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Decomposition and r-hued Coloring of $K_4(7)$-minor free graphs

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ABSTRACT

A **(k, r)-coloring** of a graph G is a proper k-vertex coloring of G such that the neighbors of each vertex of degree d will receive at least $\min\{d, r\}$ different colors. The **r-hued chromatic number**, denoted by $\chi_r(G)$, is the smallest integer k for which a graph G has a (k, r)-coloring. Let $f(r) = r + 3$ if $1 \le r \le 2$, $f(r) = r + 5$ if $3 \le r \le 7$ and $f(r) = \lfloor 3r/2 \rfloor + 1$ if $r \ge 8$. In [Discrete Math., 315-316 (2014) 47-52], an extended conjecture of Wegner is proposed that if G is planar, then $\chi_r(G) \le f(r)$; and this conjecture was verified for K_4-minor free graphs. For an integer $n \ge 4$, let $K_4(n)$ be the set of all subdivisions of K_4 on n vertices. We obtain decompositions of $K_4(n)$-minor free graphs with $n \in \{5, 6, 7\}$. The decompositions are applied to show that if G is a $K_4(7)$-minor free graph, then $\chi_r(G) \le f(r)$ if and only if G is not isomorphic to K_6.

1. Introduction

Graphs in this paper are simple and finite. Undefined terminologies and notations are referred to [3]. As in [3], $\kappa(G)$, $\kappa'(G)$, $\delta(G)$ and $\Delta(G)$ denote the vertex connectivity, the edge-connectivity, the minimum degree and the maximum degree of a graph G, respectively. A proper k-coloring is a mapping $c : V(G) \mapsto \bar{k}$ with $\bar{k} = \{1, 2, \ldots, k\}$ such that the adjacent vertices receive different colors.

For an integer $k > 0$, let $\bar{k} = \{1, 2, \ldots, k\}$. Given a graph G, if $c : V(G) \mapsto \bar{k}$ is a mapping, and if $V' \subseteq V(G)$, then define $c(V') = \{c(v) : v \in V'\}$. For an integer $r > 0$, a *(k, r)-coloring* of a graph G is a mapping $c : V(G) \mapsto \bar{k}$ satisfying both the following conditions.

(C1) $c(u) \neq c(v)$ for every edge $uv \in E(G)$;

(C2) $|c(N_G(v))| \ge \min\{d_G(v), r\}$ for any $v \in V(G)$.

For a fixed integer $r > 0$, the *r-hued chromatic number* of G, denoted by $\chi_r(G)$, is the smallest k such that G has a (k, r)-coloring. The study of (k, r)-coloring was initiated in [11] and [9]. A graph H is a **minor** of a graph G if H is isomorphic to the contraction of a subgraph of G. If G does not have a minor isomorphic to a graph J, we say that a graph G is a **J-minor**

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free graph. For a given collection $\mathcal{F}$ of graphs, define

$$EX(\mathcal{F}) = \{G: \; G \text{ does not have a minor isomorphic to a member in } \mathcal{F}\}.$$

Chen et al. first investigated the r-hued chromatic number of planar graphs [4]. Without turning to the Four Color Theorem [1,2,12], they showed that if G is a planar graph, then $\chi_2(G) \le 5$. As the 5-cycle C_5 satisfies $\chi_2(C_5) = 5$ (Theorem 2.5 of [8]), this bound is best possible. It is conjectured in [4] that C_5 is the only planar graph with 2-hued chromatic number 5. This conjecture was proved by Kim et al. [5], as an application of the Four Color Theorem. Recently Kim et al. [6] further extends this result to K_5-minor free graphs. These are summarized in the theorem below.

Theorem 1.1. *Let G be a connected graph.*
(i) (Chen et al. Theorem 1.1 of [4]) If G is a planar graph, then $\chi_2(G) \le 5$.
(ii) (Kim, Lee, and Park [5]) If G is a planar graph other than C_5*, then* $\chi_2(G) \le 4$.
(iii) (Kim, Lee and Oum [6]) If G does not have a K_5*-minor, then either* $G \cong C_5$ *or* $\chi_2(G) \le 4$.

Wegner [15] proposed the following conjecture.

Conjecture 1.2. *(G. Wegner [15]) Let*

$$\phi(r) = \begin{cases} r+5, & \text{if } 4 \le r \le 7; \\ \lfloor 3r/2 \rfloor + 1, & \text{if } r \ge 8. \end{cases}$$

If G is a planar graph with $\Delta = \Delta(G)$*, then* $\chi_\Delta(G) \le \phi(\Delta)$.

Let

$$K(r) = \begin{cases} r+3, & \text{if } 1 \le r \le 3; \\ \lfloor 3r/2 \rfloor + 1, & \text{if } r \ge 4. \end{cases} \tag{1}$$

Lih, Wang and Zhu first proved that Wegner's conjecture holds for Δ-hued chromatic number of K_4-minor free graphs. Song et al. extended it to the case when Δ is replaced by a generic r. These are summarized in Theorem 1.3.

Theorem 1.3. *Let* $G \in EX(K_4)$ *be a graph and let* $r \ge 2$ *be an integer. Then each of the following holds.*
(i) (Lih, Wang and Zhu [10]) Then $\chi_\Delta(G) \le K(\Delta(G))$.
(ii) (Song et al. [13]) $\chi_r(G) \le K(r)$.

Motivated by Conjecture 1.2 and Theorem 1.3, a conjecture on the upper bound of r-hued-chromatic number of planar graphs is proposed in [13].

Conjecture 1.4. *([13]) Let* $r \ge 1$ *be an integer and let*

$$f(r) = \begin{cases} r+3, & \text{if } 1 \le r \le 2; \\ r+5, & \text{if } 3 \le r \le 7; \\ \lfloor 3r/2 \rfloor + 1, & \text{if } r \ge 8. \end{cases}$$

If G be a connected planar graph, then $\chi_r(G) \le f(r)$.

The Four Color Theorem [1,2,12] and Theorem 1.1 indicate that Conjecture 1.4 holds when $1 \le r \le 2$. The purpose of this research is to show that Conjecture 1.4 holds in a class of graphs that properly contain all graphs in $EX(K_4)$. Thus it provides further evidence of Conjecture 1.4 and extends Theorem 1.3.

In [14], it is shown that if $r \ge 8$, then for any planar graph G, $\chi_r(G) \le 2r + 16$. This is quite far away from the expected bound stated in Conjecture 1.4.

Let H be a graph. An edge $e \in E(H)$ is said to be **subdivided** when it is deleted and replaced by a path of length two connecting its end vertices. A **subdivision** of H is a graph obtained from H by a (possibly empty) sequence of edge subdivisions. If a graph G contains subgraph J isomorphic to a subdivision of H, we call J an ***H*-subdivision**. Thus by definition, if $\Delta(H) \le 3$, then G contains an H-minor if and only if G contains an H-subdivision. For an integer $n \ge 4$, define $K_4(n)$ to be the collection of all non-isomorphic subdivisions of K_4 on n vertices. Thus $K_4(4) = \{K_4\}$ and there is only one graph in $K_4(5)$, up to isomorphism, as seen in Fig. 1. When it is understood in the context, we sometimes use $K_4(4)$ and $K_4(5)$ to represent a member in the collection.

There are three members in $K_4(6)$ up to isomorphism, as depicted in Fig. 2.

By definition, for each $n \ge 4$, we have

$$EX(K_4) \subseteq \ldots \subseteq EX(K_4(n)) \subseteq EX(K_4(n+1)) \subseteq \ldots \tag{2}$$

and for each fixed integer $n \ge 4$, $EX(K_4(n))$ contains all graphs with order less than n. Hence $\bigcup_{n=4}^{\infty} EX(K_4(n))$ contains all graphs. The following is the main result obtained in this research.

Theorem 1.5. *Let* $r \ge 2$ *be an integer. If* $G \in EX(K_4(7))$ *and G has no block isomorphic to* K_6*, then* $\chi_r(G) \le K(r)$.

In the next section, we shall prove decomposition results of graphs in $EX(K_4(n))$. This decomposition result will be applied in the proof of Theorem 1.5, to be presented in the last section.

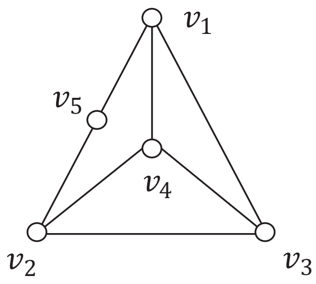

Fig. 1. A graph in $K_4(5)$.

