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On the sizes of bi-k-maximal graphs

Liqiong Xu[1] · Yingzhi Tian[2] · Hong-Jian Lai[3]

Abstract
Let $k, n, s, t > 0$ be integers and $n = s + t \geq 2k + 2$. A simple bipartite graph G spanning $K_{s,t}$ is **bi-k-maximal**, if every subgraph of G has edge-connectivity at most k but any edge addition that does not break its bipartiteness creates a subgraph with connectivity at least $k+1$. We investigate the optimal size bounds of the bi-k-maximal simple graphs, and prove that if G is a bi-k-maximal graph with $\min\{s, t\} \geq k$ on n vertices, then each of the following holds.

(i) Let m be an integer. Then there exists a bi-k-maximal graph G with $m = |E(G)|$ if and only if $m = nk - rk^2 + (r-1)k$ for some integer r with $1 \leq r \leq \lfloor \frac{n}{2k+2} \rfloor$.
(ii) Every bi-k-maximal graph G on n vertices satisfies $|E(G)| \leq (n-k)k$, and this upper bound is tight.
(iii) Every bi-k-maximal graph G on n vertices satisfies $|E(G)| \geq k(n-1) - (k^2 - k)\lfloor \frac{n}{2k+2} \rfloor$, and this lower bound is tight. Moreover, the bi-k-maximal graphs reaching the optimal bounds are characterized.

Keywords Edge connectivity · Subgraph edge-connectivity · Strength k-maximal graphs · Bi-k-maximal graphs · Uniformly dense graphs

Mathematics Subject Classification 05C35 (05C40)

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1 Introduction

Throughout this paper, $\mathbb{Z}$ denotes the set of all integers. We consider finite simple graphs, with undefined terms and notation following Bondy and Murty (2008). In particular, $\kappa'(G)$ denotes the edge connectivity of a graph G. We use G^c to denote the complement of a simple graph G. If $X \subseteq E(G^c)$, then $G + X$ is the simple graph with vertex set $V(G)$ and edge set $E(G) \cup X$. We will use $G + e$ for $G + \{e\}$. Following Bondy and Murty (2008), we often use $G[X, Y]$ to denote a bipartite graph with vertex bipartition (X, Y). The **bipartite complement** of a simple bipartite graph $G = G[X, Y]$ is defined to be the bipartite graph G^{bc} with $V(G^{bc}) = V(G)$ and

$$E(G^{bc}) = \{xy : x \in X, y \in Y \text{ and } xy \notin E(G)\}.$$

If $W \subseteq V(G)$ or if $W \subseteq E(G)$, then $G[W]$ denotes the subgraph of G induced by W. For $v \in V(G)$, we use $G - v$ for $G[V(G) - v]$. For graphs H and G, we denote $H \subseteq G$ when H is a subgraph of G. We use $G \cong H$ when the graphs G and H are isomorphic. Following Bondy and Murty (2008), for vertex subsets $X, Y \subseteq V(G)$, $E_G[X, Y]$ denotes the set of edges of G with one end in X and the other end in Y. If H, J are two subgraphs of G, then we also use $E_G[H, J]$ for $E_G[V(H), V(J)]$.

Given a graph G, Matula (1972) first explicitly studied the quantity $\bar{\kappa}'(G) = max\{\kappa(H) : H \subseteq G\}$. He called $\bar{\kappa}'(G)$ the **strength** of G. Mader (1971) and Lai (1990) considered extremal problems related to $\bar{\kappa}'(G)$. For an integer $k > 0$, a simple graph G with $|V(G)| \geq k+1$ is (strength) k**-maximal** if $\bar{\kappa}'(G) \leq k$ but for any edge $e \in E(G^c)$, $\bar{\kappa}'(G+e) > k$. Mader (1971) and Lai (1990) proved the following for a positive integer k.

Theorem 1.1 *Let $k \geq 1$ be an integer, and G be a k-maximal graph on $n > k+1$ vertices. Each of the following holds.*

(i) (Mader 1971) $|E(G)| \leq (n-k)k + \binom{k}{2}$. *Furthermore, this bound is tight.*
(ii) (Lai 1990) $|E(G)| \geq (n-1)k - \lfloor \frac{n}{k+2} \rfloor + \binom{k}{2}$. *Furthermore, this bound is tight.*

There have been quite a few studies on this topic, as seen in Anderson (2017), Anderson et al. (2017, 2018), Lai (1990), Li et al. (2019), Lin et al. (2016), Mader (1971), Matula (1968), Matula (1969), Matula (1972), Matula (1976) and Xu (2018), among others. Theorem 1.1 motivates the the current research. For integer $s \geq 1$, $t \geq 1$, define

$$\mathcal{F}(s,t) = \{G[S, T] : s = |S|, t = |T|\}.$$

Let $k > 0$ be an integer. A graph G is **bi-k-maximal** if for some positive integers s and t, we have $G \in \mathcal{F}(s,t)$ with $\bar{\kappa}'(G) \leq k$ but for any edge $e \in E(G^{bc})$, we have $\bar{\kappa}'(G+e) \geq k+1$. The graph obtained by $K_{k+1,k+1}$ deleting one edge is bi-k-maximal. Additional examples of bi-k-maximal can be found in subsections 3.1 and 3.2 of this paper. By definition, a graph $G \in \mathcal{F}(s,t)$ with $\min\{s,t\} \leq k-1$ is bi-k-maximal if and only if $G = K_{s,t}$. Thus in this paper we may assume that $\min\{s,t\} \geq k$. Then every bi-k-maximal must have at least $2k$ vertices, and $\{K_{k,n-k} : n \geq 2k\}$ is the collection of bi-k-maximal complete bipartite graphs with order at least $2k$. The main result of this paper is the following.

Theorem 1.2 *Let k and n be integers with $k \geq 2$ and $n \geq 2k+2$. Each of the following holds.*

(i) Let m be an integer. Then there exists a bi-k-maximal graph G with $m = |E(G)|$ if and only if $m = nk - rk^2 + (r-1)k$ for some integer r with $1 \leq r \leq \lfloor \frac{n}{2k+2} \rfloor$.
(ii) Every bi-k-maximal graph G on n vertices satisfies $|E(G)| \leq (n-k)k$, and this upper bound is tight.
(iii) Every bi-k-maximal graph G on n vertices satisfies $|E(G)| \geq k(n-1) - (k^2 - k)\lfloor \frac{n}{2k+2} \rfloor$, and this lower bound is tight.

Section 2 below is devoted to the study of structural properties of bi-k-maximal graphs, which would be used in our arguments. Theorem 1.2 will be proved in Section 3.

2 Properties of bi-k-maximal graphs

By the definition of bi-k-maximal graphs, we observe that when $k = 1$,

$$\text{a bipartite graph } G \text{ is bi-1-maximal if and only if } G \text{ is a tree with } |E(G)| \geq 1. \quad (1)$$

Thus throughout this section, we assume that k, s, t are integers with $\min\{s, t\} \geq k \geq 2$, and $G = G[S, T] \in \mathcal{F}(s, t) - \{K_{k,k}\}$ be a bipartite graph with $n = s + t$ vertices. Let

$$\mathcal{B}(n, k) = \{G : G \text{ is a simple bipartite graph with } |V(G)| = n \text{ and } G \text{ is bi-}k\text{-maximal}\},$$

and $\mathcal{B}(k) = \cup_{n \geq 2k} \mathcal{B}(n, k)$ be the family of all bi-k-maximal graphs. Thus for any integer $t \geq k$, we have $K_{k,t} \in \mathcal{B}(k)$. Moreover, if $H \in \mathcal{B}(k)$ and $|V(H)| \in \{2k, 2k+1\}$, then $H \in \{K_{k,k}, K_{k,k+1}\}$. It follows from definition of $\mathcal{B}(k)$ that

$$\text{for any } G \in \mathcal{B}(k),\ G \text{ is connected and } \kappa'(G) \leq \overline{\kappa}'(G) \leq k. \quad (2)$$

Lemma 2.1 *Let $G \in \mathcal{B}(n, k)$ be a graph and X be a minimum edge-cut of G with $|X| \leq k$, and let G_1 and G_2 be the two components of $G - X$. Each of the following holds.*

(i) If G is not a complete bipartite graph, then $E_{G^{bc}}[V(G_1), V(G_2)] \neq \emptyset$.
(ii) $\kappa'(G) = \bar{\kappa}'(G) = k$.

