, F_{n,k,s}\}$.*

Proof We assume that $H \ne K_{n,n}$ to show that $H = F_{n,k,s}$. Let t be the largest integer such that $K_{n,t} \subseteq H$. By Lemma 4.1, $n+s-k-1 \le t < n$. Let $V' \subset V$ be the vertex sets of H such that $H[U \cup V'] = K_{n,t}$. If $t \ge n+s-k$, since every vertex in U has degree at least $t \ge n+s-k$ in H and $\delta(H) \ge k$, we have $H = K_{n,n}$, contrary to the assumption. Hence we must have $t = n+s-k-1$.

Define $U_0 = \{u \in U : d_H(u) \ge n+s-k\}$. Since $\delta(H) \ge k$ and since every vertex in U has degree at least $n+s-k-1$ in H, it follows from the definition of the $(n+s)$-closure of G that every vertex in $V \backslash V'$ has degree exactly k in H, and is adjacent to every vertex in U_0. This implies that $|U_0| = k$, and so $H = F_{n,k,s}$. □

We need the following two lemmas to complete the proof of Theorem 1.18.

Lemma 4.3 (*Li and Ning* [12]) *If G is a balanced bipartite graph with $|V(G)| = 2n$, then $\mu(G) \le \frac{|E(G)|}{n} + n$.*

When $|V(G)| \ge 2$, let $\rho_1(G)$ and $\rho_2(G)$ denote the largest and the second largest eigenvalues of $A(G)$, respectively. Thus, $\rho_1(G) = \rho(G)$.

Lemma 4.4 (*Lai et al.* [11]) *If G is a bipartite graph with $|V(G)| \ge 2$, then $(\rho_1(G))^2 + (\rho_2(G))^2 \le |E(G)|$.*

When $0 \le \alpha \le 1$, since $A(G) + \alpha D(G) = \alpha Q(G) + (1-\alpha)A(G)$, from the properties of Rayleigh quotients we have $\Theta(G,\alpha) \le \alpha\mu(G) + (1-\alpha)\rho(G)$. Thus, the corollary below follows immediately from Lemmas 4.3 and 4.4.

Corollary 4.5 *Let α be a real number with $0 \le \alpha \le 1$. If $|V(G)| = 2n \ge 2$, then*

$$\Theta(G,\alpha) \le \alpha\mu(G) + (1-\alpha)\rho(G) \le \alpha\left(\frac{|E(G)|}{n} + n\right) + (1-\alpha)\sqrt{|E(G)|}. \quad (4.6)$$

Recall that $\Theta_0(s)$ and $\varepsilon_0(s)$ have been defined in (1.2). If $|E(G)| \le \varepsilon_0(s)$, then Corollary 4.5 implies that $\Theta(G,\alpha) \le \Theta_0(s)$. This deduces the following result.

Corollary 4.6 *Let α be a real number with $0 \le \alpha \le 1$. If $|V(G)| = 2n \ge 2$ and $\Theta(G,\alpha) > \Theta_0(s)$, then $|E(G)| > \varepsilon_0(s)$.*

Proof of Theorem 1.18 In our hypotheses, $\Theta(G,\alpha) > \Theta_0(s)$, hence, Theorem 4.2 and Corollary 4.6 imply that $\mathscr{B}_{n+s}(G)$ is isomorphic to a member in $\{F_{n,k,s}, K_{n,n}\}$. $\square$

5 The Proof of Theorem 1.14

Given two distinct vertices u, v in a graph G, if $N_G(v)\backslash(N_G(u)\cup\{u\}) \neq \emptyset \neq N_G(u)\backslash(N_G(v)\cup\{v\})$, then we construct a new graph $G' = G'(u,v)$ by replacing all edges vw by uw for each $w\in N_G(v)\backslash(N_G(u)\cup\{u\})$. This operation is called the *Kelmans transformation* from v to u (see [6]).

Lemma 5.1 (*Liu et al.* [15]) *Let G be a connected graph. If G' is a graph obtained from G by some Kelmans transformation and $\alpha \geq 0$, then $\Theta(G',\alpha) > \Theta(G,\alpha)$.*

In the discussion of Lemma 5.2 below, the notation in Definition 1.2 (vi) will be adopted.

