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A characterization of graphs with supereulerian line graphs

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ABSTRACT

The line graph $L(G)$ of a graph G is a simple graph with $E(G)$ being its vertex set, where two vertices are adjacent in $L(G)$ whenever the corresponding edges share a common vertex in G. A graph H is even if every vertex of H has even degree, and a graph is supereulerian if it has a spanning closed trail. We obtain a characterization for a graph G to have a supereulerian line graph $L(G)$, as follows: for a connected graph G with $|E(G)| \geq 3$, the line graph $L(G)$ has a spanning closed trail if and only if G has an even subgraph H (possibly null) such that both G remains connected after deleting all degree 2 vertices not in H, and every degree 2 vertex not in H must be adjacent only to vertices of degree at least 3 in G.

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1. Introduction

We follow Bondy and Murty [5] for undefined terminologies and notation, and consider loopless finite undirected graphs permitting multiple edges. A graph G with $V(G) = \emptyset$ is called a *null graph*, and a graph G is *nontrivial* if $E(G) \neq \emptyset$. In particular, we follow [5] to define paths, cycles, as well as open and closed trails. As in [5], for a vertex $v \in V(G)$, the degree of v in G, denoted by $d_G(v)$, is the number of edges incident with v in G. Throughout this paper, we write $H \subseteq G$ to mean that H is a subgraph of a graph G. For an integer $i \geq 0$, define $D_i(G) = \{v \in V(G) : d_G(v) = i\}$ to be the set of vertices of degree i in G; and define $O(G) = \cup_{j \geq 0} D_{2j+1}(G)$ to be the set of all odd degree vertices of G. A graph G is *even* if either G is the null graph, or G is nontrivial with $O(G) = \emptyset$; and G is *eulerian* if G is both connected and even. Thus a graph G is eulerian if and only if G has a closed trail C with $E(C) = E(G)$. For a subgraph H of G, let $\partial_G(H) = \{uv \in E(G) : \text{either } u \in V(H) \text{ or } v \in V(H)\}$.

A trail of G is a vertex-edge alternating sequence

$$T = v_0, e_1, v_1, e_2, \ldots, e_k, v_k \tag{1}$$

such that all the e_i's are distinct and for each $i = 1, 2, \ldots, k$, e_i is incident with both v_{i-1} and v_i. A vertex v is an internal vertex of T if $v = v_i$ for some $i \in \{1, 2, \ldots, k-1\}$. To emphasize the starting and ending elements of the trail, the trail in (1) is also called a (v_0, v_k)-trial as well as an (e_1, e_k)-trail. The trail T is a *dominating* (e_1, e_k) *-trail* of G if every edge of G is incident with an internal vertex of T.

The *line graph* of a graph G, denoted by $L(G)$, is a simple graph with $E(G)$ being its vertex set, where two vertices are adjacent in $L(G)$ whenever the corresponding edges share a common vertex in

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G. As indicated by Hemminger and Beineke [13], the problem to characterize all graphs whose line graphs have certain graphical properties, has attracted many researchers, as shown in the theorem below.

Theorem 1.1: *Let G be a connected graph with $|E(G)| \geq 3$. Each of the following holds.*

(i) *(Chartrand, Proposition 4 of [9]) $L(G)$ is eulerian if and only if for any $u, v \in V(G)$, $d_G(u) \equiv d_G(v) \pmod 2$.*
(ii) *(Hemminger and Beineke, Theorem 6.12 of [13]) $L(G)$ is bipartite if and only if G is a path or a cycle of even length.*
(iii) *(Hemminger and Beineke, Theorem 6.12 of [13]) $L(G)$ has a perfect matching if and only if $|E(G)| \equiv 0 \pmod 2$.*
(iv) *(Harary and Nash-Williams, Proposition 8 of [12]) $L(G)$ is hamiltonian (i.e., $L(G)$ has a spanning cycle) if and only if G has a closed trail C with $\partial_G(C) = E(G)$.*
(v) *(Proposition 2.2 of [17]) $L(G)$ is Hamilton-connected if and only if for any pair of edges $e_1, e_2 \in E(G)$, G has a dominating (e_1, e_2)-trail.*
(vi) *(van Blanken et al., Theorem 4 of [3]) $L(G)$ is pancyclic if and only if for any integer k with $3 \leq k \leq |E(G)|$, G has a closed trail C satisfying $|E(C)| \leq k \leq |\partial_G(C)|$.*

Investigations on sufficient and necessary conditions for $L(G)$ to be Hamilton-connected, pancyclic or pan-connected can also be found in [1,2,8,9,14,16,20], among others. These former results motivate the current research. The main purpose of this paper is to characterize graphs whose line graphs have a spanning closed trail.

A graph G is *supereulerian* if G has a spanning closed trail. Thus a graph G is supereulerian if and only if G has a spanning eulerian subgraph. By definition, eulerian graphs and hamiltonian graphs are examples of supereulerian graphs. Following Catlin [7], we use $\mathcal{SL}$ to denote the family of supereulerian graphs. The supereulerian graph problem is first introduced by Boesch et al. [4], which seeks to characterize supereulerian graphs. Pulleyblank [23] proved that determining whether a graph is supereulerian, even when restricted to planar graphs, is NP-complete. Catlin in [7] indicated that supereulerian graphs have applications in the dominating trail problem, double cycle cover problems, and the problem of determining the total interval number of a graph, among others. For more literature on supereulerian graphs and their applications, see Catlin's survey [7] and its updates in [10,15].

To state our main result, we mention one more concept for our characterization of graphs with supereulerian line graphs. Let H be an even subgraph of a graph G. We define

$$S_G(H) = D_2(G) - V(H). \tag{2}$$

Thus $S_G(H)$ is the set of vertices of degree 2 in G that are not in the subgraph H. If H is the null graph, then $S_G(H) = D_2(G)$. The main result of this paper is the following.

Theorem 1.2: *Let G be a connected graph with $|E(G)| \geq 3$. The following are equivalent.*

(i) *$L(G)$ is supereulerian.*
(ii) *G contains an even subgraph H (possibly null) such that both $G - S_G(H)$ is connected and $\cup_{u \in S_G(H)} N_G(u) \subseteq D_{\geq 3}(G)$.*

As examples, if a connected graph G does not have a vertex of degree 2, or if G has an even subgraph that contains all vertices of degree 2 in G, then by Theorem 1.2, $L(G)$ is supereulerian and so has a spanning closed trail. On the other hand, for any tree T with a vertex of degree 2, $L(T)$ is not supereulerian. In Section 2, we present preliminaries and tools that will be deployed in our arguments.

In Section 3, we shall assume Theorem 1.2 (ii) to prove Theorem 1.2 (i), while the other direction of Theorem 1.2 will be justified in the next section. Section 5 concludes this paper.

2. Preliminaries

As in [5], for a positive integer n, let P_n and C_n denote the path on n vertices and the cycle on n vertices, respectively. Let G be a graph. If Q_1 is an x_1x_2-path and Q_2 is an x_2x_3-path in G, then we use $x_1Q_1x_2Q_2x_3$ to denote the x_1x_3-walk of G by first transverse from x_1 to x_2 along Q_1 and then from x_2 to x_3 along Q_2. By definition, if Q_1 and Q_2 are internally disjoint, then $x_1Q_1x_2Q_2x_3$ is also a path. If $v \in V(G)$, then

$$N_G(v) = \{u \in V(G) : u \text{ is adjacent to } v \text{ in } G\} \text{ and } E_G(v) = \{e \in E(G) : e \text{ is incident with } v \text{ in } G\}.$$

Following [5], if $W \subset V(G)$ or $W \subseteq E(G)$, then $G[W]$ is the subgraph of G induced by W. If e is an edge of G, then let $V_G(e)$ be the set of the two vertices incident with e in G. For an edge subset $X \subseteq E(G)$, define $G - X = G[E(G) - X]$. We often use $G-e$ for $G - \{e\}$ and if $H \subseteq G$, then use $G-H$ for $G - E(H)$. If $H_1, H_2 \subseteq G$, we abbreviate $G[E(H_1) \cup E(H_2)]$ to $H_1 \cup H_2$. For $X \subseteq E(G)$, the *contraction* G/X is obtained from G by identifying the two ends of each edge in X and then by deleting the resulting loop. If $H \subseteq G$, then we write G/H for $G/E(H)$.

2.1. Collapsible graphs, reductions and supereulerian graphs

In [6], Catlin defines a graph G to be *collapsible* if for every subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a subgraph Γ_R such that $O(\Gamma_R) = R$ and $G - E(\Gamma_R)$ is connected. As shown in Proposition 1 of [15], a graph G is collapsible if and only if for every subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph L_R such that $O(L_R) = R$. Following Catlin in [7], we use $\mathcal{CL}$ to denote the family of collapsible graphs. It is shown in [7,15], $\{K_1,\ C_2,\ K_s\ (s \geq 3)\} \subseteq \mathcal{CL} \subseteq \mathcal{SL}$ and $\{C_q : q \geq 4\} \subseteq \mathcal{SL} - \mathcal{CL}$. The next theorem summarizes some properties that will be used in proving our results.