Proof Let $G = G[A, B]$ with $|A| = s$ and $|B| = t$. We may assume, by symmetry, that $s \geq t$.
(i) Since G is a bipartite graph, we let $G_1 = G_1[A_1, B_1]$ and $G_2 = G_2[A_2, B_2]$ denote the two components of $G - X$, with $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. We

argue by contradiction to prove (i) and assume that $E_{G^{bc}}[V(G_1), V(G_2)] = \emptyset$. Thus $E_G[A_1, B_2] \cup E_G[A_2, B_1] \subseteq X$, and so we have

$$|A_1||B_2| + |A_2||B_1| = |X| \leq k. \tag{3}$$

□

Claim 1 $|A_1|$, $|B_1|$, $|A_2|$, *and* $|B_2| > 0$.

If $|A_1| = 0$, then $|B_1| = 1$ as G_1 is connected. By (3), $|A_2| \leq k$. As G is not a complete bipartite graph, there exists an edge $xy \in E(G^{bc})$. Since $G \in \mathcal{B}(k)$, we have $\bar{\kappa}'(G + xy) \geq k + 1$. As $|A| = |A_2| \leq k$, implying that $\bar{\kappa}'(G + xy) \leq k$. This contradiction forces that $|A_1| > 0$. By symmetry, we conclude that Claim 1 must hold.

As $s \geq t$, we have $|A_1| + |A_2| = s \geq \frac{n}{2}$. By Claim 1, we have $|B_1| \geq 1$ and $|B_2| \geq 1$, and so $|A_1||B_2| + |B_1||A_2| \geq |A_1| + |A_2| \geq \frac{2k+1}{2} > k$.

This leads to a contradiction to (3), and so Lemma 2.1(i) holds.

As (ii) holds trivially if $G = K_{k,n-k}$, we assume that G is not a complete bipartite graph to prove (ii). By (2), $\kappa'(G) \leq \overline{\kappa}'(G) \leq k$. Assume that $X \subseteq E(G)$ is a minimum edge-cut of G. Then $|X| \leq k$. By Lemma 2.1(i), we must have $E_{G^{bc}}[V(G_1), V(G_2)] \neq \emptyset$. Pick an edge $e \in E_{G^{bc}}[V(G_1), V(G_2)]$. Since $G \in \mathcal{B}(k)$, we have $\bar{\kappa}'(G + e) > k$. Let $H \subseteq G + e$ be a subgraph such that $\kappa'(H) \geq k + 1$. By (2), $\bar{\kappa}'(G) \leq k$, and so $e \in E(H)$. It follows that $(X \cup \{e\}) \cap E(H)$ is an edge cut of H. Thus $|X| + 1 \geq |(X \cup \{e\})| \geq \kappa'(H) \geq k + 1$, implying that $|X| = k$. Thus $\kappa'(G) \geq k$. This, together with (2), implies Lemma 2.1(ii). □

Lemma 2.2 *Suppose that* $G = G[A, B] \in \mathcal{B}(k) - \{K_{k,k}\}$. *Let* $X \subseteq E(G)$ *be a minimum edge-cut of* G *such that* G_1 *and* G_2 *are the two components of* $G - X$. *One of the following must hold:*

(i) $|V(G_1)| = 1$ *and* $G_2 \in \mathcal{B}(k)$.
(ii) $G_1 \in \mathcal{B}(k)$ *and* $|V(G_2)| = 1$.
(iii) *Both* $G_1 \in \mathcal{B}(k)$ *and* $G_2 \in \mathcal{B}(k)$, *and, both* G_1 *and* G_2 *are not complete bipartite graphs.*

Proof Let X be a minimum k-edge-cut, and let G_1 and G_2 denote the two components of $G - X$. If $\min\{|V(G_1)|, |V(G_2)|\} = 1$, then we shall show that (i) or (ii) must hold; if $\min\{|V(G_1)|, |V(G_2)|\} > 1$, then we shall show that (iii) must hold.

We start with a proof for (i) and (ii). By symmetry, we shall assume that $|V(G_1)| \geq |V(G_2)| \geq 1$, and so it suffices to assume $V(G_2) = \{v\} \subset B$ to prove that Lemma 2.2(ii) must hold. Let $G_1 = G_1[A_1, B_1]$ with $A_1 = A$ and $B_1 = B - \{v\}$. By Lemma 2.1(ii), we have $\delta(G) \geq \kappa'(G) = k$. Let $s_1 = |A_1|$ and $t_1 = |B_1|$. Then $s_1 = |A| \geq k$ and $t_1 \geq k - 1$.

If $G_1 = K_{s_1,t_1}$, then $k = \bar{\kappa}'(G) \geq \kappa'(G_1) = \min\{s_1, t_1\}$. Assume first that $s_1 \geq t_1$. Then as $\kappa'(G) = k$, we have $s_1 \geq k$, $t_1 \in \{k-1, k\}$ and $N_G(v) = A$. Since $G \neq K_{k,k}$, we conclude that $t_1 = k$ and so $G_1 = K_{k,s_1} (s_1 \geq k) \in \mathcal{B}(k)$. Now assume that $s_1 < t_1$. As $\kappa'(G) = k$, we have $s_1 = k$, and so $G_1 = K_{k,t_1}$ with $t_1 \geq k$. Thus $G_1 \in \mathcal{B}(k)$. Therefore, we may assume that G_1 is not a complete bipartite graph.

Thus there exists an edge $e = xy \in E(G_1^{bc})$. We may assume that $x \in A_1$ and $y \in B_1$. Since $G \in \mathcal{B}(k)$, $G+xy$ contains a subgraph L with $\kappa'(L) \geq k+1$. Since v has degree $|X| = k$ in G, $v \notin V(L)$ and so L is a subgraph of $(G-v)+xy = G_1+xy$. Since L is a simple subgraph of G_1 with $\delta(L) \geq \kappa'(L) \geq k+1$, it follows that $|V(G_1)| \geq |V(L)| \geq 2(k+1)$, and so $G_1 \in \mathcal{B}(k)$ and Lemma 2.2(ii) holds.

It remains to prove (iii). In the rest of the proof, we assume that both $G_1 = G[A_1, B_1]$ and $G_2 = G_2[A_2, B_2]$ with $\min\{|A_1|, |B_1|, |A_2|, |B_2|\} \geq 1$ to prove (iii). If G is a complete graph $K_{s',t'}$, then since $G \not\cong K_{k,k}$ and since by Lemma 2.1(ii), $\min\{s', t'\} \geq k$, we have $\max\{s', t'\} \geq k+1$. Hence $k = |X| = |A_1||B_2| + |B_1||A_2| \geq \max\{|A_1| + |A_2|, |B_1| + |B_2|\} \geq k+1$, a contradiction. Thus with the assumption of (iii), G cannot be a complete graph. □

Claim 2 *Neither G_1 nor G_2 is a compete bipartite graph.*

By contradiction and by symmetry, we may assume that $G_1 = G[S, T] \cong K_{s_2,t_2}$ with $|S| = s_2$, $|T| = t_2$ and $1 \leq s_2 \leq t_2$. Since G is not a complete bipartite graph, by Lemma 2.1(i), $E_{G^{bc}}[V(G_1), V(G_2)] \neq \emptyset$. Let $e \in E_{G^{bc}}[V(G_1), V(G_2)]$. Since $G \in \mathcal{B}(k)$, we have $s_2 \leq k$ and that $G+e$ has a subgraph L with $\kappa'(L) \geq k+1$ and we call this subgraph $L(e)$. Since $\overline{\kappa}'(G) \leq k$, we must have $e \in E(L)$, and so $(X \cup \{e\}) \cap E(L)$ is an edge-cut of L. Since $\kappa'(L) \geq k+1$, we conclude that $X \cup \{e\} \subseteq E(L)$.

