Linear Algebra and its Applications 585 (2020) 199–208

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Generalized cospectral graphs with and without Hamiltonian cycles ☆

Fenjin Liu [a,b,*], Wei Wang [b], Tao Yu [b], Hong-Jian Lai [c]

[a] *School of Science, Chang'an University, Xi'an 710046, PR China*
[b] *School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, PR China*
[c] *Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310, USA*

A R T I C L E   I N F O

Article history:
Received 8 February 2017
Accepted 1 October 2019
Available online 9 October 2019
Submitted by R. Brualdi

MSC:
05C45
05C50
15A18

Keywords:
Spectrum
Generalized cospectral
k-(generalized) cospectrally-rooted
Hamiltonian cycle

A B S T R A C T

The *spectrum* $\sigma(G)$ of a graph G consists of all the eigenvalues (together with their multiplicities) of its adjacency matrix $A(G)$. Two graphs G and H are called *generalized cospectral* if both $\sigma(G) = \sigma(H)$ and $\sigma(\overline{G}) = \sigma(\overline{H})$, where $\overline{G}$ ($\overline{H}$) is the complement of G (H). In this paper, we generalize the notion "*cospectrally-rooted*" to "*k-cospectrally-rooted*", and obtain two equivalent statements for k-(generalized) cospectrally-rooted graphs. Furthermore, we have constructed two families of generalized cospectral graphs such that graphs in one of these two families are Hamiltonian and graphs in the other family are not Hamiltonian.

☆ This work is supported by the National Natural Science Foundation of China (Nos. 11401044, 11971376, 11501050), Postdoctoral Science Foundation of China (No. 2014M560754), Postdoctoral Science Foundation of Shaanxi, the Fundamental Research Funds for the Central Universities (No. 300102128201) and Foundation of China Scholarship Council (No. 201706565015).

* Corresponding author at: School of Science, Chang'an University, Xi'an 710046, PR China.
E-mail addresses: fenjinliu@yahoo.com (F. Liu), wang_weiw@163.com (W. Wang), newcworld001@gmail.com (T. Yu), hjlai@math.wvu.edu (H.-J. Lai).

https://doi.org/10.1016/j.laa.2019.10.001

1. Introduction

Throughout this paper, we consider undirected simple graph. Let G be a graph with vertex set $\{v_1, v_2, \dots, v_n\}$, its adjacency matrix $A(G)$ is defined to be the square matrix $(a_{ij})_{n\times n}$, in which $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. The *characteristic polynomial* of G is defined to be the characteristic polynomial of $A(G)$ and is denoted by $\phi(G; x)$, that is $\phi(G; x) = \det(xI - A(G)) = \sum_{i=0}^{n} c_i x^{n-i}$. $\phi(G; x)$ is usually abbreviated to $\phi(G)$ if there is no confusion. Since $A(G)$ is a real symmetric matrix, its eigenvalues (the roots of $\phi(G)$) are all real numbers, and can be ordered by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. These eigenvalues are called the *eigenvalues* of G, the *spectrum* $\sigma(G)$ of G is defined to be the multiset of eigenvalues of G. Two graphs G and H are called *cospectral* if they share the common spectrum, i.e., $\sigma(G) = \sigma(H)$ and thus $\phi(G) = \phi(H)$. Moreover, G and H are called *generalized cospectral* if they are cospectral with cospectral complement, i.e., $\phi(G) = \phi(H)$ and $\phi(\overline{G}) = \phi(\overline{H})$, where $\overline{G}$ ($\overline{H}$) denotes the complement of G (H). In a graph G, *contraction* of edge e with endpoints v_i, v_j is the replacement of v_i and v_j with a single vertex whose incident edges are the edges other than e that were incident to v_i or v_j. The resulting graph G/e or $G/\{v_iv_j\}$ has one less edge than G. A spanning cycle in a graph is referred as a *Hamiltonian cycle*. Let $V = \{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$ be subset of the vertex set $V(G)$, denote by $G - V$ the subgraph induced by the vertices in $V(G) \setminus V$, in particular, $G - V$ is short for $G - v_{i_1}$ when $k = 1$. For a matrix $M_{m\times n}$ and index sets $U \subseteq \{1, \dots, m\}$ and $V \subseteq \{1, \dots, n\}$, we denote by the (sub)matrix that lies in the rows of M indexed by U and columns indexed by V as $M[U, V]$. If $m = n$ and $U = V$, the principal submatrix $M[U, U]$ is abbreviated to $M[U]$. For any undefined notion or terms, we refer the readers to [4,7,8,12].