Lemma 5.2 *Let G be a graph obtained from $Z_{p,q}$ by deleting one edge. If $p\geq k+1$, $q\geq 1$, $\alpha\geq 0$ and $\delta(G)\geq k\geq 1$, then $\Theta(G,\alpha)\leq \Theta\left(Z^0_{p,q},\alpha\right)$, with equality if and only if $G = Z^0_{p,q}$.*

Proof Let $G' = Z_{p,q}$ and $G_0 = Z^0_{p,q}$. Let $e = w_0z_0 \in E(G')$, and $G = G' - e$. It suffices to show that if $G\neq G_0$, then

$$\Theta(G,\alpha) < \Theta(G_0,\alpha). \tag{5.1}$$

Let U and V be the bipartition of G' such that V contains q vertices of degree k and U contains k vertices of degree $p+q$ in G'. Let U' and V' be the vertices of degrees $p+q$ and n, respectively, in U and V of G'. Since every vertex of $V\backslash V'$ has degree k and since $G\neq G_0$, by symmetry, we may assume that $w_0\in U'$ and $z_0\in V'$.

Choose $v\in U\backslash U'$. Then, $N_G(v)\backslash(N_G(w_0)\cup\{w_0\}) = \{z_0\}$ and $N_G(w_0)\backslash(N_G(v)\cup\{v\})\neq\emptyset$. It is routine to verify that G_0 is isomorphic to the graph obtained from G by a Kelmans transformation from v to w_0. By Lemma 5.1, $\Theta(G,\alpha) < \Theta(G_0,\alpha)$, and so (5.1) holds. $\square$

Let G be a connected graph. For any real number $\alpha\geq 0$, it is well known that $A(G)$ is nonnegative and irreducible if and only if G is connected, and thus $A(G) + \alpha D(G)$ is a nonnegative irreducible matrix. This implies the existence of a unique positive unit eigenvector $f = (f(v_1), f(v_2), \ldots, f(v_n))^T$ corresponding to $\Theta(G,\alpha)$. This vector f is often called the *Perron vector* of G.

Lemma 5.3 *For any integers n, q and a real number α, define a polynomial in θ as follows:*

$$\begin{aligned}\Psi(\theta) =&\theta^4 - 2(n+q-1)\alpha\theta^3 + \Big(\alpha^2\big(n^2+4nq-3n+q^2-3q+1\big) - nq + 1\Big)\theta^2 \\ &- \alpha\Big(\alpha^2(2nq-q-n)(n+q-1) - nq(n+q-2)\Big)\theta \\ &+ \big(\alpha^2-1\big)(n-1)(q-1)\big(nq\alpha^2-1\big).\end{aligned}$$

If $2 \le q \le n$ and $\alpha \ge 0$, then $\Theta(K_{n,q} - e, \alpha)$ is the maximum root of $\Psi(\theta)$.

Proof Denote $G = K_{n,q} - e$ and $\Theta = \Theta(K_{n,q} - e, \alpha)$. Let f be the Perron vector of G, and let U and V be the two partite sets of G such that $|U| = n$ and $|V| = q$. For convenience, we suppose that $e = w_0z_0$ with $w_0 \in U$ and $z_0 \in V$.

Let $x_1 = f(w)$ for $w \in U\backslash\{w_0\}$, let $x_2 = f(w)$ for $w \in V\backslash\{z_0\}$, let $x_3 = f(w_0)$ and $x_4 = f(z_0)$. it follows from $\big(A(G) + \alpha D(G)\big)f = \Theta f$ that

$$\begin{cases} \big(\Theta - q\alpha\big)\, x_1 = (q-1)x_2 + x_4, \\ \big(\Theta - n\alpha\big)\, x_2 = (n-1)x_1 + x_3, \\ \big(\Theta - (q-1)\alpha\big)\, x_3 = (q-1)x_2, \\ \big(\Theta - (n-1)\alpha\big)\, x_4 = (n-1)x_1. \end{cases} \tag{5.2}$$