For $i \in \{1, 2\}$, define $L_i = L[V(G_i) \cap V(L)] = L_i[S_i, T_i]$ with $S_i \subseteq S$ and $T_i \subseteq T$, and denote $|V(L_1)| = \ell$ and $|S_1| = s$. Thus $\ell \geq s$ and $s_2 \geq s$. As $\delta(L) \geq \kappa'(L) \geq k+1$ and $1 \leq s \leq \ell$, it follows that

$$\begin{aligned}\ell(k+1) &\leq \sum_{v \in V(L_1)} d_{L_1}(v) = 2|E(L_1)| + |X \cup \{e\}| \\ &= 2s(\ell - s) + k + 1 \leq \frac{\ell^2}{2} + k + 1,\end{aligned} \tag{4}$$

where in the last inequality, equality holds if and only if $\ell = 2s$. By (4), we have $2(\ell-1)(k+1) \leq \ell^2$, and so $2(k+1) \leq \frac{\ell^2}{\ell-1} = \ell + 1 + \frac{1}{\ell-1} < \ell + 2$, implying that $\ell \geq 2k+1$. Since $|S_1| = s \leq s_2 \leq k$, we have $|T_1| = \ell - s \geq k+1$. If $\ell(k+1) = \frac{\ell^2}{2} + k + 1$, then by (4), we have $\ell = 2s \leq 2k$, contrary to the fact that $\ell \geq 2k+1$. Hence we must have $\ell(k+1) < \frac{\ell^2}{2} + k + 1$, which implies $2(\ell-1)(k+1) < \ell^2 - 1$, leading to $\ell \geq 2k+2$. It follows by $s \leq k$ that $|D_1| = \ell - s \geq k+2$. As $|X \cup \{e\}| \leq k+1$, there must be a vertex $v \in V(T_1)$ with $d_L(v) \leq k$, contrary to the fact that $\delta(L) \geq \kappa'(L) \geq k+1$. This justifies Claim 2.

By Claim 2, neither G_1 nor G_2 is a complete bipartite graph. Fix $i \in \{1, 2\}$. For any edge $e \in E(G_i^{bc}) \subseteq E(G^{bc})$, $G+e$ has a subgraph L with $\kappa'(L) \geq k+1$. Since X is an edge cut of G with $|X| \leq k$, we observe that $X \cap E(L) = \emptyset$. As $e \in E(L) \cap E(G_i^{bc})$, we conclude that L is a subgraph of G_i, and so $\bar{\kappa}'(G_i + e) \geq k+1$. Since $\bar{\kappa}'(G_i) \leq \bar{\kappa}'(G) \leq k$, it follows that $G_i \in \mathcal{B}(k)$. This proves (iii). □

Definition 2.3 Let $k, r \in \mathbb{Z}$ with $k > 0$ and $r \geq 2$.

(i) Let G_1 and G_2 be two vertex disjoint connected bipartite graphs such that $G_1 = G_1[A_1, B_1]$ and $G_2 = G_2[A_2, B_2]$ with $\max\{|V(G_1)|, |V(G_2)|\} \geq 2k$. A **bipartite k-edge-join** of G_1 and G_2 is a simple bipartite graph obtained from the disjoint union of G_1 and G_2 by adding k new edges $e_1, e_2, \ldots, e_k$ to the union of G_1 and G_2 such that each e_i is incident with a vertex of A_1 and a vertex of B_2 or a vertex of A_2 and a vertex of B_1. Denote by $(G_1, G_2)_k$ the set of all bipartite k-edge-joins of G_1 and G_2.

(ii) Inductively, let $G_1, G_2, \ldots, G_r$ be sequence of mutually disjoint connected bipartite graphs with $\max\{|V(G_1)|, \ldots, |V(G_r)|\} \geq 2k$ and $r \geq 3$, and assume that for some integer m with $1 \leq m \leq r-1$, the bipartite k-edge-join families $(G_1, G_2, \ldots, G_m)_k$ and $(G_{m+1}, G_{m+2}, \ldots, G_r)_k$ have been defined. A **bipartite k-edge-join of** $G_1, G_2, \ldots, G_r$ is a bipartite graph with graph of the form $G \in (H_1, H_2)_k$ for some $H_1 \in (G_1, G_2, \ldots, G_m)_k$ and $H_2 \in (G_{m+1}, G_{m+2}, \ldots, G_r)_k$. Let $(G_1, G_2, \ldots, G_r)_k$ denote the family of all bipartite k-edge-joins of $G_1, G_2, \ldots, G_r$.

Observation 2.4 below follows from Definition 2.3 and routine verifications.

Observation 2.4 Each of the following holds.

(i) $(G_1, G_2)_k = (G_2, G_1)_k$.
(ii) If $\kappa'(G_1) \geq k$, and if $\kappa'(G_2) \geq k$ or $G_2 = K_1$, then

$$\kappa'(H) = k \text{ for any } H \in (G_1, G_2)_k. \tag{5}$$

The corollaries below follows immediately from Lemma 2.2.

Corollary 2.5 *Let $G \in \mathcal{B}(k) \setminus \{K_{k,k}\}$ be a bipartite graph. Then one of the following must hold.*

(i) For some $H \in \mathcal{B}(k)$, $G \in (H, K_1)_k$.
(ii) For some non-complete bipartite graphs $H_1, H_2 \in \mathcal{B}(k)$, $G \in (H_1, H_2)_k$.

Corollary 2.6 *Suppose that $\min\{\kappa'(H_1), \kappa'(H_2)\} \geq k$ and $G \in (H_1, H_2)_k$. Then for any edge cut X of G with $|X| = k$, either X is an edge cut of H_1, or X is an edge cut of H_2, or $X = E_G[V(H_1), V(H_2)]$.*

3 The size range of bi-k-maximal bipartite graphs

The main goal of this section is to prove Theorem 1.2. Let G be a graph and let $v \in V(G)$. Define $E_G(v)$ be the set of edges in G incident with v. A minimal edge cut X is **trivial** if there exists a vertex v with $E_G(v) = X$, otherwise X is an **essential** edge cut. If $\delta(G) = \kappa'(G)$ and if every minimum edge-cut of G is trivial, then G is **super-edge-connected** (see Chen et al. 2003; Esfahanian and Hakimi 1988; Xu to appear). Throughout the rest of this paper, for an edge subset $Y \subseteq E(G)$, let $V(Y) = V(G[Y])$. For an integer $k \geq 1$ and a graph G with $\kappa'(G) \geq k$, let $\mathcal{C}_k(G)$ denote the set of all edge cuts of size k of G.

3.1 Structural properties of graphs in $\mathcal{B}(k)$

The main results of this subsection are Lemmas 3.2 and 3.3, exploring some structural properties of graphs in $\mathcal{B}(k)$ to be applied in the arguments later.

Definition 3.1 Let k, n be positive integers with $n \geq 2k + 2$.

- (i) Define $\mathcal{H}(k, n-2k)$ to be the family of simple non-complete bipartite graphs $(K_{k,k}, K_1, \ldots, K_1)_k$ of order n. Unless otherwise stated, $H(k, n-2k)$ denotes an arbitrary member in $\mathcal{H}(k, n-2k)$.
- (ii) If $G = H(k, 2)$, then G has exactly two vertices of degree k. We use $\{x_1(G), x_2(G)\}$ to denote the set of vertices of degree k in G.