Theorem 2.1: *Let G be a graph and H be a subgraph of graph G.*

- (i) *(Catlin, Theorem 3 of [8]) If G is connected, and each edge of G is in a cycle of length 2 or a complete subgraph of order at least 3, then $G \in \mathcal{CL}$.*
- (ii) *(Catlin, Lemma 3 of [6]) If $G \in \mathcal{SL}$ (resp. $\mathcal{CL}$), then $G/H \in \mathcal{SL}$ (resp. $\mathcal{CL}$).*
- (iii) *(Catlin, Theorem 8 of [6]) If $H \in \mathcal{CL}$, then $G \in \mathcal{SL}$ if and only if $G/H \in \mathcal{SL}$.*

Lemma 2.2: *Let G be a nontrivial connected graph. If $G - D_2(G)$ is connected and $\cup_{u\in D_2(G)} N_G(u) \subseteq D_{\geq 3}(G)$, then $L(G) \in \mathcal{SL}$. In particular, if $D_2(G) = \emptyset$, then $L(G) \in \mathcal{SL}$.*

Proof: Suppose $D_2(G) = \emptyset$. Since every pair of adjacent edges are incident with a vertex of degree at least three in G, every edge of $L(G)$ is in a cycle of length 3. By Theorem 2.1 (i), $L(G)$ is collapsible, and so supereulerian. Thus we assume that $D_2(G) \neq \emptyset$. Let $M = \{xy \in E(L(G)) : u \in D_2(G), E_G(u) = \{x, y\}\}$. As $\cup_{u\in D_2(G)} N_G(u) \subseteq D_{\geq 3}(G)$, M is a matching in $L(G)$ and so $V(L(G) - M) = V(L(G))$. Since $G - D_2(G)$ is connected, it follows that $L(G) - M$ is also connected, and so $L(G) - M$ is a spanning connected subgraph of $L(G)$. We need to show that $L(G) - M \in \mathcal{SL}$ to prove the lemma.

By Theorem 2.1 (i), it suffices to show that every edge in $L(G) - M$ lies in a complete subgraph of order at least 3. By contradiction, assume that there exists an edge $x'y' \in E(L(G)) - M$ which is in no cycles of length 3 in $L(G)$. As $x'y' \in E(L(G))$, there must be a vertex $v \in V(G)$ such that $v \in V_G(x') \cap V_G(y')$. Since $x'y'$ is in no cycles of length 3 in $L(G)$, we conclude that $v \in D_2(G)$ and so $x'y' \in M$ by the definition of M. This contradicts to the assumption that $x'y' \in E(L(G)) - M$, and so completes the proof of the lemma. ■

As an example to illustrate the arguments, let K_4 denote the complete graph on the four vertices $\{v_1, v_2, v_3, v_4\}$ and let $G = K_4 - v_1v_2$. Then $D_2(G) = \{v_1, v_2\}$. Let $e_1 = v_1v_3, e_2 = v_1v_4, e_3 = v_2v_3$ and $e_4 = v_2v_4$. Then $M = \{e_1e_2, e_3e_4\}$ is a matching in $L(G)$. By the definition of line graphs, $L(G) - M$ is connected, and is a disjoint union of two triangles (induced by $E_G(v_3)$ and $E_G(v_4)$, respectively). Thus by definition, $L(G) - M$ is eulerian, and so $L(G)$ is supereulerian.

Lemma 2.3: *Let L be a connected graph. Suppose $L_1, L_2, \ldots, L_k$ are pairwise edge-disjoint subgraphs of L with $L_i \in \mathcal{SL}$ for each $i \in \{1, 2, \ldots, k\}$. Each of the following holds.*

(i) *If $L = \bigcup_{i=1}^{k} L_i$, then $L \in \mathcal{SL}$.*
(ii) *If for each $e \in E(L) - \bigcup_{i=1}^{k} E(L_i)$, L has a collapsible subgraph Γ_e with $e \in E(\Gamma_e)$, then $L \in \mathcal{SL}$.*

Proof: We prove (i) by induction on k. If $k = 1$, the $L = L_1$ is supereulerian by assumption. Suppose that $k = 2$. By assumption, for each $i \in \{1, 2\}$, L_i has a spanning eulerian subgraph L'_i. As L is connected and $E(L_1) \cap E(L_2) = \emptyset$, it follows that $V(L'_1) \cap V(L'_2) = V(L_1) \cap V(L_2) \neq \emptyset$, and so $L'_1 \cup L'_2$ is a spanning connected subgraph of $L = L_1 \cup L_2$. Since $d_{L'_1 \cup L'_2}(x) = d_{L'_1}(x) + d_{L'_2}(x) \equiv 0 \pmod 2$ for each $x \in V(L)$, $L'_1 \cup L'_2$ is eulerian, and so $L \in \mathcal{SL}$.

Suppose that $k \geq 3$ and that (i) holds for smaller values of k. Since L is connected, we have $V(L_1) \cap V(L_j) \neq \emptyset$ for some $j \in \{2, 3, \ldots, n\}$. Without loss of generality, we may assume that $V(L_1) \cap V(L_k) \neq \emptyset$, and so $L_1 \cup L_k$ is connected. As shown in the case when $k = 2$, $L_1 \cup L_k \in \mathcal{SL}$. By induction, as L is the union of $k-1$ mutually edge-disjoint supereulerian subgraphs $(L_1 \cup L_k), L_2, \ldots, L_{k-1}$, we conclude that $L \in \mathcal{SL}$.

We argue by induction on $|V(L)|$ to prove (ii), and note that (ii) holds if $|V(L)| = 1$. Hence we assume that $|V(L)| > 1$ and (ii) holds for smaller values of $|V(L)|$. By (i), we may assume $L \neq \bigcup_{i=1}^{k} L_i$, and so $E_1 = E(L) - \bigcup_{i=1}^{k} E(L_i) \neq \emptyset$.

Fix an edge $e_0 \in E_1$. By assumption, L has a collapsible subgraph Γ_{e_0} with $e_0 \in E(\Gamma_{e_0})$. Let $J = L/\Gamma_{e_0}$. As $\Gamma_{e_0} \in \mathcal{CL}$, by Theorem 2.1(iii), it suffices to show that $J \in \mathcal{SL}$. For any i with $1 \leq i \leq k$, let $J_i = L_i/(E(L_i) \cap E(\Gamma_{e_0}))$. By Theorem 2.1 (ii), $J_i \in \mathcal{SL}$. As the L_i's are mutually edge-disjoint, the J_i's are also edge-disjoint supereulerian subgraphs of J. By the definition of graph contractions, $\bigcup_{i=1}^{k} J_i$ is a subgraph of J. If $J = \bigcup_{i=1}^{k} J_i$, then by (i), $J \in \mathcal{SL}$. Hence we assume that $J \neq \bigcup_{i=1}^{k} J_i$.

For any $e \in E(J) - \bigcup_{i=1}^{k} E(J_i)$, by the definition of graph contractions, we have $e \in E(L) - \bigcup_{i=1}^{k} E(L_i)$. By assumption, there exists a collapsible subgraph $\Gamma_e \subseteq L$ with $e \in E(\Gamma_e)$. Define $T_e = \Gamma_e/(E(\Gamma_e) \cap E(\Gamma_{e_0}))$. By Theorem 2.1 (ii), T_e is also collapsible. It follows by $|V(J)| < |V(L)|$ and by induction that $J = L/\Gamma_{e_0}$ is supereulerian. By Theorem 2.1 (iii), L is also supereulerian. ■

2.2. Properties of even graphs

Throughout the rest, for an integer $s > 0$, we let $\mathbb{Z}_s$ denote the additive group of integers modulo s. The following result of Veblen will be frequently applied.

Theorem 2.4 (Veblen [24]): *A graph G is even if and only if $E(G)$ can be decomposed into an edge-disjoint union of cycles.*

For any even graph D, let $\mathcal{C}(D) = \{(C^1, C^2, \ldots, C^t) :$ the C^i $(i = 1, 2, \ldots, t)$ are mutually edge-disjoint cycles of D such that $E(D) = \cup_{i=1}^{t} E(C^i)\}$. In other words, $\mathcal{C}(D)$ is the collection of all decompositions of $E(D)$ into edge-disjoint cycles. Throughout this paper, we define, for each even graph D,

$$\ell_D = \max\{t : \text{there exists a } (C^1, C^2, \ldots, C^t) \in \mathcal{C}(D)\}. \tag{3}$$

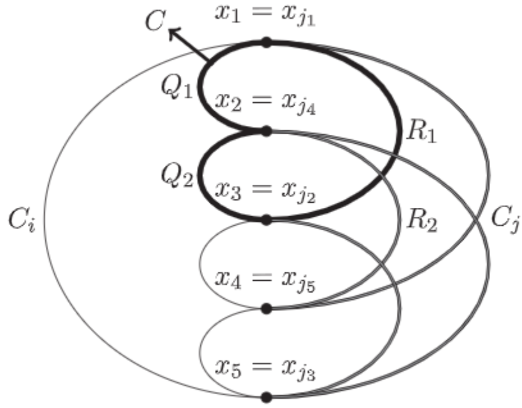

Figure 1. Illustration of the proof of Lemma 2.5 when $q = 5$ and $s = 1$.