It is well-known that the graph spectrum encodes much useful information of a graph such as the order, size, number of triangles, regularity, bipartiteness [5,6], etc. Nevertheless, the perfect matching property is not determined by the spectrum, and recently, Z.L. Blázsik et al. [1] constructed two families of cospectral b-regular ($b \geq 5$) graphs such that each graph in one of these two families has a perfect matching and graphs in the other family do not have any perfect matchings. Since Hamiltonicity [3,12] is another important property of a graph and testing whether a graph is Hamiltonian is an NP-complete problem, it is naturally asked whether a graph's Hamiltonicity can be deduced from its spectrum.

A.J. Schwenk [10,11] firstly introduced the concept of cospectrally-rooted graph, which enabled him to construct infinite pairs of cospectral trees so as to prove the famous result that almost all trees are cospectral. However, cospectral graphs constructed by the "cospectrally-rooted" approach cannot be applied to build cospectral graphs with Hamiltonian cycles since they always have a cut vertex. In order to overcome this shortcoming, we generalize the definition of cospectrally-rooted to k-cospectrally-rooted which allows us to show two families of generalized cospectral graphs where one family is Hamiltonian and the other one is not.

The rest of this paper is organized as follows: in the next section, besides the notion of cospectrally-rooted will be generalized to k-cospectrally-rooted, the two equivalent conditions will be also obtained. Take this as the foundation, a number of (generalized) cospectral graphs will be derived. In Section 3, a pair of 2-cospectrally-rooted graphs will be presented and the equivalent conditions involved in section 2 are going to be verified in details. Finally, in Section 4 generalized cospectral graphs with and without Hamiltonian cycles will be supplied.

2. *k*-cospectrally-rooted

A *k-rooted-graph* is a graph in which k vertices are labeled in a special way so as to distinguish them from other vertices. These special k vertices are called the *root vertices* of the graph. Meanwhile, the labels of these k root vertices correspond k standard orthonormal vectors $\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}$ of $\mathbb{R}^n$ which are called the *roots* of the k-rooted-graph. Without loss of generality, we always assume that the k root vertices are labeled first and the corresponding roots are $\mathbf{e}_1, \ldots, \mathbf{e}_k$. Let G and H be two k-rooted-graphs with root vertices $u_1, \ldots, u_k$; $v_1, \ldots, v_k$, respectively. (Note that G and H may be different graphs, or the same graph with two sets of different root vertices.) G and H are called *k-cospectrally-rooted* if the first k entries of the orthnormal eigenvectors of $A(G)$ respectively coincide with that of $A(H)$ for each eigenvalue λ_i $(1 \le i \le n)$. Moreover, G and H are called *k-generalized-cospectrally-rooted* if both the first k entries of the orthnormal eigenvectors of $A(G)$ respectively coincide with that of $A(H)$ for each eigenvalue λ_i $(1 \le i \le n)$ and the same results follow for $\overline{G}$ and $\overline{H}$. Early in 1978 Schwenk and Wilson [11] gave a necessary and sufficient condition for cospectrally-rooted graph.

Lemma 2.1. *[11] Let G and H be a pair of rooted graphs on n vertices whose root vertices are u_1, v_1, respectively. Suppose that $\phi(G) = \phi(H)$ with $A(G)\mathbf{x}_i = \lambda_i \mathbf{x}_i$ and $A(H)\mathbf{y}_i = \lambda_i \mathbf{y}_i$ $(i = 1, \ldots, n)$ where $\mathbf{x}_1, \ldots, \mathbf{x}_n$; $\mathbf{y}_1, \ldots, \mathbf{y}_n$ are respectively the orthonormal eigenvectors of G and H. (If these graphs have multiple eigenvalues, then the eigenvectors for the repeated eigenvalues must be selected carefully.) Then G and H are cospectrally-rooted if and only if $(\mathbf{e}_1^T \mathbf{x}_i)^2 = (\mathbf{e}_1^T \mathbf{y}_i)^2$, for each $i = 1, \ldots, n$.*

Because for a family of orthogonal vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$, substituting $\mathbf{x}_i$ into $-\mathbf{x}_i$ for any $1 \le i \le n$ does not change the orthogonal property. After properly adjusting the sign of the orthonormal eigenvectors of graph G or H, Lemma 2.1 can be strengthened by G and H are cospectrally-rooted if and only if $\mathbf{e}_1^T \mathbf{x}_i = \mathbf{e}_1^T \mathbf{y}_i$, for each $i = 1, \ldots, n$.