As an example, $\mathcal{H}(1, n-2)$ is precisely the family of trees on n vertices. Assume that $k \geq 2$. If $H \in \mathcal{H}(k, n-2k)$, then let H^1 denote the subset of $V(H)$ corresponding to the $(n-2k)$ singletons K_1's that are not in the subgraph isomorphic to $K_{k,k}$ of H. It is routine to verify that that

$$\kappa'(H) = k, \text{ for any } H \in \mathcal{H}(k, n-2k). \tag{6}$$

Lemma 3.2 *Suppose that $k \geq 2$. If $H \in \mathcal{H}(k, n-2k)$, then*

(i) $H \in \mathcal{B}(k)$, and
(ii) H is super-edge-connected.

Proof If $n = 2k + 2$, then since $\mathcal{H}(k, 2)$ contains no complete bipartite graphs, we have $\mathcal{H}(k, 2) = \{K_{k+1,k+1} - e\}$, and so Lemma 3.2 holds. Assume that $n \geq 2k+3$ and Lemma 3.2 holds for smaller values of n. Let $H_1 = H_1(k, n-2k) \in \mathcal{H}(k, n-2k)$. By Definition 2.3, there exists a graph $H_2 = H_2(k, n-2k-1) \in \mathcal{H}(k, n-2k-1)$ such that $H_1 = (H_2, K_1)_k \in \mathcal{H}(k, n-1)$. Let $V(H_1) - V(H_2) = \{v\}$.

Suppose that (ii) does not hold. Then there is a minimum edge-cut $X = E_{H_1}[V(J_1), V(J_2)]$ of H_1, where J_1 and J_2 are the two components of $H_1 - X$ with $\min\{|V(J_1)|, |V(J_2)|\} \geq 2$. By (5), $|X| = k$. Without loss of generality, assume that $v \in V(J_1)$. By (5), $\kappa'(H_2) = k$. If $E_{H_1}(v) \cap X \neq \emptyset$, then as $X \neq E_{H_1}(v)$, $X - E_{H_1}(v)$ is an edge-cut of H_2, and so $\kappa'(H_2) \leq |X - E_{H_1}(v)| < |X| = k = \kappa'(H_2)$, a contradiction. Hence we must have $E_{H_1}(v) \cap X = \emptyset$, and so $N_{H_1}(v) \subseteq V(J_1)$. It follows that $X = (V(J_1 - v), V(J_2))_{H_2}$ is an edge-cut of H_2. By induction, H_2 satisfies Lemma 3.2(ii), forcing $\min\{|V(J_1)|, |V(J_2)|\} = 1$, contrary to the assumption of $\min\{|V(J_1)|, |V(J_2)|\} \geq 2$. Thus Lemma 3.2(ii) must hold.

To prove (i), we argue by way of contradiction, and assume that $H_1 \notin \mathcal{B}(k)$. Then there is an edge $e \in E(H_1^{bc})$ such that $\overline{\kappa}'(H_1 + e) \leq k$. If $e \in E(H_2^{bc})$, then by induction, we have $H_2 \in \mathcal{B}(k)$ and so $\overline{\kappa}'(H_1 + e) \geq \overline{\kappa}'(H_2 + e) \geq k + 1$, a contradiction. Hence then $e \notin E(H_2^{bc})$. By Definition 2.3, $E(H_1) - E(H_2) = E_{H_1}(v)$, and so $|E(H_1) - E(H_2)| = k$. Since $e \notin E(H_2^{bc})$, we have $e \in E_{H_1+e}(v)$.

Let $Y = E_{H_1+e}[V(F_1), V(F_2)]$ be a minimum edge-cut of $H_1 + e$ with $|Y| = k$, where F_1 and F_2 are the two components of $(H_1 + e) - Y$. By (5), $\kappa'(H_1) = k$, and so $e \notin Y$ and $Y \neq E_{H_1}(v)$. This means $Y \subseteq E(H_1)$. Without loss of generality, assume that $v \in V(F_1)$. By (5), $\kappa'(H_2) = k$. If $Y \cap E_{H_1}(v) \neq \emptyset$, then as $Y \neq E_{H_1}(v)$,

$Y - E_{H_1}(v)$ is an edge-cut of H_2. It follows that $\kappa'(H_2) \le |Y - E_{H_1}(v)| < k = \kappa'(H_2)$, a contradiction. Hence we must have $Y \cap E_{H_1}(v) = \emptyset$, and so $Y \subseteq E(H_1 - E_{H_1}(v) = E(H_2)$. By induction, Lemma 3.2 holds for H_2, and so there exists a vertex $w \in V(H_2)$ such that $Y = E_{H_2}(w)$. As $N_{H_1}(v) \cup \{v\} \subseteq V(F_1)$, we must have $V(F_2) = \{w\}$.

Let $G' = H_1 - w$. Then $e \in E((G')^{bc})$. By Definition 2.3, $G' \in \mathcal{H}(k, (n-1)-2k)$. By induction, $G' \in \mathcal{B}(k)$, and so $\bar{\kappa}'(H_1 + e) \ge \bar{\kappa}'(G' + e) \ge k + 1$, contrary to the assumption of $\bar{\kappa}'(H_1 + e) \le k$. □

Lemma 3.3 *Let n, k be integers with $k \ge 2$ and $n \ge 2k+2$. Let $G \in \mathcal{B}(n,k) - \{K_{k,n-k}\}$. Then there exist integers $r, m_1, m_2, \ldots, m_r, m_{r+1}$ such that each of the following holds.*

(i) $1 \le r \le \lfloor \frac{n}{2k+2} \rfloor$, $m_i \ge 2$, $(1 \le i \le r)$, *and* $m_{r+1} \ge 0$;
(ii) $m_1 + m_2 + \cdots + m_{r+1} = n - 2rk$,
(iii) For each $i = 1, 2, \ldots, r$, *there exists an* $H_i = H(k, m_i) \in \mathcal{H}(k, m_i)$ *such that*

$$G \in (H_1, K_1, \ldots, K_1, H_2, K_1, \ldots, K_1, H_r, K_1, K_1, \ldots, K_1)_k,$$

with exactly m_{r+1} *singleton* K_1*'s.*

Proof We argue by induction on n. If $n = 2k + 2$, then by Corollary 2.5, we conclude that $G = H(k, 2)$, and so the theorem holds with $r = 1$ and $m_1 = 2$.

Assume that $n > 2k+2$ and the theorem holds for smaller values of n. By Corollary 2.5, either Corollary 2.5(i) or Corollary 2.5(ii) must hold. Assume first that Corollary 2.5(i) holds for G. Then there exists some $H \in \mathcal{B}(k)$ such that $G \in (H, K_1)_k$. By induction, there exist integers $r, m_1, m_2, \ldots, m_r, m'_{r+1}$ such that H satisfies Lemma 3.3 (i), (ii) and (iii). As $G \in (H, K_1)_k$, by Definition 2.3, Lemma 3.3 (i), (ii) and (iii) hold for G with $m_{r+1} = m'_{r+1} + 1$.

Now assume that Corollary 2.5(ii) holds. Then there exist non-complete bipartite graphs $H'_1, H'_2 \in \mathcal{B}(k)$ with $G \in (H'_1, H'_2)_k$. By induction, each of H'_1 and H'_2 satisfies Lemma 3.3 (i), (ii) and (iii), and so by Definition 2.3, Lemma 3.3 (i), (ii) and (iii) hold for G as well. □

Corollary 3.4 *Let k, n be integers with $k \ge 2$ and $n \ge 2k + 2$, and let $G \in \mathcal{B}(n, k) - \{K_{k,n-k}\}$. Each of the following holds.*

(i) There exist integers r, s with $1 \le r \le \lfloor \frac{n}{2k+2} \rfloor$ and $s = n - r(2k + 2)$, and graphs $H_1, H_2, \ldots H_{r+s}$ with $H_1 = H(k, 2)$ and for $2 \le i \le r + s$, we have $H_i \in \{H(k, 2), K_1\}$ such that exactly r of the H_i's are isomorphic to $H(k, 2)$ and such that $G \in (H_1, H_2, \ldots, H_{r+s})_k$.
(ii) For the integers r and s defined in (i), we have $|E(G)| = nk - rk^2 + (r - 1)k$.
(iii) The integer r is uniquely determined by the value k and the graph G. (This number r will be denoted by $r_k(G)$.)
(iv) There exist subgraphs H_1 and H_2 of G with $H_1 \in \mathcal{B}(k) - \{K_{k,j-k} : j \ge 2k\}$ such that either $H_2 = K_1$ and $G \in (H_1, K_1)_k$ or $\delta(G) \ge k+1$, $H_2 \in \mathcal{B}(k) - \{K_{k,j-k} : j \ge 2k\}$ and $G \in (H_1, H_2)_k$.