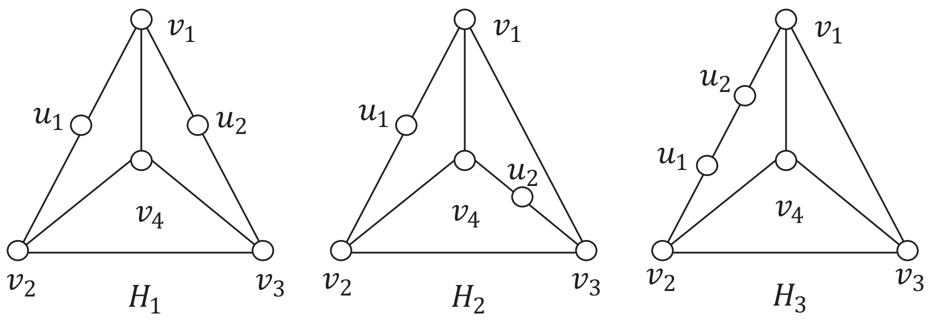

Fig. 2. The three members in $K_4(6)$.

2. Decompositions of $K_4(n)$

Throughout the rest of this paper, by $H \subseteq G$ we mean that G contains a subgraph isomorphic to H, and when there is no confusion arises, we also view that H is a subgraph of G. For a graph G and a collection $\mathcal{K}$ of subgraphs of G, we define

$$[\mathcal{K}, G] = \{H : \text{ for some } K \in \mathcal{K}, K \subseteq H \subseteq G\}.$$

When $\mathcal{K} = \{K\}$ consisting of a single subgraph of G, we often use $[K, G]$ for $[\{K\}, G]$. If for some $K \in \mathcal{K}$, H is a K-subdivision, then we also call H a $\mathcal{K}$**-subdivision**.

If X is a set of edges with end vertices in a graph G, then we use $G + X$ to denote the simple graph with vertex set $V(G)$ and edge set $E(G) \cup X$. When $X = \{e\}$, we often use $G + e$ for $G + \{e\}$. By definition, if $e \in E(G)$, then $G + e = G$. Similarly, if X is a set of edges in $E(G)$, then we use $G - X$ to denote the simple graph with vertex set $V(G)$ and edge set $E(G) - X$. When $X = \{e\}$, we often use $G - e$ for $G - \{e\}$. Throughout the discussion, a path P from a vertex u to a vertex v will be called a (u, v)**-path**.

We state the following proposition, which follows from the definition of K_4-minors.

Proposition 2.1. *Let G be a graph with a cycle C and a vertex $z \in V(G) - V(C)$. If for distinct vertices $z_1, z_2, z_3 \in V(C)$, G contains internally disjoint paths P_1, P_2 and P_3 such that each P_i is a (z, z_i)-path, then $H = G[E(C) \cup E(P_1) \cup E(P_2) \cup E(P_3)]$ is a subdivision of K_4. If $n = |V(H)|$, then H is a member in $K_4(n)$.*

We follow [3] to define the union of graphs. Let G and G' be two graphs. The **union** of G and G', denoted by $G \cup G'$, has a vertex set $V(G) \cup V(G')$ and an edge set $E(G) \cup E(G')$.

Definition 2.2. Let $k \geq 1$ be an integer, G, G', H_1, H_2, ..., H_k be vertex disjoint simple graphs.

(OP1) Suppose that $u \in V(G)$ and $u' \in V(G')$. Define $G \oplus_1 G'$ to be the simple graph obtained from $G \cup G'$ by identifying u with u' to form a new vertex, which is still denoted by u. We sometimes write $G \oplus_u G'$ for $G \oplus_1 G'$ to emphasize the vertex u.

(OP2) Suppose that $u, v \in V(G)$ and $u', v' \in V(G')$. Define $G \oplus_{u,v} G'$ to be the simple graph obtained from $G \cup G'$ by identifying u with u' to form a new vertex (again denoted by u), and v with v' to form a new vertex (again denoted by v), respectively. The vertices u, v are called the **base vertices** of $G \oplus_{u,v} G'$. Thus if either $uv \in E(G)$ or $u'v' \in E(G')$, then the edge $uv \in E(G \oplus_{u,v} G')$. If u, v are understood or not to be emphasized, we often use $G \oplus_2 G'$ for $G \oplus_{u,v} G'$.

(OP3) For each j with $1 \leq j < k$, assuming that $G(\oplus_2)_{i=1}^{j} H_i$ is obtained, we define $G(\oplus_2)_{i=1}^{j+1} H_i = (G(\oplus_2)_{i=1}^{j} H_i) \oplus_2 H_{j+1}$ in such a way that the base vertices of $G(\oplus_2)_{i=1}^{j+1} H_i$ are in $V(G)$, and for each H_i, the base vertices may be different. See Fig. 3(a).

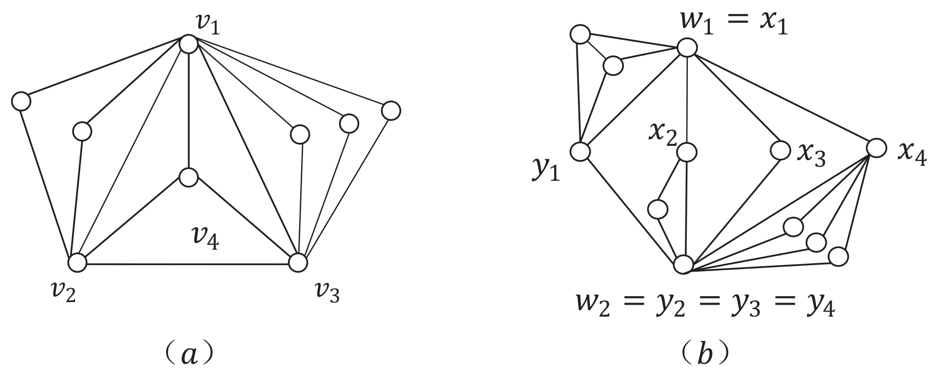

Fig. 3. (a) An example of Definition 2.2 (OP3), $K_4(\oplus_2)_{i=1}^2 H_i$, where $H_1 = K_{2,2}$ with base vertices v_1, v_2, and $H_2 = K_{2,3}$ with base vertices v_1, v_3. (b) An example of Definition 2.3 (ii), $SK_{2,4,T}$, where $T = (2, 1, 0, 3)$, $J = K_{2,4}$, $J_1 = K_{2,2}$, $J_2 = K_{2,1}$, $J_3 = K_{2,0}$, $J_4 = K_{2,3}$, and special vertices of each J_i are x_i, y_i.

We often also take the convention to assume that in (OP1), $V(G) \cap V(G') = \{u\}$, and in (OP2), $V(G) \cap V(G') = \{u, v\}$. We now can use the operations in Definition 2.2 to define some related constructions.

Definition 2.3. Let $t \geq 1$ be an integer, and $k_i \geq 0$ be an integer for $1 \leq i \leq t$.

(i) Let $K_{2,t}$ be a complete bipartite graph with w_1, w_2 being the two nonadjacent vertices in $K_{2,t}$ of degree t. The vertices w_1, w_2 are called the **special vertices** of $K_{2,t}$. Define $K'_{2,t}$ be the graph obtained by adding a maximum matching $u_1u_2, u_3u_4, \ldots, u_{t-1}u_t$ if t is even, $u_1u_2, u_3u_4, \ldots, u_{t-2}u_{t-1}$ if t is odd, among the non special vertices in $V(K_{2,t}) - \{w_1, w_2\}$. The special vertices of $K'_{2,t}$ are the special vertices of the related $K_{2,t}$.

(ii) Let $T = (k_1, k_2, \cdots, k_t)$ be a t-tuple of non-negative integers. Let J, J_1, ..., J_t be graphs such that $J \cong K_{2,t}$ with special vertices w_1 and w_2, and for $1 \leq i \leq t$, $J_i \cong K_{2,k_i}$. Define $SK_{2,t,T}$ to be a graph isomorphic to $J \oplus_2 \dot\cup_{i=1}^t J_i$ in such a way that for each j with $1 \leq j \leq t$, the base vertices x_j, y_j in $J \oplus_2 \dot\cup_{i=1}^t J_i$ are special vertices of each J_j, and $e_j = x_jy_j$ is an edge $e_j \in E(J)$ such that all the edges e_1, e_2, ..., e_t are mutually distinct and such that for distinct i and j, any vertex incident with both e_i and e_j must be in $\{w_1, w_2\}$. See Fig. 3(b).

As each x_iy_i can be any one of the two edges in a path joining the two special vertices of $J \cong K_{2,t}$, $SK_{2,t,T}$ is not unique.

2.1. *Decomposition of* $\mathcal{K}_4(5)$ *and* $\mathcal{K}_4(6)$

For an integer $i \geq 1$, define $D_i(G) = \{v \in V(G) : d_G(v) = i\}$. By definition,

$$\text{for any } H \in \mathcal{K}_4(n),\ |D_2(H)| = n - 4. \tag{3}$$

Thus up to isomorphism, there is only one graph in $\mathcal{K}_4(5)$. For an $H \in \mathcal{K}_4(6)$, the two vertices in $D_2(H)$ are of distance 1, or 2, or 3 in H. It follows that, up to isomorphism, $\mathcal{K}_4(6) = \{H_1, H_2, H_3\}$, where the H_i's are graphs depicted in Fig. 2. We first present a characterization of graphs in $EX(\mathcal{K}_4(5))$.