By multiplying $\Theta - (n-1)\alpha$ in both side of the first equation of (5.2), and then multiplying $\Theta - (q-1)\alpha$ in both side of the second equation of (5.2), it follows that

$$\begin{cases} \big(\Theta - q\alpha\big)\big(\Theta - (n-1)\alpha\big)\, x_1 = (q-1)\big(\Theta - (n-1)\alpha\big)x_2 + \big(\Theta - (n-1)\alpha\big)x_4, \\ \big(\Theta - n\alpha\big)\big(\Theta - (q-1)\alpha\big)\, x_2 = \big(\Theta - (q-1)\alpha\big)(n-1)x_1 + \big(\Theta - (q-1)\alpha\big)x_3. \end{cases} \tag{5.3}$$

By substituting the last two equation of (5.2) into (5.3), we have

$$\begin{cases} \big((\Theta - q\alpha)(\Theta - (n-1)\alpha) - (n-1)\big)\, x_1 = (q-1)(\Theta - (n-1)\alpha)x_2, \\ \big((\Theta - n\alpha)(\Theta - (q-1)\alpha) - (q-1)\big)\, x_2 = (\Theta - (q-1)\alpha)(n-1)x_1. \end{cases} \tag{5.4}$$

Now, by (5.4), Θ is equal to the maximum root of $\Psi(\theta)$, as required. □

Corollary 5.4 *Let k and s be two nonnegative integers such that $k \ge \max\{s, 1\}$. Each of the following holds.*

- (i) If $n \ge (k+1)(k-s+2)+2$, then $\rho\big(K_{n,n+s-k-1} - e\big) > \sqrt{\varepsilon_0(s)}$.
- (ii) If $n \ge 4k(k+1)$, then $\mu\big(K_{n,n+s-k-1} - e\big) > n + \frac{\varepsilon_0(s)}{n}$.

Proof In proofs below, denote $G = K_{n,n+s-k-1} - e$ and use ρ and μ for $\rho(G)$ and $\mu(G)$, respectively. Define

$$\begin{aligned}\Psi_1(\theta) =& \theta^4 - \big(n(n+s-k-1)-1\big)\theta^2 + (n+s-k-2)(n-1), \text{ and}\\ \Psi_2(\theta) =& \theta^3 - 2(2n+s-k-2)\theta^2 + \big((k-s)^2 + (n-1)(5n+5s-5k-6)\big)\theta\\ &- (n-1)(n+s-k-2)(2n+s-k-1).\end{aligned}$$

By setting $q = n+s-k-1$ and $\alpha \in \{0,1\}$ in Lemma 5.3, ρ and μ are equal to the maximum roots of $\Psi_1(\theta)$ and $\Psi_2(\theta)$, respectively. As $n \geq (k+1)(k-s+2) + 2 > k-s+4$ and by $\Psi_1(\rho) = 0$ it follows that

$$\begin{aligned}\rho^2 =& \frac{1}{2}\Big(n(n+s-k-1)-1\\ &+\sqrt{\big(n(n+s-k-1)-1\big)^2 - 4(n+s-k-2)(n-1)}\ \Big)\\ &> n(n+s-k-1)-2 \geq \varepsilon_0(s).\end{aligned}$$

This completes the proof of (i).

To prove (ii), we first prove the following claim.

Claim 1. $\Psi_2\Big(2n+s-k-2+\frac{(k+1)(k+2)}{n}\Big) < 0.$

By algebraic manipulations, we have

$$\Psi_2\left(2n+s-k-2+\frac{(k+1)(k+2)}{n}\right) = -\frac{1}{n^3}\Psi_3(n),$$

where $\Psi_3(n) = n^5 - \Big(k(k+4)-s+5\Big)n^4 - \Big((k+2)(k+1)s - (k+2)^3 + s\Big)n^3 - (k+2)(k+1)\big(2k^2+7k-s+6\big)n^2 + (k+2)^2(k+1)^2(k-s+2)n - (k+2)^3(k+1)^3$.