Lemma 2.5: *Let D be an even graph. Then for any* $(C^1, C^2, \ldots, C^{\ell_D}) \in \mathcal{C}(D)$*, and for any* $1 \leq i < j \leq \ell_D$, $|V(C^i) \cap V(C^j)| \leq 2$.

Proof: Fix a $(C^1, C^2, \ldots, C^{\ell_D}) \in \mathcal{C}(D)$. By contradiction, assume that there exist i and j with $1 \leq i < j \leq \ell_D$ such that $V(C^i) \cap V(C^j) = \{x_1, x_2, \ldots, x_q\}$ with $q \geq 3$,

$$C^i = x_1 y_1 y_2 \cdots y_{r_2} x_2 y_{r_2+1} y_{r_2+2} \cdots y_{r_3} x_3 \cdots x_{q-1} y_{r_{q-1}+1} \cdots y_{r_q} x_q y_{r_q+1} \cdots y_{r_{q+1}} x_1,$$

and

$$C^j = x_{j_1} z_1 z_2 \cdots z_{r_2'} x_{j_2} z_{r_2'+1} z_{r_2'+2} \cdots z_{r_3'} x_{j_3} \cdots x_{j_{q-1}} z_{r_{q-1}'+1} \cdots z_{r_q'} x_{j_q} z_{r_q'+1} \cdots z_{r_{q+1}'} x_{j_1},$$

where r_k, r_k' are positive integers for each $k \in \{2, 3, \ldots, q+1\}$, and $\{j_1, j_2, \ldots, j_q\}$ is a permutation of the set $\{1, 2, \ldots, q\}$. Without loss of generality, we assume $x_{j_1} = x_1$. For simplicity of notation, we let $Q_k = x_k y_{r_k+1} y_{r_k+2} \cdots y_{r_{k+1}} x_{k+1}$, $R_k = x_{j_k} z_{r_k'+1} z_{r_k'+2} \cdots z_{r_{k+1}'} x_{j_{k+1}}$ for each $k \in \mathbb{Z}_q$, where $r_1 = r_1' = 0$ when $k = 1$. Then,

$$C^i = x_1 Q_1 x_2 Q_2 x_3 \cdots x_{q-1} Q_{q-1} x_q Q_q x_1,$$

and

$$C^j = x_1 R_1 x_{j_2} R_2 x_{j_3} \cdots x_{j_{q-1}} R_{q-1} x_{j_q} R_q x_1.$$

Let $t = \min\{|j_s - j_{s+1}|\}$ where $s \in \mathbb{Z}_q$. Then $t \leq \lfloor q/2 \rfloor$. Choose $s \in \mathbb{Z}_q$ satisfying $|j_s - j_{s+1}| = t$. Take $C = (\bigcup_{k=l}^{t} Q_k) \cup R_s$, where $l = \min\{j_s, j_{s+1}\}$. Then C is a cycle of D (see Figure 1, where $C = Q_1 \cup Q_2 \cup R_1$).

We claim that $D^* = (C^i \cup C^j) \Delta C$ is connected. Since $C \subsetneq C^i \cup C^j$, $D^* = (C^i - C) \cup (C^j - C)$. Pick any two distinct vertices $y, z \in V(D^*)$. If $y, z \in V(C^i - C)$ (resp. $V(C^j - C)$), then there is a yz-path in D^* since $C^i - C$ (resp. $C^j - C$) is a path. Otherwise, we assume $y \in V(C^i - C)$ and $z \in V(C^j - C)$. Note that $d_{D^*}(x_{j_s}) = d_{C^1 \cup C^2}(x_{j_s}) - 2 = 2$, and then $x_{j_s} \in V(C^i - C) \cap V(C^j - C)$. It follows that there are yx_{j_s}-path P, and $x_{j_s}z$-path Q in D^*, then let $R = yPx_{j_s}Qz$ that is a yz-walk in D^*. Thus, D^* is connected.

Then D^* is a connected even graph as $O(D^*) = \emptyset$. Since $|\{x_1, x_2, \ldots, x_q\} \cap V(C)| = |j_s - j_{s+1}| + 1 = t + 1 \leq \lfloor \frac{q}{2} \rfloor + 1$, we have $|\{x_1, x_2, \ldots, x_q\} - V(C)| \geq q - (\lfloor \frac{q}{2} \rfloor + 1) = \lceil \frac{q}{2} \rceil - 1 \geq 1$ as $q \geq 3$. It follows that there is a vertex $x \in \{x_1, x_2, \ldots, x_q\}$ such that $d_{D^*}(x) = 4$, which shows that D^* is not a cycle. Hence, $C^i \cup C^j$ can be rewritten as $D^* \cup C$, where D^* is a connected even graph containing at least two cycles, which is a contradiction of the definition of ℓ_D. ∎

3. Sufficient condition for a supereulerian line graph

In this section, we shall assume the validity of Theorem 1.2 (ii) to prove Theorem 1.2 (i).

Proposition 3.1: *Let G be a connected graph with $|E(G)| \geq 3$. If G has an even subgraph H satisfying $D_2(G) \subseteq V(H)$, then $L(G) \in \mathcal{SL}$.*

Proof: If $D_2(G) = \emptyset$, then Lemma 2.2 implies the conclusion of the proposition. Hence we assume that $D_2(G) \neq \emptyset$. Among all even subgraphs satisfying the hypothesis of the proposition, we choose an even subgraph H of G satisfying

$$D_2(G) \subseteq V(H) \text{ with } |E(H)| \text{ minimized.} \tag{4}$$

Pick any $(H_1, H_2, \ldots, H_p) \in \mathcal{C}(H)$. We may assume that $|V(H_1)| \geq |V(H_2)| \geq \cdots \geq |V(H_p)| \geq 2$. By (4), for any $i \in \{1, 2, \ldots, p\}$, we have $V(H_i) \cap D_2(G) \neq \emptyset$.

If $|V(H_1)| = 2$, then since G is connected graph with $|E(G)| \geq 3$, any vertex $v \in D_2(G) \cap V(H_i)$ is adjacent to a vertex in $D_{\geq 3}(G)$. As $D_2(G) \subseteq V(H)$, it follows that every edge in G is incident with a vertex in $D_{\geq 3}(G)$. Consequently, every edge in $L(G)$ lies in a cycle of length 3. By Theorem 2.1 (i), $L(G)$ is collapsible, and so supereulerian. Hence we assume that $|V(H_1)| > 2$.

Define $k = \max\{i : 1 \leq k \leq p \text{ and } |V(H_i)| \geq 3\}$. For each $1 \leq i \leq k$, let $L_i = L(G)[E(H_i)]$. Since $H_1, H_2, \ldots, H_k$ are pairwise edge-disjoint cycles of length at least 3, $L_1, L_2, \ldots, L_k$ are pairwise vertex-disjoint cycles of length $|E(H_i)| \geq 3$ in $L(G)$.

As G is connected, $L(G)$ is also connected. We now apply Lemma 2.3 to show $L(G) \in \mathcal{SL}$. Since $L_i \in \mathcal{SL}$ for each $i \in \{1, 2, \ldots, k\}$, by Lemma 2.3, it suffices to show that for each $e \in E(L(G)) - \bigcup_{i=1}^{k} E(L_i)$, there is a collapsible subgraph Γ_e of $L(G)$ such that $e \in E(\Gamma_e)$. Assume that $e = xy$ for some $x, y \in E(G) = V(L(G))$. Then there is a vertex $v \in V(G)$ such that $\{x, y\} \subseteq E_G(v)$, and so $d_G(v) \geq 2$. If $d_G(v) \geq 3$, then $L(G)[E_G(v)] \cong K_{d_G(v)} \in \mathcal{CL}$. Denote by $\Gamma_e = L(G)[E_G(v)]$ and we get $e = xy \in E(\Gamma_e)$ as desired. Now we may assume that $d_G(v) = 2$, it follows that $v \in D_2(G) \subseteq V(H)$ and $E_G(v) = \{x, y\}$. Bearing in mind that $e \in E(L(G)) - \bigcup_{i=1}^{k} E(L_i)$ and by the definition of k, there must be a $j \in \{k+1, k+2, \ldots, p\}$, such that $v \in V(H_j)$. Since $|V(H_j)| = 2$ and G is a connected graph with $|E(G)| \geq 3$, hence $N_G(v)$ has only one vertex, say u. It follows that $u \in D_{\geq 3}(G)$, and so $\Gamma_e := L(G)[E_G(u)] \cong K_{d_G(u)}$ is a collapsible subgraph that contains the edge $e = xy$. By Lemma 2.3, we conclude that $L(G)$ is supereulerian. ■

Proof: **Proof of Theorem 1.2 (i), assuming the validity of Theorem 1.2 (ii):** Let H be an even subgraph of G satisfying Theorem 1.2 (ii). If $D_2(G) = \emptyset$, by Lemma 2.2, we have $L(G) \in \mathcal{SL}$. If $S_G(H) = \emptyset$, then $D_2(G) \subseteq V(H)$, and so $L(G) \in \mathcal{SL}$ by Proposition 3.1. If $S_G(H) = D_2(G) \neq \emptyset$, then, by Lemma 2.2, we have $L(G) \in \mathcal{SL}$. Therefore, we let $S'_G(H) = D_2(G) - S_G(H)$ and assume that both $S_G(H) \neq \emptyset$ and $S'_G(H) \neq \emptyset$.