Proposition 2.4. *Let G be a 2-connected simple graph. Then $G \in EX(\mathcal{K}_4(5))$ if and only if $G \in \{K_4\} \cup EX(K_4)$.*

Proof. By (2), $\{K_4\} \cup EX(K_4) \subseteq EX(\mathcal{K}_4(5))$. It remains to assume that $G \in EX(\mathcal{K}_4(5))$ and G is not isomorphic to K_4 to show that $G \in EX(K_4)$.

Argue by contradiction and assume that $G \notin EX(K_4)$. Then G contains a subgraph J that is a subdivision of K_4. Since $G \in EX(\mathcal{K}_4(5))$, we must have $|J| \leq 4$, and so $J \cong K_4$. As G is not isomorphic to K_4, there exists a vertex $v \in V(G) - V(J)$. Since G is 2-connected, by Menger's Theorem (Page 208, Theorem 9.1 of [3]), G contains two internally disjoint paths from v to two distinct vertices of J. Thus G contains a subdivision of a member in $\mathcal{K}_4(5)$, contrary to the assumption of $G \in EX(\mathcal{K}_4(5))$. This proves the proposition. □

Let t denote a positive integer and assume that $v_1v_2 \in E(K_4) \cap E(K_{2,t})$. Define

$$\mathcal{L} = \cup_{t \geq 1}\{K_4 \oplus_{v_1,v_2} K_{2,t} - v_1v_2, K_4 \oplus_{v_1,v_2} K_{2,t}\}. \tag{4}$$

In particular, $K_4 \oplus_{v_1,v_2} K_{2,1} - v_1v_2$ is the only graph in $\mathcal{K}_4(5)$, and so $\mathcal{K}_4(5) \subset \mathcal{L}$.

Proposition 2.5. *Let G be a 2-connected simple graph. Then $G \in EX(\mathcal{K}_4(6))$ if and only if $G \in EX(\mathcal{K}_4(5)) \cup \mathcal{L} \cup [\mathcal{K}_4(5), K_5]$.*

Proof. By the definition of $\mathcal{K}_4(n)$ and (2), $EX(\mathcal{K}_4(5)) \cup [\mathcal{K}_4(5), K_5] \subseteq EX(\mathcal{K}_4(6))$. As it is routine to verify that $\mathcal{L} \subseteq EX(\mathcal{K}_4(6))$, it suffices to assume $G \in EX(\mathcal{K}_4(6)) - EX(\mathcal{K}_4(5)) \cup [\mathcal{K}_4(5), K_5]$ to show that $G \in \mathcal{L}$.

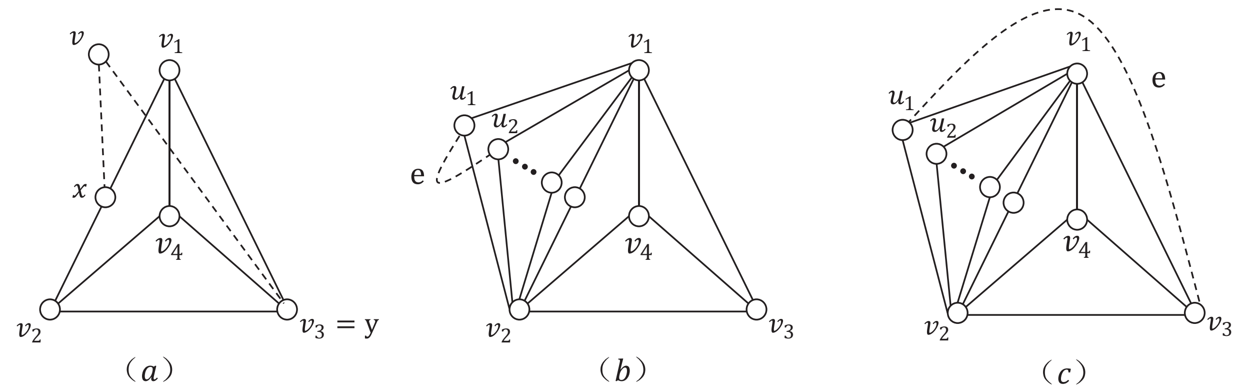

Fig. 4. Proof of Proposition 2.5: The existence of a member in $K_4(6)$.

Since $G \in EX(K_4(6)) - EX(K_4(5))$, G contains a subgraph J which is isomorphic to $K_4(5)$. We shall use the notation in Fig. 2 to label the vertices of J. In particular, v_1, v_2 are two vertices of J that are the neighbors of the only vertex of degree 2 of J.

If $V(G) = V(J)$, then $G \in [K_4(5), K_5]$. Therefore, there must be a vertex $v \in V(G) - V(J)$. By Menger's Theorem, G contains two internally disjoint paths P_1 and P_2 from v to two distinct vertices x, y of J, respectively, such that $V(P_1) \cap V(J) = \{x\}$ and $V(P_2) \cap V(J) = \{y\}$.

In our arguments that follow, we adopt the notation in Definition 2.2, with v_1, v_2 denoting the base vertices in the operation.

Claim 2.6. $\{x, y\} = \{v_1, v_2\}$.

If xy is an edge in J, then $G[E(J) \cup E(P_1) \cup E(P_2)]$ contains a $K_4(6)$-subdivision, and so $xy \notin E(J)$. If $x \in D_2(J)$ and y is not adjacent to x, then with the notation in Fig. 4(a), we may assume that $y = v_3$ and so J has a cycle $C = vxv_1v_4v_3v$ and three paths v_2x, v_2v_4, v_2v_3, forming a $K_4(6)$-subdivision also. (See Fig. 4(a) for an illustration). Either is a contradiction to the assumption of $G \in EX(K_4(6))$. Therefore, $\{x, y\} = \{v_1, v_2\}$ and the claim holds.

Since $G \in EX(K_4(6))$, we must have $P_1 = vv_1$ and $P_2 = vv_2$. It follows that $J = K_4(5) \subseteq G$. Let $t \geq 1$ be the largest positive integer such that $L_t \cong K_4 \oplus_{v_1,v_2} K_{2,t} - v_1v_2$ is a subgraph of G.

Claim 2.7. $V(G) = V(L_t)$.

If not, then there must be a vertex $v' \in V(G) - V(L_t)$. By the assumption $\kappa(G) \geq 2$, G has two internally disjoint paths Q_1 and Q_2 from v' to two distinct vertices x', y' of L_t, respectively, such that $V(Q_1) \cap V(L_t) = \{x'\}$ and $V(Q_2) \cap V(L_t) = \{y'\}$. By Claim 2.6, either $x', y' \in D_2(L_t)$ or $x', y' \in \{v_1, v_2\}$. Since t is maximized and since $G \in EX(K_4(6))$, that $x', y' \in \{v_1, v_2\}$ is not possible, and so we must have $x', y' \in D_2(L_t)$. But then $G[E(L_t) \cup E(Q_1) \cup E(Q_2)]$ contains a subdivision of $K_4(6)$, contrary to the assumption that $G \in EX(K_4(6))$. This proves Claim 2.7.

By Claim 2.7, if $E(G) = E(L_t)$, then $G = L_t \in \mathcal{L}$. Assume there exists an edge $e = zz' \in E(G) - E(L_t)$. Since G is simple, we have the following cases, either $\{z, z'\} = \{v_1, v_2\}$, $z, z' \in D_2(L_t)$, or by symmetry, $z \in D_2(L_t)$, or $z' \in V(L_t) - (D_2(L_t) \cup \{v_1, v_2\})$. For the last two cases, it is routine to verify that G has a subdivision of a member in $K_4(6)$, as illustrated in Fig. 4(b) and (c). Thus it suffices to consider the case when $\{z, z'\} = \{v_1, v_2\}$. In this case, $G \cong K_4 \oplus_{v_1,v_2} K_{2,t} \in \mathcal{L}$. This shows that we must have $G \in \mathcal{L}$, and so the proposition is justified. □

2.2. Decomposition of $K_4(7)$

In this section, we shall first show certain graph families are in $EX(K_4(7))$, and then we will prove a characterization of graphs in $EX(K_4(7))$. By (2), it is known that $EX(K_4(6)) \subset EX(K_4(7))$. Throughout this subsection, we denote $K_4(6) = \{H_1, H_2, H_3\}$, where the H_i's are depicted in Fig. 2, and the notation in Fig. 2 will be used in our arguments.

Definition 2.8. Let n, t_1, t_3, t_4 be non-negative integers with $n \geq 4$, $T = (k_1, k_2, ..., k_{t_3})$ be a t_3-tuple of positive integers. Let v_1, v_2, v_3, v_l with $l \in \{1, 4\}$ be vertices of a K_n. Suppose that $F_1 \cong K_{2,t_1}$, $F_2 \cong K'_{2,t_1}$, $F_3 \cong SK_{2,t_3,T}$, and $F_4 \cong K_{2,t_4}$ be graphs such that the special vertices of F_1, F_2, F_3 are $\{v_1, v_2\}$, and the special vertices of F_4 are $\{v_3, v_l\}$.