Proof By Definition 3.1, if $m > 2$, then $H(k, m) \in (H(k, 2), L_1, \ldots, L_{m-2})_k$ with $L_j \cong K_1$, for all $j \in \{1, 2, \ldots, m-2\}$. It follows from Lemma 3.3 that there exist integers r, s with $1 \le r \le \lfloor \frac{n}{2k+2} \rfloor$ and $s = n - r(2k+2)$, and graphs $H_1, H_2, \ldots H_{r+s}$ with

$H_1 = H(k, 2)$ and for $2 \le i \le r + s$, such that $H_i \in \{H(k, 2), K_1\}$ and that exactly r of the H_i's are isomorphic to $H(k, 2)$ and such that $G \in (H_1, H_2, \ldots, H_{r+s})_k$. This proves (i).

By (i), and by $s = n - r(2k+2)$, we have $|E(G)| = \sum_{i=1}^{r+s} |E(H_i)| + (r+s-1)k = r(k^2+2k) + (r+s-1)k = nk - rk^2 + (r-1)k$. Thus $r = \dfrac{(|V(G)| - 1)k - |E(G)|}{k^2 - k}$ is uniquely determined by the value k and the graph G. This proves (ii) and (iii).

To prove (iv), we first observe that by Lemma 2.1, we have $\delta(G) \ge \kappa'(G) = k$. If $\delta(G) = k$, then by Corollary 2.5, we must have $G \in (H_1, K_1)_k$ for some subgraph H_1 of G with $H_1 \in \mathcal{B}(k)$. Assume that $\delta(G) \ge k + 1$. Then by Lemma 3.3, we must have subgraphs H_1, H_2 of G satisfying $H_1, H_2 \in \mathcal{B}(k)$ and $G \in (H_1, H_2)_k$. □

3.2 Structural properties of graphs in $\mathcal{E}(k)$

Lemma 3.3 reveals the structure of bi-k-maximal graphs and motivates the following definition. We observed that by Definition 2.3 that $H(k, m)$ and K_1 are the building blocks of bi-k-maximal graphs. This leads to the definition of a special subfamily $\mathcal{E}(k)$ of $\mathcal{B}(k)$. The purpose of this subsection is to develop and explore some structural properties of graphs in $\mathcal{E}(k)$ to be deployed in the arguments to justify the main result of this paper in the last subsection.

Definition 3.5 Let n and k be integers with $k \ge 1$ and $n \ge 2k + 2$. Let $\mathcal{S}(n, k)$ be the set of all integral sequences $(s_1, s_2, \ldots, s_q)$ satisfying $n = s_1 + s_2 + \cdots + s_q$ such that $s_1 = 2k + 2$, and for $i = 2$, $s_i \in \{1, 2k + 2\}$. For any $s = (s_1, s_2, \ldots, s_q) \in \mathcal{S}(n, k)$, define bipartite graphs $L(s) = L(s_1, s_2, \ldots, s_q)$ as follows.

(i) Given $s = (s_1, s_2, \ldots, s_q) \in \mathcal{S}(n, k)$, define $J_i = K_1$ if $s_i = 1$ and $J_i = H(k, 2)$ if $s_i = 2k + 2$. Then the sequence of bipartite graphs $J_1, J_2, \ldots, J_q$ is called a **construction sequence** of $L(s)$.

(ii) If $i = 1$, then define $L_1 \cong H(k, 2)$.

(iii) Assume that $i \ge 2$.

(iii-A) If $s_i = 1$, then define $L_i \in (L_{i-1}, K_1)_k$.

(iii-B) If $s_i = 2k + 2$, then define $L_i \in (L_{i-1}, H(k, 2))_k$, in such a way that for each $h \in \{1, 2\}$, $d_{L_i}(x_h(L_1)) \ge k + 1$ and $d_{L_i}(x_h(J_i)) \ge k + 1$.

(iv) Define $L(s) = L_q$. Thus each $L(s)$ represents a collection of bipartite graphs.

(v) Define $\mathcal{E}(n, k) = \{L(s) : s \in \mathcal{S}(n, k)\}$ and $\mathcal{E}(k) = \left(\bigcup_{n \ge 2k+2} \mathcal{E}(n, k)\right) \cup \{K_{k,k}, K_{k,k+1}\}$.

In the discussions below, we shall adopt the notation in Definition 3.5 in our arguments involving graphs in $\mathcal{E}_k$. As an example, define

$$s'(n, k) = (s_1, s_2, \ldots, s_{n-(k+2)}), \text{ where } s_1 = k + 2 \text{ and } s_2 = s_3 = \ldots = s_{n-(k+2)} = 1.$$

Then by Definitions 3.1 and 3.5, $L(s'(n, k)) = \mathcal{H}(k, n - 2k)$.

Observation 3.6 Let $s = (s_1, s_2, \ldots, s_q) \in \mathcal{S}(n, k)$ with $q \geq 2$, and let $G \in L(s)$ with $J_1, J_2, \ldots, J_q$ being a construction sequence of G.

(i) If for some $i > 1$, $s_i = 2k + 2$, then for every j with $1 \leq j \leq i$ and with $J_j = H(k, 2)$, the degree k vertices $x_1(J_j)$ and $x_2(J_j)$ of J_j satisfies $d_{L_i}(x_1(J_j)) \geq k + 1$ and $d_{L_i}(x_2(J_j)) \geq k + 1$.

(ii) If for some i with $1 < i \leq q$, we have $s_i = 2k+2$ and if $X = E_G[V(L_{i-1}), G - V(L_{i-1})] \in \mathcal{C}_k(G)$, then $G' = G - V(L_{i-1}) \in \mathcal{S}(n - |V(L_{i-1})|, k)$, and either $G' = H(k, n - |V(L_{i-1})| - k)$ or both $d_{G'}(x_1(J_i)) \geq k + 1$ and $d_{G'}(x_2(J_i)) \geq k + 1$.

(iii) If $X = E_G[V(L_{q-1}), V(J_q)]$ and if there exists an index $i_0 < q - 1$ satisfying $V(X) \cap V(L_{q-1}) \subseteq V(L_{i_0})$, then $G \in L(s')$ where $s' = (s_1, \ldots, s_{i_0}, i_q, i_{s_0+1}, \ldots, s_{q-1})$.

(iv) If X is an edge cut of G with $|X| = k$, then there exists some i with $1 \leq i < q$ such that $X = E_G[V(L_i), V(J_{i+1})]$.

Proof By Definition 3.5 (iii-B), if $i > 1$, then Observation 3.6 (i) holds since $x_h(J_i)$ has degree k in J_i and is adjacent to a vertex in $G - V(J_i)$. Assume that $i = 1$. Let $t > 1$ be the smallest integer such that $s_t = 2k + 2$. By Definition 3.5 (iii-B), we may assume that $x_1(J_t)x_1(J_1) \in E(G)$. Since G is bipartite, it follows by Definition 3.5 (iii-B) again that we must have $x_2(J_t)x_2(J_1) \in E(G)$, and so both $d_{L_i}(x_1(J_j)) \geq k+1$ and $d_{L_i}(x_2(J_j)) \geq k + 1$, implying (i).