Denote $M = \{xy \in E(L(G)) : u \in S_G(H), \{x, y\} = E_G(u)\}$. Since $N_G(u) \subseteq D_{\geq 3}(G)$ for any vertex $u \in S_G(H)$, M is a matching in $L(G)$ and so $V(L(G) - M) = V(L(G))$. Since $G - S_G(H)$ is connected, $L(G) - M$ is connected, and so $L(G) - M$ is a spanning connected subgraph of $L(G)$. To prove Theorem 1.2(i), it suffices to show that $L(G) - M \in \mathcal{SL}$.

For any vertex u in $S_G(H)$, let $E_G(u) = \{e'_u, e''_u\}$. Obtain a new graph G' from G by splitting every vertex u in $S_G(H)$ into two new vertices u' and u'', such that u' is incident with e'_u only and u'' is incident with e''_u only. Then $|E(G')| = |E(G)| \geq 3$ and by the definition of a line graph, $L(G') \cong L(G) - M$. Since $G - S_G(H)$ is connected, we have G' is connected. By (2), $S_G(H) \cap V(H) = \emptyset$, and so H is also an even subgraph of G' with $D_2(G') = S'_G(H) \subseteq V(H)$. By Proposition 3.1, we have $L(G) - M \cong L(G') \in \mathcal{SL}$. This completes the proof of Theorem 1.2 (i). ■

4. Necessary condition for a supereulerian line graph

For sets X and Y, define $X \triangle Y = (X \cup Y) - (X \cap Y)$. Let G_1 and G_2 be two subgraphs of G. Define $G_1 \triangle G_2 = G[E(G_1) \triangle E(G_2)]$. By definition,

$$\text{If } G_1, G_2 \text{ are even, then } G_1 \triangle G_2 \text{ is also even.} \tag{5}$$

Let $\mathcal{D}_L$ be a family of spanning eulerian subgraphs of a graph L. By definition, $\mathcal{D}_L \neq \emptyset$ if and only if $L \in \mathcal{SL}$. In particular, we let $\widetilde{\mathcal{D}}_L = \{D \in \mathcal{D}_L : |E(D)| \leq |E(D')| \text{ for each } D' \in \mathcal{D}_L\}$. For convenience, if we write $D = \bigcup_{i=1}^{t} C^i \in \mathcal{D}_L$, then C^i and C^j always denote edge-disjoint cycles of L for any $\{i,j\} \subseteq \{1, 2, \ldots, t\}$ hereafter.

Lemma 4.1: *Let L be a nontrivial simple graph and $L \in \mathcal{SL}$. Let $D = \bigcup_{i=1}^{t} C^i \in \widetilde{\mathcal{D}}_L$, where $t \geq 2$. Suppose that for some $\{i,j\} \subseteq \{1, 2, \ldots, t\}$, $x \in V(C^i) \cap V(C^j)$, $x'_1 \in N_{C^i}(x)$ and $x'_2 \in N_{C^j}(x)$. If $x'_1x'_2 \notin E(C^i) \cup E(C^j)$, then $x'_1x'_2 \notin E(L)$.*

Proof: We may assume that $i = 1, j = 2$. By contradiction, assume $x'_1x'_2 \in E(L)$. Since $x'_1x, x'_2x \in E(L)$, L contains a cycle $T = xx'_1x'_2x$. Let $D' = D \triangle T$ (see Figure 2). Note that $d_{D'}(x) = d_D(x) - 2 \geq 4 - 2 = 2$ and $d_{D'}(y) = d_D(y) > 0$ for each $y \in x'_2(D) - \{x, x'_1, x'_2\}$. We have the following two claims. ■

Claim 4.1: $V(D') = V(D)$ and $|E(D')| < |E(D)|$.

If $x'_1x'_2 \in E(C^k)$ for some $k \in \{3, 4, \ldots, n\}$, say $k = 3$, then $x'_1 \in V(C^1) \cap V(C^3)$ and $x'_2 \in V(C^2) \cap V(C^3)$, and so $E(T) \subseteq E(D)$. Then $|E(D')| = |E(D) - E(T)| = |E(D)| - 3 < |E(D)|$. Since $d_{D'}(x'_1) = d_D(x'_1) - 2 \geq 2$ and $d_{D'}(x'_2) = d_D(x'_2) - 2 \geq 2$, we have $V(D') = V(D)$ and so D' is a spanning subgraph of L.

If $x'_1x'_2 \notin E(D)$, then $d_{D'}(x'_1) = d_D(x'_1) - 1 + 1 = d_D(x'_1)$ and $d_{D'}(x'_2) = d_D(x'_2) - 1 + 1 = d_D(x'_2)$. It follows that $V(D') = V(D)$. Note that $|E(D')| = |E(D) \triangle E(T)| = |E(D)| + 1 - 2 = |E(D)| - 1 < |E(D)|$.

Claim 4.2: D' is connected.

Pick any two distinct vertices $y, z \in V(D') = V(D)$. Since D is connected, there is a yz-path P in D. If $P \subseteq D'$, then we are done. Suppose $E(P) - E(D') \neq \emptyset$. Then as $P \subseteq D$, we have $E(T) \cap E(P) \neq \emptyset$. Since P is a path, $1 \leq |E(T) \cap E(P)| \leq |E(T)| - 1 = 2$.

Assume that $|E(T) \cap E(P)| = 1$. By symmetry, we may assume $xx'_1 \in E(P)$ and $P = y \cdots y'xx'_1z' \cdots z$. Let $Q = C^1 - xx'_1$. Then Q is an xx'_1-path in D'. Thus $P' = y \cdots y'xQx'_1z' \cdots z$ is a yz-walk in D'.

When $|E(T) \cap E(P)| = 2$, we may assume $\{xx'_1, xx'_2\} \subseteq E(P)$, and $P = y \cdots y'x'_1xx'_2z' \cdots z$. If $x'_1x'_2 \notin E(D)$, then $P' = y \cdots y'x'_1x'_2z' \cdots z$ is a yz-walk in D'. Otherwise, we may assume $x'_1x'_2 \in E(C^3)$ as $x'_1x'_2 \notin E(C^1) \cup E(C^2)$. Thus $Q' = C^3 - x'_1x'_2$ is an $x'_1x'_2$-path in D', and so $P' = y \cdots y'x'_1Q'x'_2z' \cdots z$ is a yz-walk in D'.

By (5), D' is even and so by Claim 2, D' is an eulerian graph. By Claim 1, $D' \in \mathcal{D}_L$ with $|E(D')| < |E(D)|$, contrary to the assumption that $D \in \widetilde{\mathcal{D}}_L$.

Recall that ℓ_D is the maximum number of edge-disjoint cycles such that $D = \bigcup_{i=1}^{\ell_D} C^i$ is decomposed into.

Lemma 4.2: *Let L be a nontrivial claw-free simple graph and $L \in \mathcal{SL}$. If $D = \bigcup_{i=1}^{\ell_D} C^i \in \widetilde{\mathcal{D}}_L$, then $|V(C^i) \cap V(C^j)| \leq 1$ for any $\{i,j\} \subseteq \{1, 2, \ldots, \ell_D\}$.*

Proof: Note that for a given $D \in \widetilde{\mathcal{D}}_L$, ℓ_D is the largest number such that $D = \bigcup_{i=1}^{\ell_D} C^i$ is an edge-disjoint union of cycles. If $\ell_D = 1$, then the proof is completed. Assume that $n \geq 2$ in the following. By

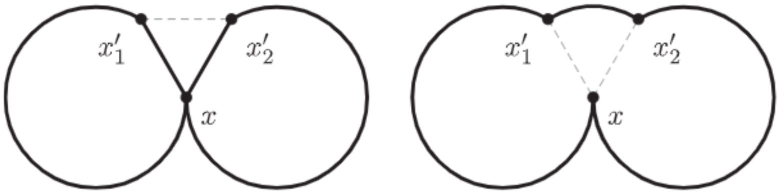

Figure 2. Illustration of the proof of Lemma 4.1.

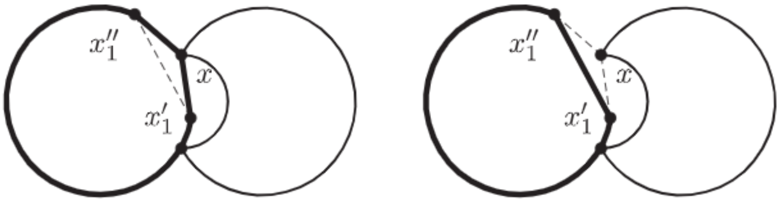

Figure 3. Illustration of the proof of Lemma 4.2

contradiction, we may assume that $|V(C^1) \cap V(C^2)| = q \geq 2$. By Lemma 2.5, $q = 2$. Let $x \in V(C^1) \cap V(C^2)$. ∎

Claim 4.3: There is a $k \in \{1, 2\}$ such that $|V(C^k)| \geq 4$ and $N_{C^k}(x) \cap V(C^1) \cap V(C^2) = \emptyset$.