Recall that $n \geq 4k(k+1)$. Thus, $\Psi_3'''(n) = 6\Big(10n^2 - 4\big(k(k+4)-s+5\big)n - (k+2)(k+1)s + (k+2)^3 - s\Big) \geq \Psi_3'''(4k(k+1)) = 6\Big(s\big(15k^2+13k-3\big) + k\big(144k^3+241k^2+22k-68\big)+8\Big) > 0$, and so $\Psi_3''(n) \geq \Psi_3''(4k(k+1)) = 2(k+1)\Big(s\big(84k^3+60k^2-35k+2\big) + k^2\big(544k^3+812k^2-154k-347\big)+76k-12\Big) > 0.$

This leads to $\Psi_3'(n) \geq \Psi_3'(4k(k+1)) = (k+1)^2\Big(s\big(208k^4+112k^3-137k^2+12k-4\big) + k\big(1024k^5+1328k^4-752k^3-791k^2+230k-84\big)+8\Big) > 0$, and so $\Psi_3(n) \geq \Psi_3(4k(k+1)) = (k+1)^3\Big(4ks\big(48k^4+16k^3-45k^2+4k-4\big) + k^2\big(768k^5+832k^4-928k^3-684k^2+215k-150\big)+4(5k-2)\Big) > 0$. This completes the proof of Claim 1.

Direct computation yields that $\Psi_2(0) = -(n-1)(n+s-k-2)(2n+s-k-1) < 0$ and $\Psi_2(n+s-k-2) = n+s-k-2 > 0$. It is observed that $\Psi_2(\theta)$ tends to infinity when θ tends to infinity. Combining this with $\Psi_2\left(2n+s-k-2+\frac{(k+1)(k+2)}{n}\right)<0$ by Claim 1, we conclude that

$$\mu > 2n+s-k-2+\frac{(k+1)(k+2)}{n} \geq n+\frac{\varepsilon_0(s)}{n},$$

and so (ii) follows. □

Proof of Theorem 1.14 Since $K_{n,n+s-k-1}-e \subset F^0_{n,k,s}$, by Corollary 5.4, $\rho\left(F^0_{n,k,s}\right) > \sqrt{\varepsilon_0(s)}$ and $\mu\left(F^0_{n,k,s}\right) > n+\frac{\varepsilon_0(s)}{n}$. Thus Theorem 1.14 follows from Lemma 5.2 and Theorem 1.18. □

6 The proofs of Theorem 1.16 and Proposition 1.17

By Definition 1.2 (vi) and (1.1), for an edge $w_0z_0 \in E(F_{n,k,s})$ with $d_{F_{n,k,s}}(w_0) = n+s-k-1$ and $d_{F_{n,k,s}}(z_0) = n$, $F^0_{n,k,s} = F_{n,k,s} - w_0z_0$. Throughout this section, we let $J = F^0_{n,k,s}$ and $J_0 = F^0_{n,k,0}$. Unless specially indicated, we assume that k and s are two nonnegative integers such that $k \geq \max\{s,1\}$.

Lemma 6.1 *If $n \geq (k+1)(k-s+2)+2$, then $\rho_2(F^0_{n,k,s}) < \sqrt{n(n+s-k-1)}$.*

Proof By Corollary 5.4 and as $K_{n,n+s-k-1}-e \subset F^0_{n,k,s} = J$, we have $\rho(J) > \sqrt{\varepsilon_0(s)}$. By Lemma 4.4, it follows that

$$\begin{aligned}\rho_2(J) &< \sqrt{|E(J)|-\varepsilon_0(s)}\\ &= \sqrt{n^2-(k+1-s)(n-k)-1-n(n+s-k-2)-(k+1)(k+2-s)}\\ &= \sqrt{n-(2k-s+3)}.\end{aligned}$$

Since $n > 2(k+1)$, $n(n+s-k-1)-\big(n-(2k-s+3)\big) = n^2-(k+2-s)n+2k-s+3 \geq 2k(k+s)+4k+s+3 > 0$. This completes the proof of the lemma. □

Lemma 6.2 *If $n \geq 2(k+1)-s$, then $\rho(F^0_{n,k,s})$ is equal to the maximum root of $\Psi_4(x)$, where $\Psi_4(x) = x^2\left(x^2-(n+s-k-2)\right)\left(x^2-k(k+1-s)\right) - \left(x^2+(n+s-k-2)(x^2-1)\right)\left(kx^2+(n-k-1)(x^2-k(k+1-s))\right)$.*

Proof By Definition 1.2 (vi) and (1.1), $J=[U,V]$ is a bipartite graph and we may assume that V contains $k+1-s$ vertices of degree k and U contains k vertices of degree n in J. Define

$$\begin{aligned} U_1 &= \{u \in U : d_J(u) = n\} \text{ and } U_2 = U \backslash U_1,\\ V_1 &= \{v \in V : n-1 \leq d_J(v) \leq n\} \text{ and } V_2 = V \backslash V_1. \end{aligned} \tag{6.1}$$

By symmetry, we may assume that $w_0 \in U_2$ and $z_0 \in V_1$.