To prove (ii), assume that for some i with $1 < i \leq q$, both $s_i = 2k + 2$ and $X = E_G[V(L_{i-1}, G - V(L_{i-1})]$ is a k-edge-cut of G. As $|X| = k$, for any j with $i < j \leq q$, no edge in $X_j = E_G[V(L_{j-1}, G - V(L_{j-1})]$ is incident with an vertex in $V(L_{i-1})$. Define $s' = (s_i, s_{i+1}, \ldots, s_q)$ and $G' = G - V(L_{i-1})$. By Definition 3.5, $s' \in \mathcal{S}(|V(G')|, k)$ and $G' \in L(s')$. If $s_{i+1} = \ldots = s_q = 1$, then $G' = H(k, n - |V(L_{i-1})| - k)$; if for some $j > i$, $s_j = 2k + 2$, then by Observation 3.6 (i), both $d_{G'}(x_1(J_i)) \geq k + 1$ and $d_{G'}(x_2(J_i)) \geq k + 1$.

Observation 3.6 (iii) follows from Definition 3.5 straightforwardly.

To prove (iv), let X is an edge cut of G with $|X| = k$ and let $i < q$ be the smallest index such that $E(L_i) \cap X \neq \emptyset$. If $i = 1$, then by Definition 3.5, $L_1 = H(k, 2)$, and $X = E_{L_1}(x_j(J_1))$ for some $j \in \{1, 2\}$. By Observation 3.6 (i), we conclude that $s = (2k + 2, 1, 1, \ldots, 1)$ and $G = H(k, n - 2k)$. Thus Observation 3.6 (iv) must hold. Assume that that $i > 1$. By the definition of i, $X \cap E(L_{i-1}) = \emptyset$. If $X \cap E(J_i) \neq \emptyset$, then $J_i \cong H(k, 2)$, and so the only edge cut of size k in J_i must be $E_{J_i}(x_j(J_i))$ with $j \in \{1, 2\}$. Since $X \cap E(J_i)$ is an edge-cut of J_i of size at most $|X| = k$, it follows that $X = E_{J_i}(x_j(J_i))$ for some $j \in \{1, 2\}$. By Observation 3.6 (i), X cannot be an edge cut of G, contrary to the assumption on X. Hence $X \cap E(J_i) = \emptyset$ and so $X = E_{L_i}[V(L_{i-1}), V(J_i)] = E_G[V(L_{i-1}), V(G) - V(L_{i-1})]$. This proves Observation 3.6 (iv) . □

Lemma 3.7 *For any $G \in \mathcal{E}_k$, we have*

$$\kappa'(G) = \bar{\kappa}(G) = k. \tag{7}$$

Proof Let $G \in \mathcal{E}_k$. Then there exists an $s = (s_1, s_2, \ldots, s_q) \in \mathcal{S}(n, k)$ such that $G \in L(s)$. If $q = 1$, then $G = H(k, 2)$, and so (7) holds. Assume that $q > 1$ and (7) holds for smaller values of q. In the rest of the proof of this lemma, we adopt the notation in Definition 3.5 and let $J_1, J_2, \ldots, J_q$ be the construction sequence of G. Define $s' = (s_1, s_2, \ldots, s_{q-1})$ and $G' = G - V(J_q)$. Then $s' \in \mathcal{S}(n - s_q, k)$, and $G' \in L(s')$. By Definition 3.5, $G \in (G', J_q)_k$. By induction, $\kappa(G') = \bar{\kappa}'(G') = k$.

To prove that $\kappa'(G) \geq k$, we assume that G has an edge-cut X with $|X| \leq k - 1$. If $X \cap E(G') \neq \emptyset$, then $X \cap E(G')$ is also an edge-cut of G', and so $k = \kappa(G') \leq |X \cap E(G')| \leq k - 1$, a contradiction. Thus $X \cap E(G') = \emptyset$. If $s_q = 1$, then $J_q = K_1$. Let w denote the only vertex in J_q. It follows that $X \subseteq E(G) - E(G') = E_G(w)$. But $|E_G(w)| = k > k - 1 \geq |X|$, implying that X is not an edge-cut of G. Hence we must have $s_q = k + 2$ and $J_q = H(k, 2)$. As $\kappa'(J_q) = k$, $X \cap E(J_q) = \emptyset$ as well. This implies that $X \subseteq E(G) - (E(G') \cup E(J_q)) = E_G[V(G'), V(J_q)]$. By Definition 3.5 and as $G = (G', J_q)_k$, we have $k = |E_G[V(G'), V(J_q)]| > |X|$, contrary to the fact that X is an edge-cut of G. This proves that $\kappa'(G) \geq k$.

Finally, we argue by contradiction to prove $\bar{\kappa}'(G) = k$, and assume that G has a subgraph H such that $\kappa(H) \geq k + 1$. By induction, $\bar{\kappa}'(G') \leq k$, and so $V(H) - V(G') \neq \emptyset$. If $V(H) \cap V(G') \neq \emptyset$, then $E_G[V(G'), V(J_q)] \cap E(H)$ is an edge-cut of H, and so $k+1 \leq \kappa(H) = |E_G[V(G'), V(J_q)] \cap E(H)| \leq |E_G[V(G'), V(J_q)]| = k$, a contradiction. This forces that $V(H) \cap V(G') = \emptyset$, and so H is a subgraph of J_m. This also leads to a contradiction, since $J_q \in \{K_1, H(k, 2)\}$ contains no nontrivial subgraph with edge-connectivity at least $k + 1$. This completes the proof of the lemma. □

Lemma 3.8 *Let $k \geq 1$ be an integer. If $G \in \mathcal{E}(n, k)$ and X is an edge cut of G with $|X| = k$, then one of the following must hold.*

(i) There exists a vertex $w \in V(G)$ such that $X = E_G(w)$.
(ii) There exists an i with $2 \leq i \leq m$, such that $X = E_G[V(L_{i-1}), G - V(L_{i-1})]$.

Proof We assume that (ii) does not hold to prove (i). We adopt the notation in Definition 3.5 in the proof. Suppose first that $q = 2$. Then $G \in (J_1, J_2)_k$ with $J_1 = H(k, 2)$ and $J_2 \in \{K_1, H(k, 2)\}$. If Lemma 3.8 (ii) does not hold, then $X \neq E_G[V(J_1), V(J_2)]$. Hence for some $i \in \{1, 2\}$, we must have both $J_i = H(k, 2)$ and $X \cap E(J_i) \neq \emptyset$, and so $X \cap E(J_i)$ is an edge-cut of J_i. It follows that $k = \kappa'(J_i) \leq |X \cap E(J_i)| \leq |X| = k$, which forces that $X \subseteq E(J_i)$ is an edge cut of J_i. By Lemma 2.1(ii), there exists a vertex $w \in V(J_i)$ such that $X = E_{J_i}(w)$. Since X is also an edge-cut of G, we have $X = E_G(w)$ and so (i) holds. Hence Lemma 3.8 holds if $q = 2$. Assume that $q \geq 3$ and Lemma 3.8 holds for smaller values of q.

Let $E' = E_G[L_{q-1}, J_q]$. By Lemma 3.7, $\bar{\kappa}'(L_{q-1}) = \kappa'(L_{q-1}) = k$. If $X \subseteq E(L_{q-1})$, then by induction, there exists a vertex $w' \in V(L_{q-1})$ such that $X = E_{L_{q-1}}(w')$. As X is an edge-cut of G, we conclude that $X = E_G(w')$, and so Lemma 3.8 (i) must hold. Hence we may assume that $X - E(L_{q-1}) \neq \emptyset$ and $X \neq E'$.