As L is simple, $N_{C^1}(x) \cap N_{C^2}(x) = \emptyset$. Since $q = 2$, we have $|V(C^1)| \geq 4$ or $|V(C^2)| \geq 4$. By symmetry, assume $|V(C^1)| \geq 4$. If $|V(C^2)| = 3$, then $N_{C^1}(x) \cap V(C^2) = \emptyset$ and so $N_{C^1}(x) \cap V(C^1) \cap V(C^2) = \emptyset$. If $|V(C^2)| \geq 4$, then as $|(N_{C^1}(x) \cup N_{C^2}(x)) \cap V(C^1) \cap V(C^2)| \leq |V(C^1) \cap V(C^2) - x| = q - 1 = 1$, this claim holds also.

Let $N_{C^1}(x) = \{x_1', x_1''\}$ and $N_{C^2}(x) = \{x_2', x_2''\}$. Assume that $|V(C^1)| \geq 4$ and $\{x_1', x_1'', x_2'\} \cap (V(C^1) \cap V(C^2)) = \emptyset$ without loss of generality. It follows that $\{x_1'x_2', x_1''x_2'\} \cap (E(C^1) \cup E(C^2)) = \emptyset$. By Lemma 4.1, we have $\{x_1'x_2', x_1''x_2'\} \cap E(L) = \emptyset$. Since L is a claw-free simple graph, we have $x_1'x_1'' \in E(L)$, and so G has a cycle $T = xx_1'x_1''x$. Take $D' = D \Delta T$ (see Figure 3).

Note that $|E(D')| = |E(D)| - 1$ if $x_1'x_1'' \notin E(D)$; $|E(D')| = |E(D)| - 3$ if $x_1'x_1'' \in E(D)$. To obtain a contradiction, we are to show $D' \in \mathcal{D}_L$.

Note that $d_{D'}(x) = d_D(x) - 2 \geq 2$, $d_{D'}(y) = d_D(y)$ for each $y \in V(D) - \{x, x_1', x_1''\}$, If $x_1'x_1'' \notin E(D)$, then $d_{D'}(x_1') = d_D(x_1') - 1 + 1 = d_D(x_1')$ and $d_{D'}(x_1'') = d_D(x_1'')$. If $x_1'x_1'' \in E(D)$, then $d_{D'}(x_1') = d_D(x_1') - 2 \geq 2$ and $d_{D'}(x_1'') = d_D(x_1') - 2 \geq 2$. It shows $V(D') = V(D)$ and $d_{D'}(z) \equiv 0 \pmod 2$ for each $z \in V(D')$. If D' is connected, then D' is eulerian, and so $D' \in \mathcal{D}_L$ as $V(D') = V(D)$. It suffices to show D' is connected.

Case 1. $x_1'x_1'' \notin E(D)$. Let $S = (C^1 \cup C^2) \Delta T$. Then S is a subgraph of D'. Since $D'/S = D/(C^1 \cup C^2)$ is connected, if we can show S is connected, then D' is connected. Since $E(C^2) \cap E(T) = \emptyset$, we have $S = (C^1 \Delta T) \cup C^2$. As $C^1 \Delta T$ and C^2 are both cycles of L and $|V(C^1 \Delta T) \cap V(C^2)| = 1$, it follows that S is connected.

Case 2. $x_1'x_1'' \in E(D)$. Since D is an edge-disjoint union of cycles of L and $x_1'x_1'' \notin E(C^1 \cup C^2)$, we have $x_1'x_1'' \in E(C^k)$ for some $k \in \{3, 4, \ldots, \ell_D\}$, say $k = 3$. Let $S = (C^1 \cup C^2 \cup C^3) \Delta T$ (see Figure 4). Then S is a subgraph of D'. Since $D'/S = D/(C^1 \cup C^2 \cup C^3)$ is connected, if we can show S is connected, then D' is connected. Note that $S = (C^1 \Delta T) \cup C^2 \cup (C^3 \Delta T)$ as $T \subseteq C^1 \cup C^3$. Since$(C^1 \Delta T) \cup (C^3 \Delta T)$ and C^2 are both cycles of L and $|V(C^1 \Delta T) \cap V(C^2)| = 1$, we conclude that S is connected.

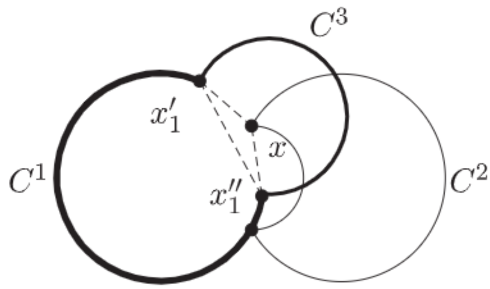

Figure 4. Illustration of Case 2 in the proof of Lemma 4.2.

Lemma 4.3: *Let L be a nontrivial claw-free simple supereulerian graph, and $D = \bigcup_{i=1}^{\ell_D} C^i \in \widetilde{\mathcal{D}_L}$ where $\ell_D \geq 2$. Suppose for some $\{i,j\} \subseteq \{1,2,\ldots,\ell_D\}$, $V(C^i) \cap V(C^j) = \{x\}$, $N_{C^i}(x) = \{x'_i, x''_i\}$ and $N_{C^j}(x) = \{x'_j, x''_j\}$. Then, $\{x'_ix'_j, x'_ix''_j, x''_ix'_j, x''_ix''_j\} \cap E(L) = \emptyset$ and $\{x'_ix''_i, x'_jx''_j\} \subseteq E(L)$. Furthermore, $x \notin V(C^k)$ for any $k \in \{1,2,\ldots,\ell_D\} - \{i,j\}$.*

Proof: Since $V(C^i) \cap V(C^j) = \{x\}$ and L is simple, we have $\{x'_ix'_j, x'_ix''_j, x''_ix'_j, x''_ix''_j\} \cap (E(C^i) \cup E(C^j)) = \emptyset$. Then, by Lemma 4.1, $\{x'_ix'_j, x'_ix''_j, x''_ix'_j, x''_ix''_j\} \cap E(L) = \emptyset$. It follows $\{x'_ix''_i, x'_jx''_j\} \subseteq E(L)$ as L is a claw-free simple graph.

Suppose that $x \in V(C^i) \cap V(C^j) \cap V(C^k)$ for some $k \in \{1,2,\ldots,\ell_D\} - \{i,j\}$. Denote by $N_{C^k}(x) = \{x'_k, x''_k\}$. By Lemma 4.2, $\{x\} = V(C^i) \cap V(C^k)$. It follows $yz \notin E(L)$ for any $y \in N_{C^s}(x)$ and any $z \in N_{C^t}(x)$ where $\{s,t\} \subseteq \{i,j,k\}$, and so $L[\{xx'_i, xx'_j, xx'_k\}] \cong K_{1,3}$, a contradiction. ■

A *cactus closed chain* form by q cycles is the graph obtained from a cycle of length q by replacing every edge with a cycle.

Proof: **Proof of Theorem 1.2 (i) implying Theorem 1.2 (ii):** We shall prove the result by induction on $m = |E(G)|$. If $|E(G)| = 3$, then since $L(G) \in \mathcal{SL}$, we have $L(G) \cong C_3$ as $L(G)$ is simple. Then G is isomorphic to any graph with $|E(G)| = 3$ except P_4. In this case, Theorem 1.2 (i) and (ii) can be routinely verified.

Assume $m \geq 3$ and Theorem 1.2 (i) implying Theorem 1.2 (ii) for smaller values of m. Note that $L(G)$ is a nontrivial claw-free simple supereulerian graph. By Lemma 4.2, there exists a member

$$D = \bigcup_{i=1}^{\ell_D} C^i \in \widetilde{\mathcal{D}_{L(G)}} \text{ such that } |V(C^i) \cap V(C^j)| \leq 1, \forall \{i,j\} \subseteq \{1,2,\ldots,\ell_D\}. \tag{6}$$

Case 1. $\ell_D = 1$.

In this case, $L(G)$ is hamiltonian since D is a spanning connected cycle of $L(G)$. Then by Theorem 1.1 (iv), G has a dominating eulerian subgraph H. If $S_G(H) = \emptyset$, then we are done. Suppose that $S_G(H) \neq \emptyset$. We shall verify that H satisfies Theorem 1.2 (ii).

Since H is dominating in G, for any vertex $u \in S_G(H)$, $N_G(u) \subseteq V(H)$. If u is adjacent to a vertex in $D_2(G) \cap V(H)$, then since H is eulerian, $u \in V(H)$ also, contrary to the assumption that $u \notin S_G(H)$. Hence, $N_G(u) \subseteq D_{\geq 3}(G)$.