In fact, we can generalize Lemma 2.1 to k-cospectrally-rooted graphs. The next theorem gives another two equivalent conditions.

Theorem 2.2. *Let G and H be two k-cospectrally-rooted graphs on n vertices with root vertices $u_1, \ldots, u_k$; $v_1, \ldots, v_k$, respectively. Suppose that $A(G)$ has orthonormal eigenvec-*

tors $\mathbf{x}_1, \ldots, \mathbf{x}_n$*;* $A(H)$ *has orthonormal eigenvectors* $\mathbf{y}_1, \ldots, \mathbf{y}_n$ *such that* $A(G)\mathbf{x}_i = \lambda_i \mathbf{x}_i$ *and* $A(H)\mathbf{y}_i = \lambda_i \mathbf{y}_i$ *for* $i = 1, \ldots, n$*. (If these graphs have multiple eigenvalues, then the eigenvectors for the repeated eigenvalues must be selected carefully.) Then the following statements (ii) and (iii) are equivalent to each other and that they imply statements (i) and (iv):*

(i) $\phi(G) = \phi(H)$ *and* $\phi(G - u_i) = \phi(H - v_i)$ *for each* $i = 1, \ldots, k$*.*
(ii) $\mathbf{e}_i^T \mathbf{x}_j = \mathbf{e}_i^T \mathbf{y}_j$*, for each* $i = 1, \ldots, k$*;* $j = 1, \ldots, n$*.*
(iii) *There exists an* $n \times n$ *orthogonal matrix* Q *of the form*

$$Q = \begin{pmatrix} I_k & O^T \\ O & Q_1 \end{pmatrix}, \tag{1}$$

such that $Q^T A(G) Q = A(H)$ *where* I_k *is the identity matrix of order* k*,* O *is the zero matrix of order* $(n-k) \times k$*.*
(iv) $\phi(G) = \phi(H)$ *and* $\phi(G - U) = \phi(H - V)$ *where* $U = \{u_{i_1}, \ldots, u_{i_r}\}$*,* $V = \{v_{i_1}, \ldots, v_{i_r}\}$*,* $1 \le r \le k$ *and* $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, k\}$*.*

Proof. (ii)⇒(iii) Suppose that

$$X = (\mathbf{x}_1 \quad \mathbf{x}_2 \quad \cdots \quad \mathbf{x}_n), \quad Y = (\mathbf{y}_1 \quad \mathbf{y}_2 \quad \cdots \quad \mathbf{y}_n).$$

It follows that $X^T A(G) X = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) = Y^T A(H) Y$. Denote by $Q := XY^T$, since Y consists of orthonormal eigenvectors, we have $YY^T = I$, by this fact we obtain

$$\begin{aligned} Q^T A(G) Q &= (XY^T)^T A(G)(XY^T) \\ &= Y(X^T A(G) X)Y^T \\ &= Y \mathrm{diag}(\lambda_1, \ldots, \lambda_n) Y^T \\ &= Y(Y^T A(H) Y)Y^T \\ &= A(H). \end{aligned}$$

Since $(\mathbf{e}_i^T \mathbf{x}_j) = (\mathbf{e}_i^T \mathbf{y}_j)$, for each $i = 1, \ldots, k$ and for each $j = 1, \ldots, n$ we have $\mathbf{e}_i^T X = \mathbf{e}_i^T Y$ and $\mathbf{e}_i^T Q = \mathbf{e}_i^T XY^T = \mathbf{e}_i^T YY^T = \mathbf{e}_i^T$ for each $i = 1, \ldots, k$. We conclude that Q is of the form shown in Eq. (1).