Let f be the Perron vector of J, and let $\rho = \rho(J)$. We shall adopt the following notation in the rest of the arguments:

$$\begin{cases} x_1 = f(w) \quad \text{if} \quad w \in U_1, \\ x_2 = f(w) \quad \text{if} \quad w \in V_2, \\ x_3 = f(w) \quad \text{if} \quad w \in V_1 \backslash \{z_0\}, \\ x_4 = f(w) \quad \text{if} \quad w \in U_2 \backslash \{w_0\}, \\ x_5 = f(w_0), \ \text{and } x_6 = f(z_0). \end{cases} \tag{6.2}$$

As $\big(A(J)\big)f = \rho f$, it follows that

$$\begin{cases} \rho x_1 = (k+1-s)x_2 + (n+s-k-2)x_3 + x_6, \\ \rho x_2 = kx_1, \\ \rho x_3 = kx_1 + (n-k-1)x_4 + x_5, \\ \rho x_4 = (n+s-k-2)x_3 + x_6, \\ \rho x_5 = (n+s-k-2)x_3, \\ \rho x_6 = kx_1 + (n-k-1)x_4. \end{cases} \tag{6.3}$$

The first four equations of (6.3) imply that

$$x_4 = \left(1 - \frac{k(k+1-s)}{\rho^2}\right) x_1. \tag{6.4}$$

The equations on x_3, x_5 and x_6 of (6.3) lead to

$$x_3 = \frac{\rho^2}{\rho^2 - (n+s-k-2)} x_6. \tag{6.5}$$

It follows from (6.5) and the first two equations of (6.3) that

$$x_6 = \frac{\big(\rho^2 - (n+s-k-2)\big)\big(\rho^2 - k(k+1-s)\big)}{\rho\big(\rho^2 + (n+s-k-2)(\rho^2-1)\big)} x_1. \tag{6.6}$$

With algebraic manipulations and utilizing (6.4), (6.6) and the sixth equation of (6.3), ρ is equal to the maximum root of $\Psi_4(x)$, as desired. $\square$

Proof of Proposition 1.17 Let $\Psi_4(x)$ as defined in Lemma 6.2.

To complete the proof, by Lemma 6.1, it suffices to show that

$$\Psi_4\left(\sqrt{n(n+s-k-1)}\right) > 0. \tag{6.7}$$

Let $\Phi_2(n) = \Psi_4\left(\sqrt{n(n+s-k-1)}\right)$. Algebraic manipulation yields $\Phi_2(n) = 2n^4 - \left((k^2+4)(k+1-s)+2\right)n^3 + \left(2k^3(k-2s+2) + 2k^2(s-1)^2 - (2s-5)(k-s) - 2(s-3)\right)n^2 - (k-s+1)(k-s+2)\left(k^2(k-s) - 2k+1\right)n - k(k+1)(k-s+1)(k-s+2)$.

Case 1. $s \geq 1$.

Let $\Phi_3(n) = \Big(2k^3(k-2s+2) + 2k^2(s-1)^2 - (2s-5)(k-s) - 2(s-3)\Big)n^2 - (k-s+1)(k-s+2)\Big(k^2(k-s) - 2k+1\Big)n - k(k+1)(k-s+1)(k-s+2)$ and $\Phi_4(n) = 2n^4 - \big((k^2+4)(k+1-s)+2\big)n^3$.

When $2n \geq (k^2+4)(k+1)$, we have $\Phi_3'(n) \geq \Phi_3'\Big(\frac{1}{2}(k^2+4)(k+1)\Big) = 2k^5(k-s)^2 + 2k^4(k-s)(3k-s) + k^3(k-s)(13k-6s+28) + k^3(s^2-7s+13) + 7k^2(s-1)s + 22k^2(k-s) + k^2(s^3-4s+4) + 5k(2s^2-8s+7k) + 45k - 25s + 7s^2 + 22 > 0$. This implies that $\Phi_3(n) \geq \Phi_3\Big(\frac{1}{2}(k^2+4)(k+1)\Big) = \frac{1}{4}(k+1)\Big(2k^7(k-s)^2 + 2k^6(k-s)(3k-s) + 4k^5(k-s)(5k-3s) + k^4(k-s)(49k-12s) + k^3(k-s)(83k-16s) + 12k^3\big(s(s-1) + 12(k-s)\big) + 2k^2s\big(k^2s^2 - 2k + 4s^2\big) + 2k^2(11s^2 - 77s + 91k) + 4(46k^2 - 41ks + 6s^2) + 44(ks^2 - 2s + 4k) + 80\Big) > 0$.