To prove that $G - S_G(H)$ is connected, we let G^1 be the component of $G - S_G(H)$ that contains H. As H is dominating in G, every vertex in $G - S_G(H) \cup V(H)$ must be adjacent to a vertex in H, and so $G - S_G(H) = G^1$ is connected.

Case 2. $\ell_D \geq 2$.

For the given D satisfies (6), we define a new graph L' with $V(L') = \{C^1, C^2, \ldots, C^{\ell_D}\}$ (viewing every cycle C^i of D as a vertex in L'), and there are k edges joining C^i and C^j in L' if and only if

Figure 5. Illustration of Subcase 2.1.1 in the proof of Theorem 1.2 (i) implying Theorem 1.2 (ii) when $s = 5$.

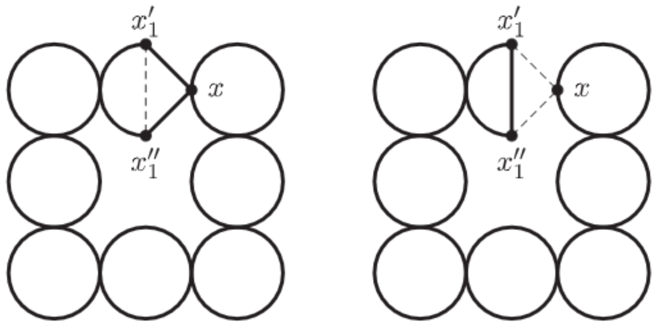

Figure 6. Illustration of Subcase 2.1.2 in the proof of Theorem 1.2 (i) implying Theorem 1.2 (ii) when $s = 8$.

$|V(C^i) \cap V(C^j)| = k$ in D, where $\{i, j\} \subseteq \{1, 2, \ldots, \ell_D\}$. By (6), $k \leq 1$ and so L' is a simple graph. Since $D \in \widetilde{\mathcal{D}_{L(G)}} \subseteq \mathcal{D}_{L(G)}$ is connected, L' is also connected.

Subcase 2.1. L' has a cycle of length s, say $C_{L'}$, where $3 \leq s \leq \ell_D$. Assume that $V(C_{L'}) = \{C^1, C^2, \ldots, C^s\}$. By the definition of L' and the furthermore part of Lemma 4.3, there is a cactus closed chain form by cycles $C^1, C^2, \ldots, C^s$ in $L(G)$.

Subcase 2.1.1. $|V(C^i)| = 3$ for each $i \in \{1, 2 \ldots, s\}$. Thus, there exists a cycle $C \subset \bigcup_{i=1}^{s} C^i$, such that $V(C) = \bigcup_{i=1}^{s} V(C^i)$ and $E(C) \subsetneqq \bigcup_{i=1}^{s} E(C^i)$ (see Figure 5). Take $D' = C \cup (\bigcup_{i=s+1}^{\ell_D} C^i)$ if $s < \ell_D$; otherwise, $D' = C$. Since C and $D'/C \cong D/(\bigcup_{i=1}^{s} C^i)$ are both connected, we have D' is connected. Note that $d_{D'}(v) \equiv 0 \pmod 2$ for each $v \in V(D')$, then D' is eulerian and so $D' \in \mathcal{D}_{L(G)}$ as $V(D') = V(D) = V(L(G))$. However, $|E(D')| = |E(D)| - s$, which contradicts the choice of $D \in \widetilde{\mathcal{D}_{L(G)}}$.

Subcase 2.1.2. $|V(C^i)| \geq 4$ for some $i \in \{1, 2, \ldots, s\}$. Without loss of generality, suppose that $|V(C^1)| \geq 4$ and $V(C^1) \cap V(C^2) \neq \emptyset$. Then we may assume that $V(C^1) \cap V(C^2) = \{x\}$, $N_{C^1}(x) = \{x_1', x_1''\}$. Then by Lemma 4.3, we have $x_1'x_1'' \in E(L)$. Denote by $T = xx_1'x_1''x$ the cycle of L. Let $D' = D \triangle T$ (see Figure 6).

Note $x_1'x_1'' \notin E(D)$ (otherwise, there is C^j for some $j \in \{2, 3, \ldots, \ell_D\}$ such that $x_1'x_1'' \in E(C^j)$, which shows $|V(C^1) \cap V(C^j)| \geq 2$, a contradiction). Then $D' = (C^1 \triangle T) \cup (\bigcup_{i=2}^{\ell_D} C^i)$. Since $L' - C^1C^2$ is connected (notice that C^1C^2 is an edge in L'), D' is connected. Note that $d_{D'}(x) = d_D(x) - 2 \equiv 0 \pmod 2$ and $d_{D'}(y) = d_D(y) \equiv 0 \pmod 2$ for each $y \in V(D') - \{x\}$, then D' is an eulerian subgraph of $L(G)$. It shows $D' \in \mathcal{D}_{L(G)}$ as $V(D') = V(D)$. However, $|E(D')| = |E(D)| - 1$, which contradicts the choice of $D \in \widetilde{\mathcal{D}_{L(G)}}$.

Subcase 2.2. L' is a tree. Without loss of generality, suppose that C^1 is a leaf of L', and $N_{L'}(C^1) = \{C^2\}$. So $|V(C^1) \cap V(C^2)| = 1$ and $V(C^1) \cap V(C^j) = \emptyset$ for each $j \in \{3, 4, \ldots, n\}$. Let $V(C^1) \cap V(C^2) = \{x\}$, $N_{C^1}(x) = \{x_1', x_1''\}$ and $N_{C^2}(x) = \{x_2', x_2''\}$. Let $V_G(x) = \{u_1, u_2\}$. By Lemma 4.3, we have $\{x_1'x_2', x_1'x_2'', x_1''x_2', x_1''x_2''\} \cap E(L(G)) = \emptyset$ and $\{x_1'x_1'', x_2'x_2''\} \subseteq E(L(G))$. Then we may assume that $V_G(x) \cap V_G(x_1') \cap V_G(x_1'') = \{u_1\}$ and $V_G(x) \cap V_G(x_2') \cap V_G(x_2'') = \{u_2\}$. So $d_G(u_1) \geq 3$. Suppose that $V_G(x_1') - \{u_1\} = \{v_1\}$ and $V_G(x_1'') - \{u_1\} = \{v_2\}$ where v_1 and v_2 can be the same vertex (see Figure 7).

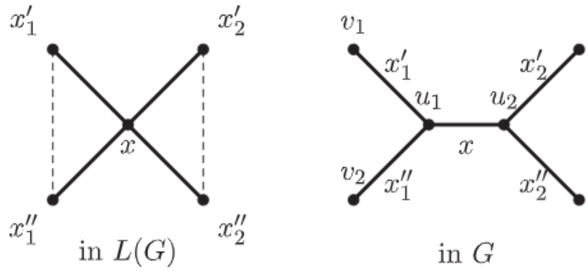

Figure 7. Illustration of Subcase 2.2 in the proof of Theorem 1.2(i) implying Theorem 1.2(ii).

Subcase 2.2.1. $|V(C^1)| = 3$. Let $D^* = D - \{x_1', x_1''\} = \bigcup_{i=2}^{\ell_D} C^i$. Then D^* is a spanning connected even subgraph of $L(G) - \{x_1', x_1''\}$, which shows $L(G) - \{x_1', x_1''\} \in \mathcal{SL}$. Note that $L(G) - \{x_1', x_1''\} \cong L(G - \{x_1', x_1''\})$. Let $G^* = G - \{x_1', x_1''\}$. Since G^* is a nontrivial connected graph with $|E(G^*)| < |E(G)| = m$. Then by the induction hypothesis, there is an even subgraph H^* (possibly null) in G^* such that $G^* - S_{G^*}(H^*)$ is connected and $N_{G^*}(u) \subseteq D_{\geq 3}(G^*)$ for any vertex $u \in S_{G^*}(H^*) = D_2(G^*) - V(H^*)$.

Let $H = H^*$. Then H is also an even subgraph in G, and

$$S_G(H) = (S_{G^*}(H^*) - \{u_1\}) \cup (\{v_1, v_2\} \cap D_2(G)).$$

■

Claim 4.4: H satisfies Theorem 1.2 (ii).

First, to prove that for each vertex $u \in S_G(H)$, $N_G(u) \subseteq D_{\geq 3}(G)$, we pick $u \in S_G(H)$. Suppose $u \in S_{G^*}(H^*) - \{u_1\}$. If $S_{G^*}(H^*) \cap \{v_1, v_2\} = \emptyset$, then $S_{G^*}(H^*) - \{u_1\} = S_{G^*}(H^*) - \{u_1, v_1, v_2\}$. For each vertex $u \in S_{G^*}(H^*) - \{u_1, v_1, v_2\}$, we have $N_G(u) = N_{G^*}(u) \subseteq D_{\geq 3}(G^*) \subseteq D_{\geq 3}(G)$. If $S_{G^*}(H^*) \cap \{v_1, v_2\} \neq \emptyset$, assume $v_1 \in S_{G^*}(H^*)$, then $N_G(v_1) = N_{G^*}(v_1) \cup \{u_1\} \subseteq D_{\geq 3}(G)$ since $d_G(u_1) = d_{G^*}(u_1) + 2 \geq 3$ as G^* is connected. Now, we assume $u \in \{v_1, v_2\} \cap D_2(G)$, say $u = v_1 \in D_2(G)$. Then $v_1 \in D_1(G^*)$. Let $N_{G^*}(v_1) = \{w\}$. It follows $N_G(v_1) = \{u_1, w\} \subseteq D_{\geq 3}(G)$ as $d_G(w) = d_{G^*}(w) \geq 3$. Hence we conclude that $\bigcup_{u \in S_G(H)} N_G(u) \subseteq D_{\geq 3}(G)$.