(iii)⇒(ii) Since $Q^T = \begin{pmatrix} I_k & O^T \\ O & Q_1^T \end{pmatrix}$, we see that $e_i^T Q^T = e_i^T$ for each $i = 1, \ldots, k$. Suppose that $A(G)$ has orthonormal eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ and $X = (\mathbf{x}_1 \quad \mathbf{x}_2 \quad \cdots \quad \mathbf{x}_n)$, then

$$\begin{aligned}\mathrm{diag}(\lambda_1,\dots,\lambda_n) &= X^T A(G) X \\ &= X^T (Q A(H) Q^T) X \\ &= (X^T Q) A(H) (Q^T X) \\ &:= Y^T A(H) Y.\end{aligned}$$

It follows that $e_i^T Y = e_i^T Q^T X = e_i^T X$ for for each $i = 1, \dots, k$.

(iii)⇒(iv) Let the root vertex set of G and H be $S = \{u_1, \dots, u_k\}$, $T = \{v_1, \dots, v_k\}$, respectively. Denote by $A_1 = A(G)[S]$, $A_2 = A(G)[G - S]$, $B^T = A(G)[S, G - S]$; $\widetilde{A}_1 = A(H)[T]$, $\widetilde{A}_2 = A(H)[H - T]$, $\widetilde{B}^T = A(H)[T, H - T]$, then

$$A(G) = \begin{pmatrix} A_1 & B^T \\ B & A_2 \end{pmatrix}, \quad A(H) = \begin{pmatrix} \widetilde{A}_1 & \widetilde{B}^T \\ \widetilde{B} & \widetilde{A}_2 \end{pmatrix}.$$

Since there exists an orthogonal matrix $Q = \begin{pmatrix} I_k & O^T \\ O & Q_1 \end{pmatrix}$ such that $Q^T A(G) Q = A(H)$, we have $\widetilde{A}_1 = A_1$, $\widetilde{B}^T = B^T Q_1$ and $Q_1^T A_2 Q_1 = \widetilde{A}_2$. Furthermore, it is obvious that $A_1[S - U] = \widetilde{A}_1[T - V]$, and $B^T[S - U, G - S]Q_1 = (B^T Q_1)[S - U, G - S] = \widetilde{B}^T[T - V, H - V]$, set $\widetilde{Q} = \begin{pmatrix} I_{k-r} & O^T \\ O & Q_1 \end{pmatrix}$, it follows that $\widetilde{Q}^T A(G - U)\widetilde{Q} = A(H - V)$. Therefore $\phi(G - U) = \phi(H - V)$.

(iv)⇒(i) it is clearly established. □

(i)⇒(ii) are not generally true since such counterexample exits. However, we do not know whether (iv) implies (ii).

The *k-coalescence* $G \circledast \Gamma$ of two graphs G and Γ is obtained from G and Γ by identifying k vertices u_i $(i = 1, \dots, k)$ of G to k vertices w_i $(i = 1, \dots, k)$ of Γ, respectively. It immediately yields a corollary below.

Corollary 2.3. *Let G and H be two k-cospectrally-rooted graphs with root vertices $u_1, \dots, u_k$; $v_1, \dots, v_k$, respectively. Suppose Γ is any a graph with arbitrary k vertices $w_1, \dots, w_k$. Then the k-coalescence $G \circledast \Gamma$ about vertices $u_1, \dots, u_k$ and $w_1, \dots, w_k$ is cospectral with the k-coalescence $H \circledast \Gamma$ about vertices $v_1, \dots, v_k$ and $w_1, \dots, w_k$.*

Proof. Since G and H are k-cospectrally-rooted, Theorem 2.2 (iii) implies that there exists an orthogonal matrix $Q = \begin{pmatrix} I_k & O^T \\ O & Q_1 \end{pmatrix}$ such that $Q^T A(G) Q = A(H)$. Set $Q^* = \begin{pmatrix} I_{|V(\Gamma)|} & O^T \\ O & Q_1 \end{pmatrix}$, it is a routine to verify that $(Q^*)^T A(G \circledast \Gamma) Q^* = A(H \circledast \Gamma)$. Thus $G \circledast \Gamma$ and $H \circledast \Gamma$ have the same spectrum. □

An orthogonal matrix Q is *regular* if it has all row and column sums 1. The following result, due to Johnson and Newman [9], gives another equivalent condition for generalized cospectral graphs.

Theorem 2.4. *[2,9] Let G and H be two graphs with adjacency matrices $A(G)$ and $A(H)$, respectively, then the following are equivalent:*

(i) *G and H are cospectral with cospectral complements.*
(ii) *There exists a regular orthogonal matrix Q, such that $A(G) = Q^T A(H)Q$.*

If we consider the generalized spectrum, then Theorem 2.2 and Corollary 2.3 have following enhanced version.