When $2n \geq (k^2+4)(k+1)$, we have $2n - \big((k^2+4)(k+1-s)+2\big) \geq s(k^2+4) - 2 \geq k^2+2 > 0$, and so $\Phi_4(n) > 0$. As $\Phi_2(n) = \Phi_3(n) + \Phi_4(n) > 0$, it follows that (6.7) must hold.

Case 2. $s = 0$.

Define $\Phi_5(n) = 2n^3 - \Big(k^2(k+1) + 2k + 4\Big)n^2 + (k+2)\big(k^3 - k + 1\big)n + k(k+2)(k+1)$. As $s = 0$, $\Phi_2(n) = (n-k-1)\Phi_5(n)$. Since $2n \geq (k^2+4)(k+1) > k^2(k+1) + 2k + 4$, we have $\Phi_5(n) > (k+2)\big(k^3 - k + 1\big)n + k(k+2)(k+1) > k(k+2)(k+1) > 0$. Thus, $\Phi_2(n) > 0$ and so (6.7) holds. □

Lemma 6.3 *If* $n \geq 3k(k+1)$ *and* $k \geq 2$, *then* $\rho(F^0_{n,k,0}) > \sqrt{n(n-k-2) + (k+2)^2}$.

Proof Throughout this proof, we let $\rho = \rho(J_0)$ to simplify the notation.

By Lemma 6.2, ρ is equal to the maximum root of $\Psi_5(x)$, where $\Psi_5(x) = x^2\Big(x^2 - (n-k-2)\Big)\Big(x^2 - k(k+1)\Big) - \Big(x^2 + (n-k-2)(x^2-1)\Big)\Big(kx^2 + (n-k-1)(x^2 - k(k+1))\Big)$. To show that $\rho > \sqrt{n(n-k-2) + (k+2)^2}$, it suffices to prove $\Psi_5(\sqrt{n(n-k-2) + (k+2)^2}) < 0$.

Denote by $\Phi_6(n) = \Phi(\sqrt{n(n-k-2) + (k+2)^2}) = -n^5 + (k^2+6k+10)n^4 - \big(3k(k^2+5k+13) + 37\big)n^3 + (k+2)(4k^3 + 17k^2 + 45k + 44)n^2 - (k^3(3k^2 + 22k + 78) + 174k^2 + 219k + 112)n + (k+2)(k^5 + 7k^4 + 26k^3 + 64k^2 + 85k + 44)$.

When $n \geq 3k(k+1)$ and $k \geq 2$, we have $\Phi''''_6(n) = 24\big(-5n + k^2 + 6k +$

$10\big) \leq -24(14k^2+9k-10)<0$ and thus $\Phi'''_6(n) \leq \Phi'''_6(3k(k+1)) = -6\big(k^2(78k^2-87)+9k(11k^2-9)+37\big)<0$. Once again, since $n \geq 3k(k+1)$ and $k \geq 2$, we have $\Phi''_6(n) \leq \Phi''_6(3k(k+1)) = -2\big(k^4(216k^2-274)+k^3(405k^2-673)+65k^2+199k-88\big)<0$ and so $\Phi'_6(n) \leq \Phi'_6(3k(k+1)) = -\big(k^3(297k^5+729k^4-375k^3-1899k^2-575k+303)+159k^2(k-1)+309k(k^2-1)+112\big)<0$.