Lemma 2.9. *Then each of the followings holds.*

(i) If $|\{v_1, v_2, v_3, v_l\}| \geq 3$ *and* $l \in \{1, 4\}$, *then* $K_n \oplus_{v_1,v_2} F_1 \oplus_{v_3,v_l} F_4 \in EX(K_4(n+3))$.

(ii) $K_n \oplus_{v_1,v_2} F_2 \oplus_{v_1,v_2} F_3 \in EX(K_4(n+3))$.

(iii) $K_n \oplus_{v_1,v_2} F_1 \in EX(K_4(n+2))$.

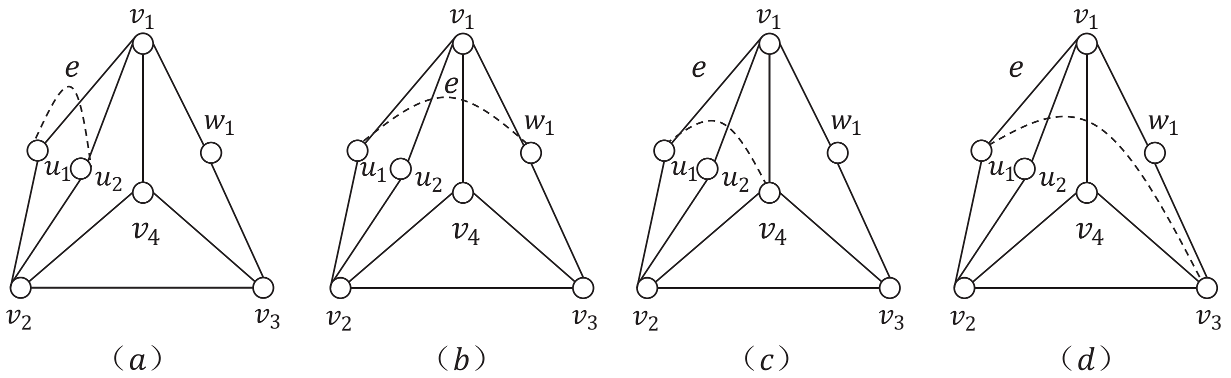

Fig. 5. Proof of Case 1 in Lemma 2.11: $N_1 + e$ contains a $K_4(7)$-subdivision.

Proof. As (ii) and (iii) can be proved in a similar way, we only prove (i). Let $G = K_n \oplus_{v_1,v_2} F_1 \oplus_{v_3,v_l} F_4$. Let L be a $K_4(n+k)$-subdivision of G with k maximized. By Definition 2.3 and the definition of $K_4(n)$-minor, $|V(L) \cap D_2(F_1)| = 1$ and $|V(L) \cap D_2(F_4)| = 1$. It follows that $|V(L)| = |V(K_n)| + 2 = n + 2$, which implies Lemma 2.9(i). □

Definition 2.10. We continue using the notation in Definition 2.8. Let $t_2 > 0$ be an integer and $F_1' \cong K_{2,t_2}$ with special vertices $\{v_3, v_l\}$ with $l \in \{1, 4\}$. Let K denote the complete graph K_n in Definition 2.8 with $V(K) = \{v_1, v_2, ..., v_n\}$.

(i) Define $L_1 := L_1(t_1, t_2) = K_4 \oplus_{v_1,v_2} F_1 \oplus_{v_3,v_\ell} F_1'$, $N_1 := N_1(t_1, t_2) = L_1(t_1, t_2) - \{v_1v_2, v_3v_\ell\}$, and $\mathcal{L}_1 = \{G \in [N_1, L_1] : |V(G)| \geq 6\}$.

(ii) Define $L_2 := L_2(t_1, t_3, T) = K_4 \oplus_{v_1,v_2} F_2 \oplus_{v_1,v_2} F_3$, where $T = (k_1, k_2, ..., k_{t_3})$ is a t_3-tuple with $k_1 \geq k_2 \geq ... \geq k_{t_3} \geq 0$, $N_2 := N_2(t_1, t_3, T) = K_4 \oplus_{v_1,v_2} F_1 \oplus_{v_1,v_2} F_3 - v_1v_2 - \bigcup_{i=1}^{t_3} e_i$, where the e_i's are defined in Definition 2.3. Define $\mathcal{L}_2 = \{G \in [N_2, L_2] : |V(G)| \geq 6\}$.

(iii) Define $L_3 := L_3(t_1) = K_5 \oplus_{v_1,v_2} F_1$, $N_3 := N_3(t_1) = L_3 - \{v_1v_2, v_1v_3, v_2v_5\}$, and $\mathcal{L}_3 = \{G \in [N_3, L_3] : |V(G)| \geq 6\}$.

In Definition 2.10(i), as the vertex v_ℓ varies, each pair (N_1, L_1) represents a family of such pairs with N_1 spans L_1. Similarly, as remarked after Definition 2.3, each pair of (N_2, L_2) represents a family of such pairs with N_2 spans L_2. In the arguments that follow, and in particular in Lemma 2.11, we always use (N_i, L_i) to denote an arbitrary member in the corresponding family.

Lemma 2.11. *Let G be a graph and adopt the notation $K_4(6) = \{H_1, H_2, H_3\}$ in Fig. 2. Each of the following holds.*

(i) $\{G \in \mathcal{L}_1 : |V(G)| = 6\} = \{H_1, H_2\}$ and $\{G \in \mathcal{L}_2 : |V(G)| = 6\} = \{H_3\}$.

(ii) $\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \subseteq EX(K_4(7))$.

(iii) Suppose that $|V(G)| \geq 7$. For $i = 1, 2, 3$, if $G = N_i$ and $e = xy \notin E(L_i)$ with $x, y \in V(L_i)$, then $G + e$ contains a $K_4(7)$-subdivision.

Proof. Lemma 2.11(i) follows from definition of $K_4(6)$. By Lemma 2.9, we conclude that $\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \subseteq EX(K_4(7))$. It remains to prove Lemma 2.11 (iii) and so we assume that $|V(G)| \geq 7$. We continue using the notation in Definition 2.10, and so $K = K_s$ with $V(K) = \{v_1, v_2, ..., v_s\}$ and $s \in \{4, 5\}$. We shall show that for any $e = xy \notin E(L_i)$, with $x, y \in V(L_i)$, we have $G + e \notin EX(K_4(7))$ by finding a $K_4(7)$-subdivision in $G + e$.

Case 1. $G = N_1$

Let $G = N_1 = (t_1, t_2) = L_1(t_1, t_2) - \{v_1v_2, v_3v_\ell\}$ and $e = xy \notin E(L_1)$. As $|V(G)| \geq 7$, we may assume that $t_1 \geq 2$ and $t_2 \geq 1$. Let $D_2(F_1) - \{v_1, v_2\} = \{u_1, u_2, ..., u_{t_1}\}$ and $D_2(F_1') - \{v_3, v_\ell\} = \{w_1, w_2, ..., w_{t_2}\}$. By symmetry, for each possible case, the Table 1 identifies a cycle C, a vertex $z \in V(G) - V(C)$ and internally disjoint (z, z_1)-paths for distinct $z_1, z_2, z_3 \in V(C)$. Thus by Proposition 2.1, $(G + e)[V(C) \cup \{z\}]$ contains a $K_4(7)$-subdivision (see Fig. 5 and Fig. 6 for an illustration, and Table 1 for the $K_4(7)$-subdivision). This proves Case 1.

Case 2. $G = N_2$

Let $G = K_4 \oplus_{v_1,v_2} F_1 \oplus_{v_1,v_2} F_3$ and $e = xy \notin E(L_2)$. As $|V(G)| \geq 7$, we may assume that $t_3 > 0$, and either $t_1 > 0$ or $k_1 \geq 2$ or both $(t_1, k_1) = (0, 1)$ and $t_3 > 1$. Let w_1, w_2 be the special vertices of the K_{2,t_3} in the definition of $F_3 \cong SK_{2,t_3,T}$, where where $w_1 = v_1$, $w_2 = v_2$ and $T = (k_1, k_2, ..., k_{t_3})$ is a t_3-tuple of non-negative integers. For each $j \in \{1, 2, ..., t_3\}$, let x_j, y_j be the special vertices of $J_j \cong K_{2,k_j}$ in Definition 2.3(ii) and $D_2(J_j) = \{w_1^j, w_2^j, ..., w_{k_j}^j\}$. Denote $D_2(F_1) - \{v_1, v_2\} = \{u_1, u_2, ..., u_{t_1}\}$. By symmetry, for each possible case, Table 2 identifies a cycle C, a vertex $z \in V(G) - V(C)$, a vertex $z \in V(G) - V(C)$ and internally disjoint (z, z_1)-paths for distinct $z_1, z_2, z_3 \in V(C)$. Thus by Proposition 2.1, $(G + e)[V(C) \cup \{z\}]$ contains a $K_4(7)$-subdivision (see Fig. 7 for an illustration, and Table 2 for the $K_4(7)$-subdivision). This proves Case 2.