We are to show that $G - S_G(H)$ is connected. As $V_G(x) \cap V_G(x_2') \cap V_G(x_2'') = \{u_2\}$. We have $d_{G^*}(u_2) \geq 3$, which implies $u_2 \notin S_{G^*}(H^*)$. Since $u_2 \in N_{G^*}(u_1)$ and $G^* - S_{G^*}(H^*)$ is connected, $G^* - (S_{G^*}(H^*) - \{u_1\})$ is also connected. It follows that $G - (S_{G^*}(H^*) - \{u_1\})$ is connected. If $\{v_1, v_2\} \cap D_2(G) = \emptyset$, then $S_G(H) = S_{G^*}(H^*) - \{u_1\}$, which implies $G - S_G(H)$ is connected. Hence we assume $\{v_1, v_2\} \cap D_2(G) \neq \emptyset$. Let $G_1^* = G^* - (S_{G^*}(H^*) - \{u_1\})$.

If $|\{v_1, v_2\} \cap D_2(G)| = 1$, say $v_1 \in D_2(G)$, then either $v_1 (= v_2) \notin V(G^*)$ or $v_1 \in D_1(G^*)$ and $v_2 \notin D_1(G^*)$. When $v_1 (= v_2) \notin V(G^*)$, $N_G(v_1) = \{u_1\}$ and so $G - [(S_{G^*}(H^*) - \{u_1\}) \cup \{v_1\}] = G - S_G(H)$ is connected. Assume that $v_1 \in D_1(G^*)$ and $v_2 \notin D_1(G^*)$, denote $N_{G^*}(v_1) = \{w_1\}$. Since $w_1 \notin S_{G^*}(H^*)$ and $d_{G_1^*}(v_1) = d_{G^*}(v_1) = 1$, $G_1^* - v_1 = G^* - [(S_{G^*}(H^*) - \{u_1\}) \cup \{v_1\}] = G^* - S_G(H)$ is connected, implying that $G - S_G(H)$ is connected.

If $|\{v_1, v_2\} \cap D_2(G)| = 2$, then $\{v_1, v_2\} \subseteq D_1(G^*)$. Assume $N_{G^*}(v_1) = \{w_1\}$ and $N_{G^*}(v_2) = \{w_2\}$. Since $w_1 \notin S_{G^*}(H^*)$, $w_2 \notin S_{G^*}(H^*)$, $d_{G_1^*}(v_1) = d_{G^*}(v_1) = 1$ and $d_{G_1^*}(v_2) = d_{G^*}(v_2) = 1$, $G_1^* - \{v_1, v_2\} = G^* - [(S_{G^*}(H^*) - \{u_1\}) \cup \{v_1, v_2\}] = G^* - S_G(H)$ is connected, which implies $G - S_G(H)$ is connected, and justifies the claim.

Subcase 2.2.2. $|V(C^1)| \geq 4$. Recall that $V(C^1) \cap V(C^2) = \{x\}$ and $V(C^1) \cap V(C^j) = \emptyset$ for every $j \in \{3, 4, \ldots, n\}$. Let $T = xx_1'x_1''x$. Take $D^* = D \Delta T = (C^1 \Delta T) \cup (\bigcup_{i=2}^{\ell_D} C^i)$ as $x_1'x_1'' \notin E(D)$. Since $(C^1 \Delta T) \cap (\bigcup_{i=2}^{\ell_D} C^i) = \emptyset$, there are two connected components of D^*, say $D_1^* = C^1 \Delta T$ and $D_2^* = \bigcup_{i=2}^{\ell_D} C^i$.

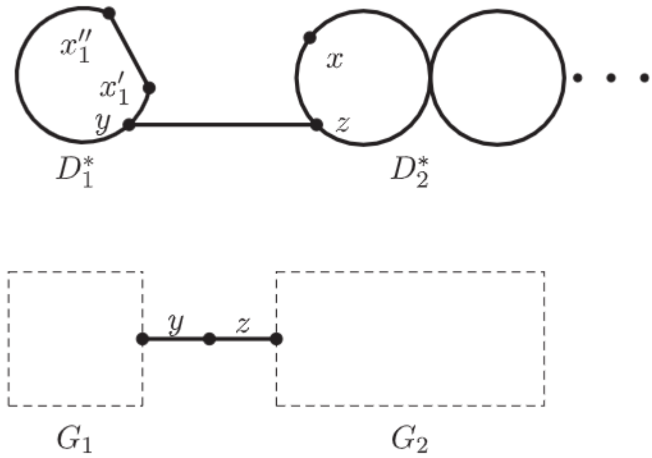

Figure 8. Illustration of Subcase 2.2.2 in the proof of Theorem 1.2(i) implying Theorem 1.2(ii).

Let $X_1 = V(D_1^*) = V(C^1) - \{x\}$ and $X_2 = V(D_2^*) = \bigcup_{i=2}^{\ell_D} V(C^i)$. Then $X_1 \cap X_2 = \emptyset$ and $X_1 \cup X_2 = V(D) = E(G)$. Let $G_i = G[X_i]$ for each $i = 1, 2$. Then $G_1 \subseteq G$, $G_2 \subseteq G$ and $G_1 \cup G_2 = G$ (see Figure 8).

Since $|V(C^1)| \geq 4$ and $\ell_D \geq 2$, D_1^* and D_2^* are two spanning even subgraphs of $L(G)[X_1]$ and $L(G)[X_2]$, respectively. Then, for each $i = 1, 2$, $L(G_i) = L(G)[X_i] \in \mathcal{SL}$. It follows that G_i is a nontrivial connected graph with $|E(G_i)| = |X_i| < |E(G)| = m$. By induction, there is an even subgraph H_i (possibly null) in G_i such that $G_i - S_{G_i}(H_i)$ is connected and $N_{G_i}(u) \subseteq D_{\geq 3}(G_i)$ for any vertex $u \in S_{G_i}(H_i)$.

Take $H = H_1 \cup H_2$. Since $X_1 \cap X_2 = \emptyset$, we have $E(H_1) \cap E(H_2) = \emptyset$, and then H is also an even subgraph in G. We have the following claims.

Claim 5. $v \in V(G_i) - V(G_j)$ if and only if $d_{G_i}(v) = d_G(v)$ for each $\{i, j\} = \{1, 2\}$.

For a given $\{i, j\} = \{1, 2\}$, and a given $v \in V(G_i)$. Suppose $v \in V(G_i) - V(G_j)$. Then $d_{G_i}(v) = d_G(v)$. Suppose $d_{G_i}(v) = d_G(v)$, and $v \in V(G_i) \cap V(G_j)$. Then $d_G(v) = d_{G_i}(v) + d_{G_j}(v) > d_{G_i}(v) = d_G(v)$, a contradiction.

Claim 6. Let $S_1 = S_{G_1}(H_1) \cup S_{G_2}(H_2) - V(G_1 \cap G_2)$ and $S_2 = V(G_1 \cap G_2) \cap D_2(G)$. Then, $S_G(H) = S_1 \cup S_2$.

If there is a vertex $u \in S_1 \cap S_2$, then $u \notin V(G_1 \cap G_2)$ as $u \in S_1$. But as $u \in S_2$ implying $u \in V(G_1 \cap G_2)$, a contradiction. Hence we have $S_1 \cap S_2 = \emptyset$.

If $u \in S_1 - S_G(H)$, then $u \notin V(G_1 \cap G_2)$ as $u \in S_1$. Assume $u \in V(G_1) - V(G_2)$. Note that $u \in S_1$, then $u \in S_{G_1}(H_1)$ and $d_G(u) = d_{G_1}(u) = 2$ by Claim 5. Since $u \notin S_G(H)$, we have $u \in V(H)$, and then $u \in V(H_1)$, implying $u \notin S_{G_1}(H_1)$, a contradiction. So, $S_1 \subseteq S_G(H)$.

If $u \in S_2 - S_G(H)$, then $u \in D_1(G_1) \cap D_1(G_2)$ as $u \in S_2$. Since $u \in D_2(G)$, we have $u \in V(H)$. By the definition of H, $u \in V(H_1) \cup V(H_2)$. By symmetry, we may assume $u \in V(H_1)$. Then $d_{C^1}(u) \geq 2$, which contradicts the fact of $u \in D_1(G_1)$. Thus, $S_2 \subseteq S_G(H)$, and then $S_1 \cup S_2 \subseteq S_G(H)$.