This implies that $\Phi_6(n) \leq \Phi_6(3k(k+1)) = -\big(162k^{10}+486k^9-198k^8-1827k^7-$-

. Denote by $\Phi_7(k) = 162k^4+486k^3-198k^2-1827k-1339$. Since $\Phi_7(2) = 695 > 0$ and $\Phi_7(k) = 175k^3-1339+k^2(311k-198)+k(162k^3-1827) > 0$ for $k \geq 3$, we have $\Phi_6(n)<0$, as desired. □

Lemma 6.4 *If* $n \geq 2(k+2)^2$ *and* $k \geq 1$, *then* $\mu(F^0_{n,k,0}) > 2n-k-1.5$.

Proof Throughout this proof, we let $\mu = \mu(J_0)$ to simplify the notation. Define U_1, U_2, V_1 and V_2 as in (6.1) with $s=0$ and let $w_0 \in U_2$ and $z_0 \in V_1$. Moreover, we let f be the Perron vector of J_0 and we also adopt the same notation from (6.2). As $\big(Q(J_0)\big)f = \mu f$, it follows that

$$\begin{cases} (\mu-n)\,x_1 = (k+1)x_2+(n-k-2)x_3+x_6, \\ (\mu-k)\,x_2 = kx_1, \\ (\mu-n)\,x_3 = kx_1+(n-k-1)x_4+x_5, \\ (\mu-n+k+1)\,x_4 = (n-k-2)x_3+x_6, \\ (\mu-n+k+2)\,x_5 = (n-k-2)x_3, \\ (\mu-n+1)x_6 = kx_1+(n-k-1)x_4. \end{cases} \tag{6.8}$$

The first four equations of (6.8) imply that

$$x_4 = \frac{(\mu-n)(\mu-k)-k(k+1)}{(\mu-k)(\mu-n+k+1)}x_1. \tag{6.9}$$

The equations on x_3, x_5 and x_6 of (6.8) lead to

$$x_3 = \frac{(\mu-n+1)(\mu-n+k+2)}{(\mu-n)(\mu-n+k+2)-(n-k-2)}x_6. \tag{6.10}$$

It follows from (6.10) and the first two equations of (6.8) that

$$x_6 = \frac{(\mu^2-(k+n)\mu+k(n-k-1))(\mu^2-(2n-k-2)\mu+(n-1)(n-k-2))}{(\mu-k)((n-k-1)\mu^2-(2n^2-(3k+5)n+(k+2)^2)x+(n-1)(n-k-1)(n-k-2))}x_1. \tag{6.11}$$

With algebraic manipulations and utilizing (6.9), (6.11) and the sixth equation of (6.8), μ is equal to the maximum root of $\Psi_6(x)$, where $\Psi_6(x) = x^4-\big(4n-4-k\big)x^3+\big(5n^2-(k+11)n-2k^2+6\big)x^2-\big(2n^3+(2k-7)n^2-(6k^2+7k-7)n+2k^3+8k^2+6k-2\big)x+2k(n-1)(n-k-1)(n-k-2)$.

Since $\Psi_6(2n-k-1.5)=\frac{-1}{16}\Phi_8(n)$, where $\Phi_8(n)=16n^3+4(4k^2-4k-19)n^2-4(4k^3-25k-23)n-8k^3-36k^2-66k-33=4n(4n^2-19n-4k^3+25k+23)+16k(k-1)n^2-8k^3-36k^2-66k-33$. Note that $n\geq 2(k+2)^2$. Thus, $\Phi_8''(n)=8(12n+4k^2-4k-19)\geq\Phi_8''(2(k+2)^2)=8(28k^2+92k+77)>0$ and so $\Phi_8'(n)\geq\Phi_8'(2(k+2)^2)=4(64k^4+428k^3+1076k^2+1193k+487)>0$. This implies that $\Phi_8(n)\geq\Phi_8(2(k+2)^2)=192k^6+1952k^5+8272k^4+18624k^3+23348k^2+15294k+4031$. Now, we can conclude that $\Psi_6(2n-k-1.5)<0$, completing the proof of this result. $\square$

Lemma 6.5 (*Li and Ning* [13]) *Let G be a balanced bipartite graph on $2n$ vertices. If $\delta(G)\geq k\geq 1$, $n\geq 2k+3$ and $|E(G)|>n(n-k-2)+(k+2)^2$, then G is traceable unless $G\subseteq F_{n,k,0}$ or $k=1$ and $G\subseteq K_{n-1,n-1}\cup K_2$.*