Theorem 2.5. *Let G and H be two k-generalized-cospectrally-rooted graphs with root vertices $u_1, \dots, u_k$; $v_1, \dots, v_k$, respectively. Suppose that $A(G)$ has orthonormal eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_n$; $A(H)$ has orthonormal eigenvectors $\mathbf{y}_1, \dots, \mathbf{y}_n$ such that $A(G)\mathbf{x}_i = \lambda_i \mathbf{x}_i$ and $A(H)\mathbf{y}_i = \lambda_i \mathbf{y}_i$ for $i = 1, \dots, n$. Further suppose that $A(\overline{G})$ has orthonormal eigenvectors $\overline{\mathbf{x}}_1, \dots, \overline{\mathbf{x}}_n$; $A(\overline{H})$ has orthonormal eigenvectors $\overline{\mathbf{y}}_1, \dots, \overline{\mathbf{y}}_n$ such that $A(\overline{G})\overline{\mathbf{x}}_i = \overline{\lambda_i}\overline{\mathbf{x}}_i$ and $A(\overline{H})\overline{\mathbf{y}}_i = \overline{\lambda_i}\overline{\mathbf{y}}_i$ for $i = 1, \dots, n$. (If these graphs have multiple eigenvalues, then the eigenvectors for the repeated eigenvalues must be selected carefully.) Then the following statements (ii) and (iii) are equivalent to each other and that they imply statements (i) and (iv):*

(i) $\begin{cases} \phi(G) = \phi(H) \\ \phi(G - u_i) = \phi(H - v_i) & \textit{for each } i = 1, \dots, k, \end{cases}$ *and*
$\begin{cases} \phi(\overline{G}) = \phi(\overline{H}) \\ \phi(\overline{G - u_i}) = \phi(\overline{H - v_i}) & \textit{for each } i = 1, \dots, k. \end{cases}$

(ii) $\mathbf{e}_i^T \mathbf{x}_j = \mathbf{e}_i^T \mathbf{y}_j$ *and* $\mathbf{e}_i^T \overline{\mathbf{x}}_j = \mathbf{e}_i^T \overline{\mathbf{y}}_j$ *for each $i = 1, \dots, k$; $j = 1, \dots, n$.*

(iii) *There exists an $n \times n$ regular orthogonal matrix Q of the form*

$$Q = \begin{pmatrix} I_k & O^T \\ O & Q_1 \end{pmatrix},$$

such that $Q^T A(G)Q = A(H)$ where I_k is the identity matrix of order k, O is the zero matrix of order $(n-k) \times k$.

(iv) $\begin{cases} \phi(G) = \phi(H) \\ \phi(G - U) = \phi(H - V) \end{cases}$ *and* $\begin{cases} \phi(\overline{G}) = \phi(\overline{H}) \\ \phi(\overline{G - U}) = \phi(\overline{H - V}) \end{cases}$
where $U = \{u_{i_1}, \dots, u_{i_r}\}$, $V = \{v_{i_1}, \dots, v_{i_r}\}$ and $\{i_1, \dots, i_r\} \subseteq \{1, \dots, k\}$.

Similarly, by Theorem 2.5, we can construct many generalized cospectral graphs as follows.

Corollary 2.6. *Let G and H be two k-generalized-cospectrally-rooted graphs with root vertices $u_1, \dots, u_k$; $v_1, \dots, v_k$, respectively. Suppose Γ is any a graph with arbitrary k vertices $w_1, \dots, w_k$. Then the k-coalescence $G \circledast \Gamma$ about vertices $u_1, \dots, u_k$ and $w_1, \dots, w_k$*

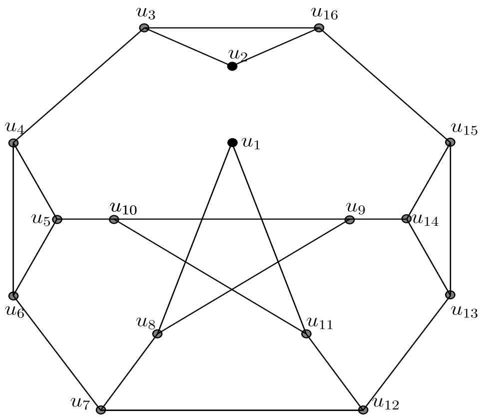

Fig. 1. G.