For each vertex $u \in S_G(H) - S_2$, since $d_G(u) = 2$, we have $u \notin V(G_1 \cap G_2)$. Assume $u \in V(G_1) - V(G_2)$. Since $u \in S_G(H)$, $u \notin V(H)$ and then $u \notin V(H_1)$. It shows $u \in S_{G_1}(H_1)$ as $d_{G_1} = d_G(u) = 2$ by Claim 5. Then $u \in (S_{G_1}(H_1) \cup S_{G_2}(H_2)) - V(G_1 \cap G_2) = S_1$. Thus, $S_G(H) = S_1 \cup S_2$. This justifies Claim 6.

Claim 7. H satisfies Theorem 1.2 (ii).

We first show that for each vertex $u \in S_G(H)$, $N_G(u) \subseteq D_{\geq 3}(G)$. If $u \in S_1$, then either $u \in V(G_1)$ or $u \in V(G_2)$. By symmetry, we may assume $u \in V(G_1) - V(G_2)$. Then, by Claim 5, $N_G(u) = N_{G_1}(u) \subseteq D_{\geq 3}(G_1) \subseteq D_{\geq 3}(G)$ as $G_1 \subseteq G$. If $u \in S_2$, then $u \in D_1(G_1)$ and $u \in D_1(G_2)$, and so $N_G(u) = N_{G_1}(u) \cup N_{G_2}(u) \subseteq D_{\geq 3}(G_1) \cup D_{\geq 3}(G_2) \subseteq D_{\geq 3}(G)$.

It remains to prove that $G - S_G(H)$ is connected. Pick $v, w \in V(G - S_G(H))$.

Suppose first that $v, w \in V(G_i)$ for some $i = 1, 2$. If $v, w \notin S_{G_i}(H_i)$, then there is a vw-path in $G_i - S_{G_i}(H_i)$. Note that each vertex $u \in S_2$, $u \in D_1(G_1) \cap D_1(G_2)$. Then this vw-path is a subgraph of $G_i - (S_{G_i}(H_i) \cup S_2) \subseteq G - S_G(H)$.

If $\{v, w\} \cap S_{G_i}(H_i) \neq \emptyset$, then assume $v \in S_{G_i}(H_i)$. Since $v \notin S_G(H)$, we have $v \in V(G_1 \cap G_2)$. Note that $S_{G_i}(H_i)$ is a stable set, then $v' \notin S_{G_i}(H_i)$ where $v' \in N_{G_i}(v)$. Then, if $w \notin S_{G_i}(H_i)$, then there is a $v'w$-path P in $G - S_G(H)$, and then take $R = vv'Pw$ that is a vw-walk in $G - S_G(H)$; otherwise, there is a $v'w'$-path Q in $G - S_G(H)$ where $w' \in N_{G_i}(w)$ as $w' \notin S_{G_i}(H_i)$, then take $R = vv'Qw'w$ and then we are done.

Now we may assume $v \in V(G_1)$ and $w \in V(G_2)$. Note that $u_1 \notin D_2(G)$, then $u_1 \notin S_G(H)$. Since $x_1' \in X_1$, $u_1 \in V(G_1)$. Similarly, $x \in X_2$, then $u_1 \in V(G_2)$. It shows $x \in V(G_1 \cap G_2)$. By the previous discussion, there are vx-path P and xw-path Q in $G - S_G(H)$. Take $R = vPxQw$, which is a vw-walk in $G - S_G(H)$ and then we are done.

Hence, $G - S_G(H)$ is connected and so H satisfies Theorem 1.2 (ii).

5. Conclusion

As indicated by Faudree et al. in [11], line graphs, being the core in the family of claw-free graphs, are a class of graphs that have been considered as having particular interests. Hemminger and Beineke also commented in [13] that 'it has been shown that several (but not all) NP-complete problems for graphs are polynomial time problems for line graphs'. Boesch, Suffel and Tindell in [4] investigate the Chinese Postman Problem within the simple graph, and defined it as the 'subeulerian problem'. Seeking the dual of the subeulerian problem, they introduced the supereulerian problem, which has becoming an intensively studied area in graph theory. In this paper, we have shown a characterization of graphs whose line graphs are supereulerian, which is a natural research objective. While a Hamilton cycle (spanning connected 2-regular subgraph) is seemingly similar to a spanning closed trial (spanning connected subgraph with even degree in each vertex), characterizing graphs with supereulerian line graphs seems more complicated than characterizing graphs with hamiltonian line graphs. This might be due to the fact that a closed trails in $L(G)$ permits that a vertex can be revisited more than once. In the root graph G, this means that an edge in G can be 'revisited' more than once, which increases the complexity in the discussion. The notion of supereulerian graphs have been extended to spanning trailable graphs and strongly spanning trailable graphs in [21,22], and (s, t)-supereulerian graphs in [18,19]. it is also natural and of interests to characterize graphs whose line graphs are spanning trailable, strongly spanning trailable or (s, t)-supereulerian, respectively. The current methods used in this paper do not seem to be directly applicable to such problems.

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References

[1] A. Benhocine and J.-L. Fouquent, *Pancyclic and pandconnected line graphs*, J. Graph Theory 11 (1987), pp. 385–398.

[2] A. Benhocine and J.-L. Fouquent, *The Chv́ atal-Erdös condition for pancyclic line-graphs*, Discrete Math. 66 (1987), pp. 21–26.

[3] E. van Blanken, J. van den Heuvel and H.J. Veldman, *Pancyclicity of hamiltonian line graphs*, Discrete Math. 138 (1995), pp. 379–385.

[4] F.T. Boesch, C. Suffel and R. Tindell, *The spanning subgraphs of eulerian graphs*, J. Graph Theory1 (1977), pp. 79–84.

[5] J.A. Bondy, U.S.R. Murty, *Graph Theory*, Springer, New York, 2008.

[6] P.A. Catlin, *A reduction method to find spanning Eulerian subgraphs*, J. Graph Theory 12 (1988), pp. 29–44.

[7] P.A. Catlin, *Supereulerian graphs: A survey*, J. Graph Theory 16 (1992), pp. 177–196.

[8] P.A. Catlin, T. Iqbalunnisa, T.N. Janakiraman and N. Srinivasan, *Hamilton cycles and closed trails in iterated line graphs*, J. Graph Theory 14 (1990), pp. 347–364.

[9] G. Chartrand, *On Hamiltonian line graphs*, Trans. Am. Math. Soc. 134(3) (1968), pp. 559–566.

[10] Z.-H. Chen and H.-J. Lai, *Reduction techniques for super-Eulerian graphs and related topics-a survey*, in *Combinatorics and Graph Theory'95, Vol. 1 (Hefei)*, World Scientific Publishing, River Edge, NJ, 1995, pp. 53–69.

[11] R. Faudree, E. Flandrin and Z. Ryjáček, *Claw-free graphs, a survey*, Discrete Math. 164 (1997), pp. 87–147.

[12] F. Harary and C.St.J.A. Nash-Williams, *On eulerian and hamiltoninan graphs and line graphs*, Canad. Math. Bull. 8 (1965), pp. 701–709.

[13] R.L. Hemminger and L.W. Beineke, *Line graphs and line digraphs*, in *Selected Topics in Graph Theory*, L.W. Beineke and R.J. Wilson, eds., Academic Press, New York, 1978, pp. 271–305.

[14] H.-J. Lai and Y. Shao, *Problems related to hamiltonian line graphs*, AMS/IP Stud. Adv. Math. 39 (2007), pp. 149–159.

[15] H.-J. Lai, Y. Shao and H. Yan, *An update on supereulerian graphs*, WSEAS Trans. Math. 12 (2013), pp. 926–940.

[16] H.-J. Lai, Y. Shao, G. Yu and M. Zhan, *Hamiltonian connectedness in 3-connected line graphs*, Discrete Appl. Math. 157(5) (2009), pp. 982–990.

[17] H.-J. Lai, Y. Shao and M. Zhan, *Every 4-connected line graph of a quasi claw-free graph is hamiltonian connected*, Discrete Math. 308 (2008), pp. 5312–5316.

[18] L. Lei, X. Li and B. Wang, *On (s,t)-supereulerian locally connected graphs*, in *International Conference on Computational Science*, 2007, pp. 384–388

[19] L. Lei, X. Li, B. Wang and H.-J. Lai, *On (s,t)-supereulerian graphs in locally highly connected graphs*, Discrete Math. 310 (2010), pp. 929–934.

[20] H. Li, H.-J. Lai, Y. Wu and S. Zhu, *Panconnected index of graphs*, Discrete Math. 340 (2017), pp. 1092–1097.

[21] P. Li, K. Wang, M. Zhan and H.-J. Lai, *Strongly spanning trailable graphs with short longest paths*, Ars Combinatoria 137 (2018), pp. 3–39.

[22] J. Liu, A. Yu, K. Wang and H.-J. Lai, *Degree sum and hamiltonian-connected line graphs*, Discrete Math. 341 (2018), pp. 1363–1379.

[23] W.R. Pulleyblank, *A note on graphs spanned by Eulerian graphs*, J. Graph Theory 3 (1979), pp. 309–310.

[24] O. Veblen, *An application of modular equations in analysis situs*, Ann. Math. 14 (1912–1913), pp. 86–94.