If $X \cap E' \neq \emptyset$, then since $X \neq E'$, either $X \cap E(L_{q-1})$ is an edge-cut of L_{q-1}, whence $\kappa'(L_{q-1}) \leq |X \cap E(L_{q-1})| < k$, a contradiction; or both $J_q = H(k, 2)$ and $X \cap E(J_q)$ is an edge-cut of J_q, whence $k = \kappa'(H(k, 2)) \leq |X \cap E(H(k, 2))| < k$, another contradiction. Hence we must have $X \cap E' = \emptyset$. It follows that both $J_q = H(k, 2)$ and $X \subseteq E(J_q)$. Again X must be an edge-cut of J_q, and so by Lemma 3.7, there exists a vertex $w'' \in V(J_q)$ such that $X = E_{J_q}(w'')$. As $x_1(J_q)$ and $x_2(J_q)$ are the only two vertices of degree k in J_q, by Definition 3.5 (iii-B), $d_G(x) \geq k+1$ for any $x \in \{x_1(J_q), x_2(J_q)\}$, and so no such vertices would exist. This justifies Lemma 3.8. □

Lemma 3.9 *Let $k \geq 2$ be an integer. Then $\mathcal{E}(k) \subseteq \mathcal{B}(k)$.*

Proof Suppose n and k be positive integers with $n \geq 2k+2$, and let $G \in \mathcal{E}(n,k)$. By Definition 3.5, there exists an $s = (s_1, s_2, \ldots, s_q) \in \mathcal{S}(n,k)$ such that $G \in L(s)$. We shall adopt the notation in Definition 3.5 and use $L_1, L_2, \ldots, L_q$ to denote the construction sequence in the process to build L_q. We argue by induction on n to prove the Lemma. If $n = 2k+2$, then by Definition 3.5, $G = H(k,2) \in \mathcal{B}(k)$. Assume that $n > 2k+2$ and that $\mathcal{E}(n,k) \subseteq \mathcal{B}(k)$ for smaller values of n.

By contradiction, assume that $G \in \mathcal{E}(n,k) - \mathcal{B}(k)$. Choose an $s = (s_1, s_2, \ldots, s_q) \in \mathcal{S}(n,k)$ with q minimized so that $G \in L(s)$, and let $J_1, J_2, \ldots, J_q$ to denote the corresponding graphs as in Definition 3.5. Since $n > 2k+2$, we have $q \geq 2$, and so $G \in (L_{q-1}, J_q)_k$. In the rest of the proof, we let G_1 and G_2 be the two components of $G - X$.

We will prove the Lemma by finding a contradiction. Since $G \in \mathcal{E}(n,k) - \mathcal{B}(k)$, there exists an edge $e = uv \in E(G^{bc})$, we have

$$\bar{\kappa}'(G+e) \leq k. \tag{8}$$

If $u, v \in V(L_{q-1})$, then by induction, $L_{q-1} \in \mathcal{B}(k)$ and so $\bar{\kappa}'(G+e) \geq \bar{\kappa}'(L_{q-1}+e) \geq k+1$, contrary to (8). Similarly, when $s_q = 2k+2$ and so $J_q = H(k,2)$, we cannot have $u, v \in V(J_q)$. Hence we may assume that

$$u \in V(L_{q-1}) \text{ and } v \in V(J_q). \tag{9}$$

By (8), there exists an edge-cut X of $G+e$ with $|X| = k$. If $e \in X$, then $X - e$ is an edge-cut of G with $\kappa'(G) \leq |X - e| = k-1$, contrary to Lemma 3.7. Hence $e \notin X'$, and so X is an edge-cut of G. By Lemma 3.8, either Lemma 3.8 (i) or Lemma 3.8 (ii) must hold.

Assume Lemma 3.8 (i) holds. Then for some vertex $w \in V(G)$, we have $X = E_G(w)$. As $s = (s_1, s_2, \ldots, s_q) \in \mathcal{S}(n,k)$ and $G \in L(s)$, either $w \in \{x_1(J_1), x_2(J_1)\}$, or for some $\ell > 1$, $s_\ell = 1$ and $V(J_\ell) = \{w\}$. If $w \in \{x_1(J_1), x_2(J_1)\}$, then by Observation 3.6(ii) and Lemma 3.2, $G = H(k, n-k) \in \mathcal{B}(k)$, contrary to the assumption that $G \in \mathcal{E}(n,k) - \mathcal{B}(k)$. Therefore, we there must be an $\ell > 1$ such that $s_\ell = 1$ and $V(J_\ell) = \{w\}$. Let $s' = (s_1, \ldots, s_{\ell-1}, s_{\ell+1}, \ldots, s_q)$. Since $s \in \mathcal{S}(n,k)$, we conclude that $s' \in \mathcal{S}(n-1,k)$. Since $d_G(w) = k$, it follows by Definition 3.5 that $G - w \in L(s')$. By induction, $G - w \in \mathcal{B}(k)$, and so $\bar{\kappa}'(G+e) \geq \bar{\kappa}'((G-w)+e) \geq k+1$, contrary to (8).

Therefore, Lemma 3.8 (ii) must hold, and so for some i with $2 \leq i \leq q$,

$$\begin{aligned} X &= E_G[V(L_{i-1}), V(G) - V(L_{i-1})] \\ &= E_{G+e}[V(L_{i-1}), V(G+e) - V(L_{i-1})]. \end{aligned} \tag{10}$$

Let $G' = G - V(L_{i-1})$. By Observation 3.6 (iii), $G' \in \mathcal{S}(n - |V(L_{i-1})|, k)$. By (8), $e \in E((G')^{bc})$. By induction, $G' \in \mathcal{B}(k)$, and so we have $\bar{\kappa}'(G + e) \geq \bar{\kappa}'(G' + e) \geq k + 1$, contrary to (8). This completes the proof of the lemma. □

3.3 Justification of Theorem 1.2

We are now ready to prove Theorem 1.2. By (1), it suffices to study the bi-k-maximal graphs for $k \geq 2$. For integers k and n with $k \geq 2$ and $n \geq 2k + 2$, define

$$F_B(n, k) = \max\{|E(G)| : G \in \mathcal{B}(n, k)\} \text{ and } f_B(n, k) = \min\{|E(G)| : G \in \mathcal{B}(n, k)\};$$

In order to recursively characterize the extremal graphs in $\{G : G \in \mathcal{B}(n, k) \text{ and } |E(G)| = f_B(n, k)\}$, we recall that by Corollary 3.4, $r_k(G)$ is uniquely determined by k and G, and make the following definition.

Definition 3.10 Let k and n be integers with $k \geq 2$ and $n \geq 2k + 2$. Let $\mathcal{B}_L(k)$ be the family of graphs satisfying the following.

(BL1) $H(k, 2) \in \mathcal{B}_L(k)$.
(BL2) For any $H_1, H_2 \in \mathcal{B}_L(k) \cup \{K_1\}$, a graph $G \in (H_1, H_2)_k$ is in $\mathcal{B}_L(k)$ if and only if both $r_k(G) = \lfloor \frac{|V(G)|}{2k+2} \rfloor$ and $\lfloor \frac{|V(H_1)|}{2k+2} \rfloor + \lfloor \frac{|V(H_2)|}{2k+2} \rfloor = \lfloor \frac{|V(G)|}{2k+2} \rfloor$.

Let $\mathcal{B}_L(n, k) = \{G \in \mathcal{B}_L(k) : |V(G)| = n\}$. We will prove a slightly extended version of Theorem 1.2 as follows.

Theorem 3.11 *Let k and n be integers with $k \geq 2$ and $n \geq 2k+2$. Define $r_0 = \lfloor \frac{n}{2k+2} \rfloor$. Each of the following holds.*

(i) Let m be an integer. There exists a graph $G \in \mathcal{B}(n, k)$ with $m = |E(G)|$ if and only if $m = nk - rk^2 + (r - 1)k$ for some integer r with $1 \leq r \leq \lfloor \frac{n}{2k+2} \rfloor$,
(ii) $F_B(n, k) = (n - k)k$. Moreover, a graph $G \in \mathcal{B}(n, k)$ with $|E(G)| = F_B(n, k)$ if and only if $G \in \{K_{k,n-k}\} \cup \mathcal{H}(k, n - k)$,
(iii) $f_B(n, k) = k(n - 1) - (k^2 - k)\lfloor \frac{n}{2k+2} \rfloor$. Moreover, a graph $G \in \mathcal{B}(n, k)$ with $|E(G)| = f_B(n, k)$ if and only if $G \in \mathcal{B}_L(n, k)$.