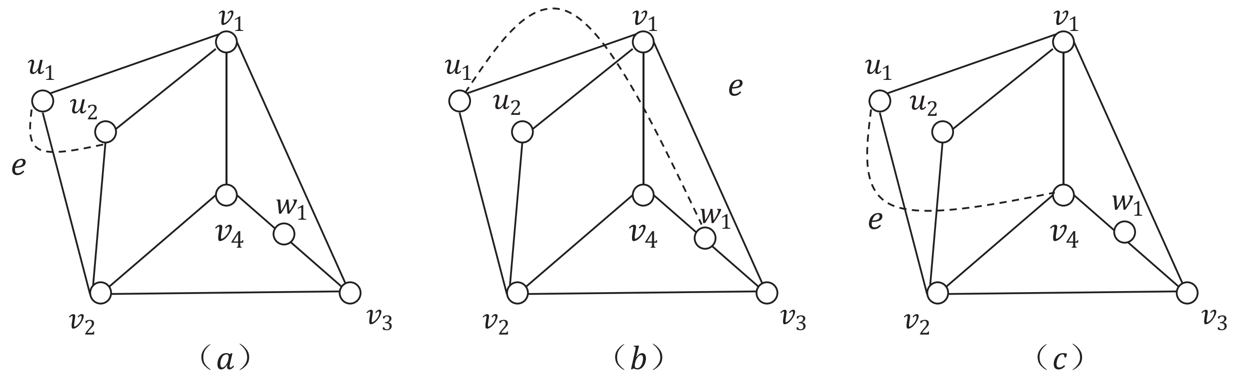

Fig. 6. Proof of Case 1 in Lemma 2.11: $N_1 + e$ contains a $K_4(7)$-subdivision.

Table 1
$N_1 + e$ has a $K_4(7)$-subdivision.

Figure	z	z_1	z_2	z_3	C
4(a)	v_4	v_1	v_2	v_3	$u_1u_2v_2v_3w_1v_1u_1$
4(b)	v_4	v_1	v_2	v_3	$u_1w_1v_3v_2u_2v_1u_1$
4(c)	u_1	v_1	v_2	v_4	$v_1u_2v_2v_4v_3w_1v_1$
4(d)	v_4	v_1	v_2	v_3	$u_1v_2u_2v_1w_1v_3u_1$
5(a)	u_1	v_1	u_2	v_2	$v_1u_2v_2v_3w_1v_4v_1$
5(b)	v_4	v_1	w_1	v_2	$u_1v_1u_2v_2v_3w_1u_1$
5(c)	u_1	v_1	v_4	v_2	$v_1u_2v_2v_4w_1v_3v_1$

Table 2
$N_2 + e$ contains a $K_4(7)$-subdivision.

Figure	z	z_1	z_2	z_3	C
6(a)	v_3	v_1	v_2	v_4	$w_1^1v_4v_2y_1w_2^1v_1w_1^1$
6(b)	v_3	v_1	v_2	v_4	$u_1v_4v_2y_1w_1^1v_1u_1$
6(c)	v_3	v_1	v_2	v_4	$w_1^1u_1v_2v_4v_1w_2^1y_1w_1^1$
6(d)	v_3	v_1	v_2	v_4	$y_1w_1^2y_2v_2v_4v_1w_1^1y_1$
6(e)	v_3	v_1	v_2	v_4	$w_1^1w_2^1y_1v_2v_4v_1w_1^1$
6(f)	v_3	v_1	v_2	v_4	$w_1^1v_4v_1u_1v_2y_1w_1^1$
6(g)	v_3	v_1	v_2	v_4	$y_1u_1v_2v_4v_1w_1^1y_1$
6(h)	v_3	v_1	v_2	v_4	$w_1^1w_1^2y_2v_2v_4v_1w_1^1$

Case 3. $G = N_3$

We adopt the notation in Definition 2.10 and use v_4 to denote the only vertex of degree 4 in N_3 not adjacent to vertices in $D_2(N_3)$. Denote $D_2(F_1) - \{v_1, v_2\} = \{u_1, u_2, ..., u_{t_1}\}$. If $e = u_1u_2$, then let $z = v_4$ and $C = v_1u_1u_2v_2v_3v_5$; if $e = u_1v_j$ with $j \in \{3, 4, 5\}$, then let $z = u_1$ and $C = v_1u_2v_2v_3v_4v_5$. In any case, it follows by Proposition 2.1 that $N_3 + e$ contains a $K_4(7)$-subdivision. This proves Case 3.

The proof of these cases justifies Lemma 2.11 (iii). □

Theorem 2.12. *Let G be a 2-connected simple graph. The the following are equivalent.*
(i) $G \in EX(K_4(7))$.
(ii) $G \in EX(K_4(6)) \cup \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \cup [K_4(6), K_6]$.

Proof. By Lemma 2.11 (i), (ii) and (2), we conclude that

$$EX(K_4(6)) \cup \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \cup [K_4(6), K_6] \subseteq EX(K_4(7)). \tag{5}$$

Thus it remains to prove (i) implies (ii). We shall assume $G \in EX(K_4(7)) - EX(K_4(6)) \cup [K_4(6), K_6]$ to prove that $G \in \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$. In the arguments below, we continue using the notation in Definition 2.10, and denote $K_4(6) = \{H_1, H_2, H_3\}$ as in Fig. 2 (together with the notation in Fig. 2 (together with the notation in Fig. 2).

As $G \in EX(K_4(7)) - EX(K_4(6)) \cup [K_4(6), K_6]$, we must have $|V(G)| \geq 7$ and G has a $K_4(6)$-subdivision H as a subgraph. As any $K_4(6)$-subdivision on at least 7 vertices must be a $K_4(7)$-subdivision, and as $G \in EX(K_4(7))$, we conclude that $H \in K_4(6) = \{H_1, H_2, H_3\}$.

By Lemma 2.11(i), $H_1, H_2 \in \mathcal{L}_1$ and $H_3 \in \mathcal{L}_2$; by Definition 2.10, H_1, H_2 are subgraphs of N_3. Thus G has a subgraph in $\cup_{i=1}^3 \mathcal{L}_i$. Let J denote a subgraph of G that is isomorphic to a member in $\cup_{i=1}^3 \mathcal{L}_i$ with $|V(J)| + |E(J)|$ maximized. Since $J \in \cup_{i=1}^3 \mathcal{L}_i$, we may assume that $G \neq J$ as otherwise, Theorem 2.12(ii) holds.

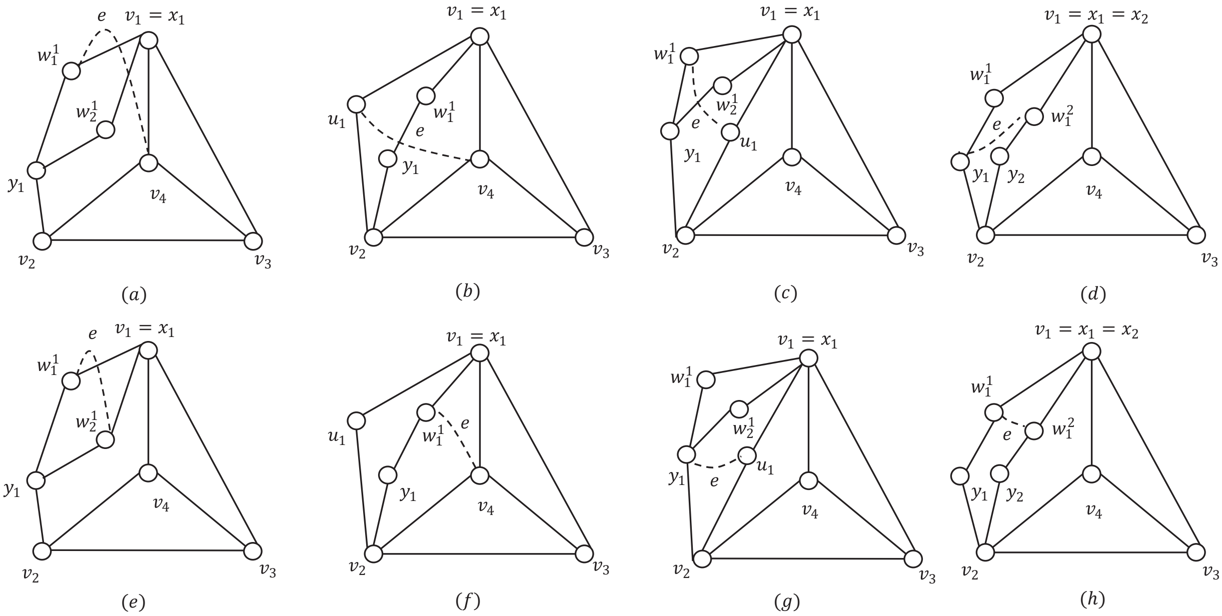

Fig. 7. Proof of Case 2 in Lemma 2.11: $N_2 + e$ contains a $K_4(7)$-subdivision for the cases in Table 2.