Proof of Theorem 1.16 Let G denote the balanced bipartite graph satisfying the hypotheses of Theorem 1.16. By (4.6) and Lemmas 6.3–6.4, we have

$$n+\frac{n(n-k-2)+(k+2)^2}{n}\leq 2n-k-1.5<\mu(J_0)\leq\frac{|E(J_0)|}{n}+n, \tag{6.12}$$

$$\sqrt{n(n-k-2)+(k+2)^2}<\rho(J_0)\leq\sqrt{|E(J_0)|}. \tag{6.13}$$

Each of (6.12) and (6.13) implies that $|E(J_0)|>n(n-k-2)+(k+2)^2$. By Lemma 6.5, by the assumption that either $\rho(G)\geq\rho\left(F^0_{n,k,s}\right)$ or $\mu(G)\geq\mu\left(F^0_{n,k,s}\right)$, and as $\delta(G)\geq 2$, we conclude that either G is traceable or $G\subseteq F_{n,k,0}$. It follows by Lemma 5.2 that $G\in\left\{F_{n,k,0},\ F^0_{n,k,0}\right\}$. $\square$

7 The proof of Theorem 1.21

Throughout this section, we assume that $G=[U,V]$ is an almost balanced bipartite graph with $|U|=|V|+1=n$. Let v_0 be a vertex not in $V(G)$ and define a balanced bipartite graph G^{v_0} from G by adding v_0 and n edges joining v_0 to all vertices of U.

Lemma 7.1 *If G^{v_0} is $(q,\ q)$-Hamiltonian, then G is $(q,\ q)$-traceable.*

Proof As the case when $q=0$ follows from definition immediately, we assume that $q\geq 1$. Let S be an arbitrary set of $2q$ vertices of G such that $|S\cap U|=q=|S\cap V|$. Choose a vertex $v\in S\cap V$. Let $S_1=\left(S\backslash\{v\}\right)\cup\{v_0\}$ and $V_1=V\cup\{v_0\}$. Then $|S_1\cap U|=q=|S_1\cap V_1|$. Since G^{v_0} is $(q,\ q)$-Hamiltonian, $G^{v_0}\left[V(G^{v_0})\backslash S_1\right]$ contains a Hamiltonian cycle, and hence $G\left[V(G)\backslash S\right]$ is traceable, as $G^{v_0}\left[(V(G^{v_0})\backslash S_1)\backslash\{v\}\right]=G\left[V(G)\backslash S\right]$. By the arbitrariness of S, G is $(q,\ q)$-traceable. $\square$

Proof of Theorem 1.21 (i): We first show that, under the assumption of Theorem 1.21, we have

$$|E(G)| > n(n+q-k-2) + (k+1)(k+1-q). \tag{7.1}$$

Assume that $|E(G)| \leq n(n+q-k-2) + (k+1)(k+1-q)$. By (1.3) and Corollary 4.5, we have $\Omega(\alpha) \geq \alpha\left(\frac{|E(G)|}{n} + n\right) + (1-\alpha)\sqrt{|E(G)|} \geq \Theta(G, \alpha)$, contrary to the assumption that $\Omega(\alpha) < \Theta(G, \alpha)$. Hence (7.1) follows.

From (7.1), it follows that $|E(G^{v_0})| > n(n+q-k-1) + (k+1)(k+1-q) = \varepsilon_0(q+1)$. By Theorem 4.2 and Proposition 1.13, either G^{v_0} is (q, q)-Hamiltonian or $G^{v_0} \subseteq F_{n,k,q+1}$. It follows by Lemma 7.1 that either G is (q, q)-traceable or $G \subseteq Z_{n+q-k-1,k-q}$. □

Proof of Theorem 1.21 (ii): By Corollary 5.4, we have $\rho\left(K_{n,n+q-k-1} - e\right) > \sqrt{\varepsilon_0(q)} > \Omega(0)$ and $\mu\left(K_{n,n+q-k-1} - e\right) > n + \frac{\varepsilon_0(q)}{n} > \Omega(1)$. Note that $K_{n,n+q-k-1} - e \subset Z^0_{n+q-k-1,k-q}$. Thus, Theorem 1.21 (i) implies that G is (q, q)-traceable unless $G \subseteq Z_{n+q-k-1,k-q}$. Now, the result follows from Lemma 5.2. □

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