Proof For given values k and n, define

$$\mathcal{M}(n, k) = \{nk - rk^2 + (r - 1)k : \text{ where } r = 1, 2, \ldots, r_0\}.$$

We first show that for any integer r with $1 \leq r \leq r_0$, there exists a graph $G \in \mathcal{B}(n, k)$ such that $|E(G)| = nk - rk^2 + (r - 1)k$. Since $K_{k,n-k} \in \mathcal{B}(n, k)$ and $|E(K_{k,n-k})| = k(n - k) \in \mathcal{M}(n, k)$ with $r = 1$. Assume that $2 \leq r \leq r_0$. Let $\ell = n - (2k + 2)r$, $s_1 = s_2 = \ldots s_r = k + 2$, $s_{r+1} = \ldots = s_\ell = 1$, and $s = (s_1, s_2, \ldots, s_\ell)$. For any $G \in L(s)$, by Definition 3.5, $|V(G)| = (2k + 2)r + \ell = n$, and $|E(G)| = \sum_{i=1}^{r} |E(H(k, 2))| + (\ell + r - 1)k = r(k^2 + 2k) + (n - (2k + 2)r + r - 1)k = nk - rk^2 + (r - 1)k$. As $L(s) \subseteq \mathcal{E}(n, k)$, by Lemma 3.9, $G \in \mathcal{B}(n, k)$, and so any value in $\mathcal{M}(n, k)$ is the size of a bi-k-maximal graph of size n.

Conversely, let $G \in \mathcal{B}(n,k)$. We shall show that $|E(G)| \in \mathcal{M}(n,k)$. As $K_{k,n-k} \in \mathcal{B}(n,k)$ and $|E(K_{k,n-k})| \in \mathcal{M}(n,k)$, we assume that $G \in \mathcal{B}(n,k) - \{K_{k,n-k} : n \geq 2k\}$. By Corollary 3.4(ii), $|E(G)| = nk - rk^2 + (r-1)k \in \mathcal{M}(n,k)$. This proves (i).

To prove (ii) and (iii), we first observe that by Theorem 3.11 (i), we have $F_B(n,k) = (n-k)k$ and $f_B(n,k) = k(n-1) - (k^2-k)\lfloor\frac{n}{2k+2}\rfloor$. As $\mathcal{B}(2k+2,k) = \{H(k,2), K_{k,k+2}\}$. Theorem 3.11 (ii) and (iii) hold when $n = 2k+2$. We assume that $n > 2k+2$ and Theorem 3.11 holds for smaller values of n. It is routine by direct computation to conclude that every graph $G \in \mathcal{H}(k,n-k) \cup \{K_{k,n-k}\}$ satisfies $|E(G)| = F_B(n,k)$ and every graph $G \in \cup_{s \in \mathcal{S}'(n,k)} L(s)$ satisfies $|E(G)| = f_B(n,k)$.

To complete the proof for (ii), let $G \in \mathcal{B}(n,k) - \{K_{n,n-k}\}$ be a graph with $|E(G)| = k(n-k)$. Since $|E(G)| = rk^2 + (n-2kr)k + (r-1)k = nk - rk^2 + (r-1)k = nk - k^2$, then $r = 1$, which implies that $G \in \{K_{k,n-k}\} \cup \mathcal{H}(k,n-k)$, and so (ii) must hold.

To complete the proof for (iii), as graphs in $\{K_{k,n-k}\}$ are bi-k-maximal graphs, (iii) holds if $2k \leq n \leq 2k+2$. Assume that (iii) holds for smaller values of n, and $n > 2k+2$. Let $G \in \mathcal{B}(n,k) - \{K_{k,n-k}\}$ be a graph with $|E(G)| = k(n-1) - (k^2-k)\lfloor\frac{n}{2k+2}\rfloor$. Since $|E(G)| = nk - rk^2 + (r-1)k$, it follows that we must have $r = r_0$. By Corollary 2.5, either (i) or (ii) of Corollary 2.5 must hold.

Case 1 *Corollary 2.5 (i) holds and for some $H \in \mathcal{B}(k)$, $G \in (H, K_1)_k$.*

Then $nk - r_0k^2 + (r_0-1)k = |E(G)| = |E(H)| + k$, and so $|E(H)| = (n-1)k - r_0k^2 + (r_0-1)k$. Since $r_0 = \lfloor\frac{n}{2k+2}\rfloor$, we can express that $n = r_0(2k+2) + h$, where $0 \leq h \leq 2k+1$.

Assume $h = 0$ and let $r_0' = \lfloor\frac{n-1}{2k+2}\rfloor$. As $h = 0$, we have $r_0' = r_0 - 1$ and $n - 1 = r_0'(2k+2) + 2k + 1$. Since $H \in \mathcal{B}(k)$, by Theorem 3.11 (i), for some $1 \leq r \leq r_0'$, $|E(H)| = (n-1)k - rk^2 + (r-1)k$. As $G \in (H, K_1)_k$ and $|E(G)| = k(n-1) - (k^2-k)r_0$, we have

$$\begin{aligned} k(n-1) - (k^2-k)r_0 &= |E(G)| = |E(H)| \\ &+ k = (n-1)k - rk^2 + (r-1)k + k = (n-1)k - r(k^2-k), \end{aligned}$$

which forces $r_0 = r \leq r_0' = r_0 - 1$, a contradiction. This implies that it is impossible to have $h = 0$ in this case.

Hence we must have $h > 0$. Thus $(n-1) = r_0(2k+2) + (h-1)$, and so $r_0 = \lfloor\frac{n-1}{2k+2}\rfloor$. As $|V(H)| = n-1$, it follows by $|E(H)| = (n-1)k - r_0k^2 + (r_0-1)k$ and by induction that for some $s_1 \in \mathcal{S}'(n-1,k)$, we have $H \in L(s_1)$. As $G \in\in (H, K_1)_k$, by Definition 3.5, we have $G \in L(s)$, for some $s \in\in \mathcal{S}'(n,k)$. This justifies (iii) in Case 1.

Case 2 *Corollary 2.5 (ii) holds and for some non-complete bipartite graphs $H_1, H_2 \in \mathcal{B}(k)$, $G \in (H_1, H_2)_k$.*

Let $n_1 = |V(H_1)|$, $n_2 = |V(H_2)|$, $r_1 = \lfloor\frac{n_1}{2k+2}\rfloor$ and $r_2 = \lfloor\frac{n_2}{2k+2}\rfloor$. By the conclusion of Theorem 3.11 (i) and by the assumption of of Theorem 3.11 (iii), it follows by

$n_1 + n_2 = n$ and $r_1 + r_2 = \lfloor \frac{n_1}{2k+2} \rfloor + \lfloor \frac{n_2}{2k+2} \rfloor \le r_0$ that

$$k(n-1) - (k^2 - k)r_0 = |E(G)| = k + \sum_{i=1}^{2} |E(H_i)|$$

$$= k + \sum_{i=1}^{2} [n_i k - r_i k^2 + (r_i - 1)k] \ge nk - r_0(k^2 - k) - k.$$

This forces that $r_1 + r_2 = r_0$ and for $i \in \{1, 2\}$, $|E(H_i)| = n_i k - r_i k^2 + (r_i - 1)k = f_B(n_i, k)$. By induction, we have $H_i \in \mathcal{B}_L(k, n_i) \subseteq \mathcal{B}_L(k)$. Since $G \in (H_1, H_2)_k$ and since $r_1 + r_2 = r_0$, it follows by Definition 3.10 that $G \in \mathcal{B}_L(k, n)$. This completes the proof of Case 2, as well as the theorem. □

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