If $|V(G)| = |V(J)|$, then there must be an edge $e \in E(G) - E(J)$. Since $J \in \cup_{i=1}^{3} \mathcal{L}_i$, it follows by Definition 2.10 that there must be some $i \in \{1, 2, 3\}$ and a pair (N_i, L_i) such that $J \in [N_i, L_i]$. By Lemma 2.11, either $J + e \in \cup_{i=1}^{3} \mathcal{L}_i$, contrary to the choice of J; or $J + e$ contains a $K_4(7)$-subdivision, contrary to the assumption that $G \in EX(K_4(7))$. Hence we must have $|V(G)| > |V(J)|$. Pick a vertex $z \in V(G) - V(J)$. By $\kappa(G) \geq 2$, G contains two internally disjoint paths P_1 and P_2 such that for two distinct vertices $x_1, x_2 \in V(J)$ and $i \in \{1, 2\}$, P_i is a (z, x_i)-path with $V(J) \cap V(P_i) = \{x_i\}$.

By Lemma 2.11(iii) with the edge e in Lemma 2.11(iii) being replaced by the (x_1, x_2)-path $P = G[E(P_1) \cup E(P_2)]$, we conclude that since G does not have a $K_4(7)$ minor, we must have either $x_1x_2 \in E(N_i)$ or $x_1x_2 \in E(L_i) - E(N_i)$. If $x_1x_2 \in E(N_i)$, then by the definition of N_i in Definition 2.10, subdividing an edge in N_i would result in a $K_4(7)$-minor, leading to the contradiction that G has a $K_4(7)$-minor. Hence we must have $x_1x_2 \in E(L_i) - E(N_i)$, and so $x_1x_2 \in E(K)$ for some $K \subseteq L_i$, where K is either a K_4 or a K_5 in Definition 2.10. But then, $(J - x_1x_2) \cup E(P)$ either violates the maximality of J, or yields a $K_4(7)$-minor in G. In either case, a contradiction is found. This proves the theorem. □

3. Proof of Theorem 1.5

Recall that $K(r)$ is defined in (1). In this section, we shall show that if G is a $K_4(7)$-subdivision free graph, then for any $r \geq 1$, $\chi_r(G) \leq K(r)$ unless $r = 2$ and G has a block isomorphic to K_6. The following lemma suggests that it suffices to verify the statement for 2-connected graphs. A proof of Lemma 3.1 can also be found in [7]. We present its short proof here for the sake of completeness.

Lemma 3.1. *Let $G = G_1 \oplus_1 G_2$. Then $\chi_r(G) \leq \max\{\chi_r(G_1), \chi_r(G_2), r + 1\}$.*

Proof. Let $V(G_1) \cap V(G_2) = \{w\}$ and let $k = \max\{\chi_r(G_1), \chi_r(G_2), r + 1\}$. For $i \in \{1, 2\}$, G_i has a proper (k, r)-coloring $c_i : V(G_i) \to \bar{k}$. By permuting the colors in G_1, we may assume that $c_1(w) = c_2(w)$. Define $c : V(G) \to \bar{k}$ as

$$c(v) = \begin{cases} c_1(v), & \text{if } v \in V(G_1); \\ c_2(v), & \text{if } v \in V(G_2). \end{cases} \tag{6}$$

Then for any $v \in V(G) - \{w\}$, both (C1) and (C2) in the definition of r-hued colorings are satisfied at v. We are to prove (C1) and (C2) are satisfied at w. In particular, $c(w) \neq c(u)$ for any $u \in N(w)$. Let $h_1 = |N_{G_1}(w)|$ and $h_2 = |N_{G_2}(w)|$. If $\max\{h_1, h_2\} \geq r$ or $|c(N(w))| \geq r$, then (C2) is satisfied at w under the coloring c. Therefore we assume that both $\max\{h_1, h_2\} \leq r - 1$ and $|c(N(w))| \leq r - 1$. Hence $|c(N_{G_1}(w))| = |N_{G_1}(w)| = h_1$ and $|c(N_{G_2}(w))| = |N_{G_2}(w)| = h_2$.

Let $t = \min\{h_2, r - h_1\}$. Since $k \geq r + 1$, and $t \leq r - h_1 < k - h_1 = k - |N_{G_1}(w)|$, there exist t distinct colors $\{\alpha_1, \alpha_2, ..., \alpha_t\} \subset \bar{k} - c(N_{G_1}[w])$. As $t \leq h_2$, there exist t distinct vertices $\{u_1, u_2, ..., u_t\} \subseteq N_{G_2}(w)$. Define a permutation on colors in $\bar{k}$ by

$$\pi = \begin{pmatrix} c_2(w) & c_2(u_1) & c_2(u_2) & \dots & c_2(u_t) & \dots \\ c_2(w) & \alpha_1 & \alpha_2 & \dots & \alpha_t & \dots \end{pmatrix}$$

and define a new coloring $c' : V(G) \to \bar{k}$ as follows:

$$c'(v) = \begin{cases} c_1(v), & \text{if } v \in V(G_1); \\ \pi(c_2(v)), & \text{if } v \in V(G_2) - \{w\}. \end{cases} \tag{7}$$

Then $|c'(w)| = |c'_{G_1}(w) \cup c'_{G_2}(w)| \geq h_1 + t = h_1 + \min\{h_2, r - h_1\} = \min\{h_1 + h_2, r\}$, and so c' is a proper (k, r)-coloring of G, hence Proposition 3.1 holds. □

Lemma 3.2. *Let $r \geq 2$ be an integer and let G be a 2-connected graph. Each of the following holds.*

(i) If $G \in EX(K_4(5))$, then $\chi_r(G) \leq K(r)$.

(ii) If $G \in EX(K_4(6))$, then $\chi_r(G) \leq K(r)$.

(iii) If $r \geq 3$ and $G \in [K_4(6), K_6]$, then $\chi_r(G) \leq K(r)$ and if $G \in [K_4(6), K_6] - \{K_6\}$, then $\chi_2(G) \leq K(2)$.

Proof. By Proposition 2.4, $EX(K_4(5)) = \{K_4\} \cup EX(K_4)$. As $\chi_r(K_4) = 4 \leq K(r)$, Lemma 3.2(i) follows from Theorem 1.3(ii).

By Proposition 2.5, $EX(K_4(6)) - EX(K_4(5)) \subseteq \mathcal{L} \cup [K_4(5), K_5]$, where $\mathcal{L}$ is defined in (4). Since for any $r \geq 2$, $K(r) \geq 5$, it suffices to assume that $G \in \mathcal{L}$ with $|V(G)| \geq 6$ to show that $\chi_r(G) \leq K(r)$. Let $\{v_1, v_2, v_3, v_4\}$ denote the vertices of the K_4 in the definition of $K_4 \oplus_{v_1,v_2} K_{2,t}$. By (4), $G \in \{K_4 \oplus_{v_1,v_2} K_{2,s} - v_1v_2, K_4 \oplus_{v_1,v_2} K_{2,s}\}$ for some integer s. Denote $D_2(G) = \{u_1, u_2, \dots, u_s\}$. Define $c : V(G) \to \overline{r+2}$ as follows:

$$\begin{array}{lcll} c(z) & = & i + (r-2), & \text{if } z = v_i \text{ and } 1 \leq i \leq 4. \\ c(z) & \equiv & j \pmod{r-2}, & \text{if } z = u_j \text{ and } 1 \leq j \leq s. \end{array}$$

Note that $c(z)$ is equal to $r-2$ if $j \pmod{r-2} \equiv 0$. It is routine to verify that c is a proper coloring, and every v_i has $\min\{r, d_G(v_i)\}$ different colors assigned to its neighbors and every u_j has 2 colors in its neighbors. Thus c is a $(r+2, r)$-coloring of G, and so $\chi_r(G) \leq r+2 \leq K(r)$. This proves Lemma 3.2(ii).

We adopt the notation in Fig. 2 for the graphs in $K_4(6)$ and let $G \in [K_4(6), K_6]$. Thus G has a spanning subgraph $H \in K_4(6) = \{H_1, H_2, H_3\}$. Using the notation in Fig. 2, we denote $V(G) = \{v_1, v_2, v_3, v_4, u_1, u_2\}$, with u_1 and u_2 denoting the only two vertices in $D_2(H)$, where we label u_1 to be the vertex with $u_1v_1 \in E(G)$ and $u_1v_3 \notin E(H)$. In the following proof, we view that G is a spanning subgraph of K_6. Since $G \neq K_6$, G must have a pair of vertices $x, y \in V(G)$ with $xy \notin E(G)$. If every edge in $E(K_6) - E(G)$ has the form v_iv_j for some $1 \leq i < j \leq 4$, then we by symmetry assume that $v_1v_2 \notin E(G)$. Define $(c(u_1)c(u_2), c(v_3), c(v_4)) = (1, 2, 3, 4)$ and $c(v_1) = c(v_2) = 5$. As in this case, u_i is adjacent to every vertex of $V(G) - \{u_i\}$ and each v_j is adjacent to two vertices colored differently, it follows by definition that c is a (5,2)-coloring of G. Hence we assume, by symmetry, that $xy \in \{u_1v_3, u_1u_2\}$ (or we can convert the other cases to these two by relabeling the vertices).

Initially define $c : V(G) \to \bar{5}$ by starting from $c(v_i) = i$, $1 \leq i \leq 4$, with $c(u_1)$, $c(u_2)$ to be assigned. If $H \in \{H_1, H_2\}$, then u_2 is adjacent to two different colored vertices. Extend c by assigning

$$(c(u_1), c(u_2)) = \begin{cases} (3, 2) & \text{if } u_1v_3 \notin E(G); \\ (5, 5) & \text{if } u_1u_2 \notin E(G). \end{cases}$$

If $H = H_3$, then u_2 is adjacent to u_1 and v_1, and $u_1v_3 \notin E(G)$. Extend c by defining $c(u_1) = 3$ and $c(u_2) = 5$. In any case, c is a (5,2)-coloring of G. This justifies Lemma 3.2(iii) and completes the proof of the lemma. □

Lemma 3.3. *Let $r \geq 2$ be an integer, and $G \in EX(K_4(7))$ be a 2-connected graph. Each of the following holds.*

(i) $\chi_2(G) \leq K(2)$ if and only if G is not isomorphic to K_6.

(ii) If $r \geq 3$, then $\chi_r(G) \leq K(r)$.

Proof. By Theorem 2.12, $EX(K_4(7)) - (EX(K_4(6)) \cup [K_4(6), K_6]) \subseteq \mathcal{L}_1 \cup \mathcal{L}_3 \cup \mathcal{L}_3$. Thus by Lemma 3.2, it suffices to assume that $G \in \mathcal{L}_1 \cup \mathcal{L}_3 \cup \mathcal{L}_3$ to prove that $\chi_r(G) \leq K(r)$. To proceed the proof, we continue adopting the notation in Definition 2.8 and Definition 2.10. The lemma will then be proved after we justify each of the following three claims.

Claim 3.4. *If $G \in \mathcal{L}_1$, then $\chi_r(G) \leq r + 2 \leq K(r)$.*

By Definition 2.10, there exist integers $t_1 \geq 2$ and $t_2 \geq 2$ such that G is spanned by an $N_1(t_1, t_2)$ with possibly $v_1v_2, v_3v_\ell \in E(G)$. Recall that $V(K_4) = \{v_1, v_2, v_3, v_4\}$ and denote $D_2(F_1) - \{v_1, v_2, v_3, v_4\} = \{u_1, u_2, \dots, u_{t_1}\}$ and $D_2(F_1') - \{v_1, v_2, v_3, v_4\} = \{u_{t_1+1}, u_{t_1+2}, \dots, u_{t_1+t_2}\}$. By the definition of $\mathcal{L}_1$, we assume that each u_j, $(1 \leq j \leq t_1)$, is not incident with v_3 and each $u_{t_1+j'}$, $(1 \leq j' \leq t_2)$, is not incident with v_2.

For the case when $r = 2$, we define $c_1 : V(G) \to \overline{r+2}$ by letting $c_1(v_i) = i$ for $1 \leq i \leq 4$, $c_1(u_j) = 3$, for $1 \leq j \leq t_1$ and $c_1(u_{t_1+j'}) = 2$, for $1 \leq j' \leq t_2$.

Assume that $r \geq 3$. Define $c : V(G) \to \overline{r+2}$ as follows:

$$c_1(z) = i + (r-2), \qquad \text{if } z = v_i, \text{ for } 1 \leq i \leq 4;$$

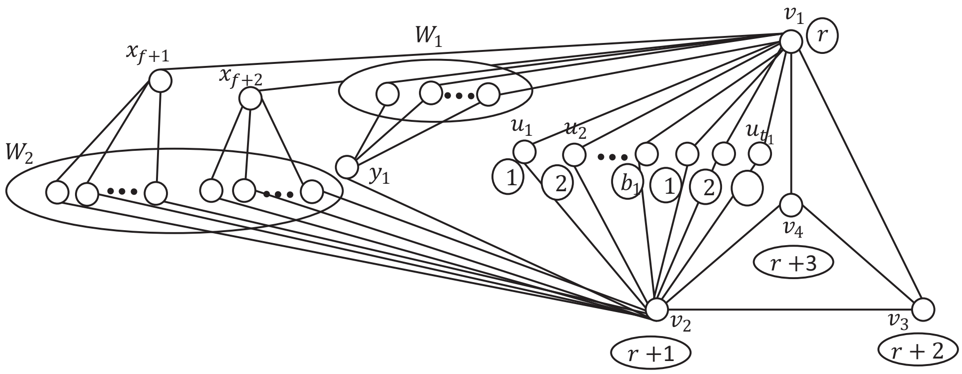

Fig. 8. Coloring in Step 1 of Claim 3.5: Numbers in the circles are colors.

$$c_1(z) \equiv j \pmod{r-2}, \qquad \text{if } z = u_j \text{ and } 1 \le j \le t_1 + t_2.$$

Note that $c_1(z)$ is equal to $r-2$ if $j \pmod{r-2} \equiv 0$. It is routine to verify that c_1 satisfies both (C1) and (C2) and is an $(r+2, r)$-coloring of G, independent of whether the edges v_1v_2, v_3v_ℓ are in $E(G)$ or not. Thus $\chi_r(G) \le r+2 \le K(r)$.

Claim 3.5. *If $G \in \mathcal{L}_2$, then $\chi_r(G) \le r+3 \le K(r)$.*

By Definition 2.10, every graph in $\mathcal{L}_2$ is a planar graph. Thus by Theorem 1.1(i), if $G \in \mathcal{L}_2$, then $\chi_2(G) \le K(2)$. Therefore, we assume that $r \ge 3$ and $|V(G)| \ge K(3) + 1 = 7$, and continue using the notation in Definition 2.10. Let $G \in [N_2(t_1, t_3, T), L_2(t_1, t_3, T)]$ with $t_1 \ge 0$ and $t_3 > 0$, where $T = (k_1, k_2, ..., k_{t_3})$ with $k_1 \ge k_2 \ge ... \ge k_{t_3} \ge 0$. Recall that $V(K_4) = \{v_1, v_2, v_3, v_4\}$, $F_1 \cong K_{2,t_1}$, $F_2 \cong K'_{2,t_1}$ and $F_3 \cong SK_{2,t_3,T}$, and $L_2 := L_2(t_1, t_3, T) = K_4 \oplus_{v_1,v_2} F_2 \oplus_{v_1,v_2} F_3$. Let $D_2(F_1) = \{u_1, u_2, ..., u_{t_1}\}$, and for each $j \in \{1, 2, ..., t_3\}$, let x_j, y_j be the special vertices of $J_j \cong K_{2,k_j}$ in Definition 2.3(ii) with $x_1 = x_2 = ... = x_f = v_1$, $y_{f+1} = y_{f+2} = ... = y_{t_3} = v_2$, and $D_2(J_j) = \{w_1^j, w_2^j, ..., w_{k_j}^j\}$. See Fig. 8.

We shall construct an $(r+3, r)$-coloring c of G in the following steps. Before the coloring, we define these sequences

$$W_1 = w_1^1, w_2^1, ..., w_{k_1}^1, w_1^2, w_2^2, ..., w_{k_2}^2, ..., w_1^f, w_2^f, ..., w_{k_f}^f, \tag{8}$$
$$W_2 = w_1^{f+1}, w_2^{f+1}, ..., w_{k_{f+1}}^{f+1}, w_1^{f+2}, w_2^{f+2}, ..., w_{k_{f+2}}^{f+2}, ..., w_1^{t_3}, w_2^{t_3}, ..., w_{k_{t_3}}^{t_3}.$$

We are to define the mapping $c_2 : V(G) \to \overline{r+3}$ in the following steps then verify that c_2 satisfies the hued coloring conditions (C1) and (C2) after that.

Step 1. In this step, we color the vertices in $V(K_4) \cup V(F_1)$.

Let $b_1 = \min\{t_1, r-1\}$, Define

$$c_2(z) = i + (r-1), \qquad \text{if } z = v_i \text{ and } 1 \le i \le 4;$$
$$c_2(z) \equiv j \pmod{r-1}, \qquad \text{if } z = u_j,\ 1 \le j \le t_1. \tag{9}$$

Note that $c_2(z)$ is equal to $r-1$ if $j \pmod{r-1} \equiv 0$. If $t_1 \ge r-1$, then $b_1 = \min\{t_1, r-1\} = r-1$. (See Fig. 8 for an example of the coloring in this case.) Hence the coloring of all the vertices colored in Step 1 would satisfy both (C1) and (C2). We then color $c(\{y_1, y_2, ..., y_f\}) = \{r+2\}$, $c(\{x_{f+1}, x_{f+2}, ..., x_{t_3}\}) = \{r+3\}$. For the other vertices, we color the vertices in the sequence W_1 subsequently using colors $1, 2, ..., r-2, r-1, 1, 2, ..., r-2, r-1, ...,$ and do the same to color the vertices in the sequence W_2, to obtain a coloring $c_2 : V(G) \to \overline{r+3}$. It is routine to verify that c is an $(r+3, r)$-coloring of G. Therefore, in the following, we assume that $b_1 = t_1 < r-1$, and continue with the following steps by extending the partial coloring defined in (9).

Step 2. In this step, we color the vertices $\{y_1, y_2, ..., y_f\}$ and $\{x_{f+1}, x_{f+2}, ..., x_{t_3}\}$. Define

$$c_2(z) \equiv t_1 + j \pmod{r-1}, \qquad \text{if } z = y_j \text{ and } 1 \le j \le f;$$
$$c_2(z) \equiv t_1 + j \pmod{r-1}, \qquad \text{if } z = x_{f+j} \text{ and } 1 \le j \le t_3 - f. \tag{10}$$

Note that $c_2(z)$ is equal to $r-1$ if $t_1 + j \pmod{r-1} \equiv 0$. Let $s_2 = r-1-t_1$, $b'_2 = \min\{f, s_2\}$, and $b''_2 = \min\{t_3 - f, s_2\}$. Thus by definition, there exists an index j with $f \ge j > b'_2$ only if $b'_2 = s_2$, which implies $\overline{r-1} \cup \{r+2, r+3\} \subseteq c(N_G(v_1))$. Similarly, there exists an index j with $t_3 - f \ge j > b''_2$ only if $b''_2 = s_2$, which implies $\overline{r-1} \cup \{r+2, r+3\} \subseteq c(N_G(v_2))$. (See Fig. 9 for an example of the coloring in this case.)

Step 3. Continuing the coloring process in Steps 1 and 2, we in this step shall color all the other vertices, listed in the two sequence W_1 and W_2 defined in (8).

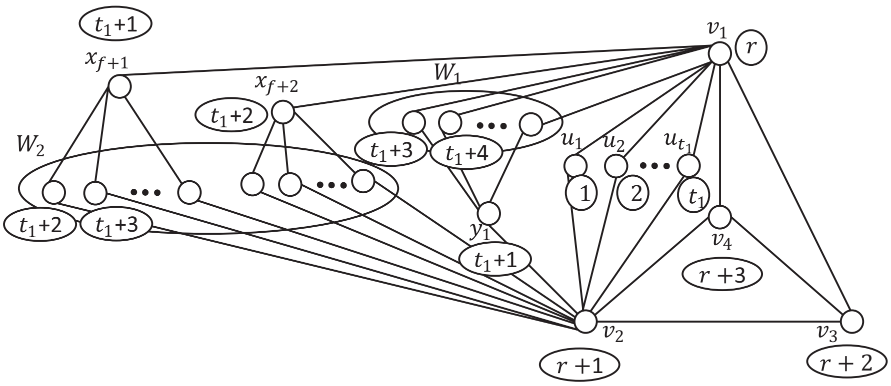

Fig. 9. Coloring in Steps 2 and 3 of Claim 3.5: Numbers in the circles are colors.

For vertices in the sequences W_1, we define an infinite color vector $C_1 = (b''_2 + 1, b''_2 + 2, \dots, r-1, r+2, r+3, 1, 2, \dots, r-1, r+2, r+3, 1, 2, \dots, r-1, r+2, r+3, \dots)$, and let c^1_i denote the ith component of C_1. To color $w^1_1, w^1_2, \dots, w^1_{k_1}$, if $c(y_1)$ is in the colors of $\{c^1_1, c^1_2, \dots, c^1_{k_1}\}$, then permute the all the components in the first k_1 components of C_1 that is equal to $c(y_1)$ by the first color in $(c^1_{k_1+1}, c^1_{k_1+2}, \dots)$ not equal to $c(y_1)$ and obtain a new color vector, still denoted by C_1 (to simplify notation). After this adjustment, we color $w^1_1, w^1_2, \dots, w^1_{k_1}$ using the first k_1 components of (the possibly adjusted) C_1.

Assuming that $j > h \geq 1$ and we have colored the vertices $w^1_1, w^1_2, \dots, w^1_{k_1}, w^2_1, w^2_2, \dots, w^2_{k_2}, \dots, w^h_1, w^h_2, \dots, w^h_{k_h}$, using the first $h' := k_1 + k_2 \dots + k_h$ of (currently adjusted) C_1, To color $w^{h+1}_1, w^{h+1}_2, \dots, w^{h+1}_{k_{h+1}}$, if $c(y_{h+1})$ is in $\{c^1_{h'+1}, c^1_{h'+2}, \dots, c^1_{h'+k_{h+1}}\}$, then permute the all the components in this group of colors that is equal to $c(y_{h+1})$ by the first color in $(c^1_{h'+k_{h+1}+1}, c^1_{h'+k_{h+1}+2}, \dots)$ not equal to $c(y_{h+1})$ and obtain a new color vector, again denoted by C_1 (to simplify notation). After this adjustment, we color $w^{h+1}_1, w^{h+1}_2, \dots, w^{h+1}_{k_{h+1}}$ using the colors from the $(h'+1)$th to the $(h'+k_{h+1})$th components of (the possibly adjusted) C_1. Thus the coloring of vertices in W_1 is done by such an inductive color assignment.

Similarly, for vertices in the sequence W_2, we define an infinite color vector $C_2 = (b'_2 + 1, b'_2 + 2, \dots, r-1, r+2, r+3, 1, 2, \dots, r-1, r+2, r+3, \dots 1, 2, \dots, r-1, r+2, r+3, \dots)$, and using the same coloring process described in the previous paragraph with the color vector C_1 replaced by the color vector C_2 to complete the definition of the coloring c.

Now the coloring of G is completed. By definition, c_2 is a proper $(r+3)$-coloring. In Step 1, (C2) is satisfied for vertices in $\{v_3, v_4\} \cup \{u_1, u_2, \dots, u_{t_1}\}$. After the completion of Step 3, every vertex in the sequence of W_1 and W_2 has two different colors in their neighbors, and so (C2) is satisfied for each of these vertices. For each vertex $y \in \{y_1, y_2, \dots, y_f\}$, by Step 3, at least $\min\{r, d_G(y)\}$ different colors in $\overline{r+3} - \{r, c(y)\}$ appear in $c(N_G(y))$. Similarly, For each vertex $x \in \{x_{f+1}, x_{f+2}, \dots, x_{t_3}\}$, by Step 3, at least $\min\{r, d_G(x)\}$ different colors in $\overline{r+3} - \{r+1, c(x)\}$ appear in $c(N_G(x))$. For v_1, we have $\{1, 2, \dots, b'_2\} \cup \{r+2, r+3\} \subseteq c(N_G(v_1))$ in Steps 1 and 2, and in Step 3, at least $\min\{d_G(v_1) - b'_2, r - b'_2\}$ different colors in $\{b'_2 + 1, \dots, r-1\}$ are in $c(N_G(v_1))$. This implies that (C2) is satisfied at v_1. Similarly, (C2) is satisfied at v_2. Thus c is indeed an $(r+3, r)$-coloring of G, which completes the proof of the claim.

Claim 3.6. *If $G \in \mathcal{L}_3$, then $\chi_r(G) \leq r+3 \leq K(r)$.*

By Definition 2.10, for an integer $t_1 \geq 2$, G is a spanning subgraph of $L_3(t_1) = K_5 \oplus_{v_1, v_2} F_1$ where $F_1 = K_{2,t_1}$ and $V(K_5) = \{v_1, v_2, v_3, v_4, v_5\}$. Let $D_2(F_1) = \{u_1, u_2, \dots, u_{t_1}\}$. For $r = 2$, define $c_3 : V(G) \to \overline{r+2}$ as $c_3(v_i) = i$ and $c(u_j) = 3$ for all i, j.

For $r \geq 3$, define $c_3 : V(G) \to \overline{r+3}$ as follows:

$$c_3(z) = i + (r-2), \qquad \text{if } z = v_i \text{ and } 1 \leq i \leq 5;$$

$$c_3(z) \equiv j \pmod{r-2}, \qquad \text{if } z = u_j \text{ and } 1 \leq j \leq t_1.$$

Note that $c_3(z)$ is equal to $r-2$ if $j \pmod{r-2} \equiv 0$. It is routine to verify that c_3 is an $(r+3, r)$-coloring of G, and so $\chi_r(G) \leq r+3$. □

Proof of Theorem 1.5. By Lemma 3.1, it suffices to prove Theorem 1.5 within 2-connected graphs. By Theorem 2.12, it remains to prove Lemmas 3.2 and 3.3. Thus the validity of these lemmas completes the proof of Theorem 1.5. □

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