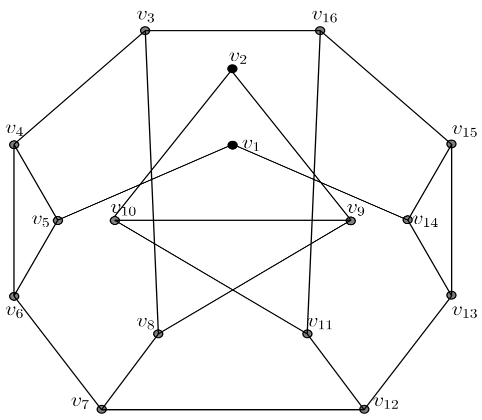

Fig. 2. H.

is generalized cospectral with the k-coalescence $H \circledast \Gamma$ *about vertices* $v_1, \ldots, v_k$ *and* $w_1, \ldots, w_k$.

3. Examples

We give three examples in this section, where the first is a pair of 2-generalized-cospectrally-rooted graphs of order 16; the second is an illustration of 2-coalescence of two graphs; and the third schematically shows the contraction graph obtained from Example 2 by contracting a path P_3 and three triangles. These graph operations will play a significant role in section 4.

Example 1. Let G and H be two graphs shown in Fig. 1 and Fig. 2. Then G and H are 2-generalized-cospectrally-rooted graphs with rooted vertices u_1, u_2; v_1, v_2, respectively.

Example 2. Let Γ (see Fig. 3) be the unicyclic graph obtained from the triangle C_3 by appending a pendent vertex. Let G and H be two graphs shown in Fig. 1 and Fig. 2. Then according to Corollary 2.6, the 2-coalescence $G \circledast \Gamma$ about vertices u_1, u_2 and w_1, w_4 (see Fig. 3) is generalized cospectral with the 2-coalescence $H \circledast \Gamma$ about vertices v_1, v_2 and w_1, w_4.

Example 3. Let $G \circledast \Gamma$ be the graph shown in Fig. 4. Then $(G \circledast \Gamma)/\{w_2w_3, w_3w_4\}$, $(G \circledast \Gamma)/(\{w_2w_3, w_3w_4\} \cup \{u_2u_3, u_2u_{16}, u_3u_{16}, u_4u_5, u_4u_6, u_5u_6, u_{13}u_{14}, u_{13}u_{15}, u_{14}u_{15}\})$ are represented in Fig. 5 and Fig. 6, respectively.

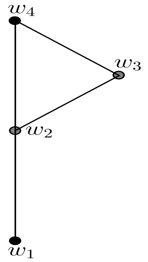

Fig. 3. Γ.

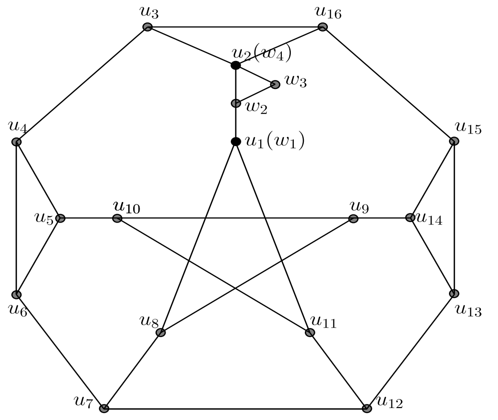

Fig. 4. $G \circledast \Gamma$.

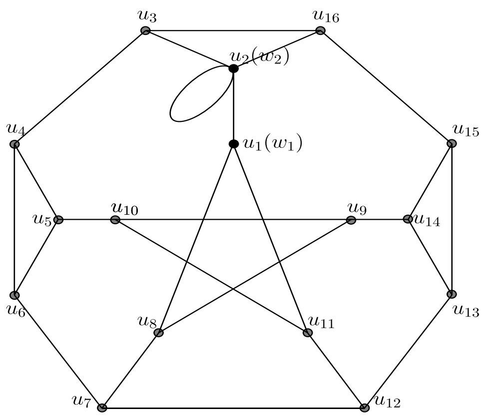

Fig. 5. $(G \circledast \Gamma)/P_3$.

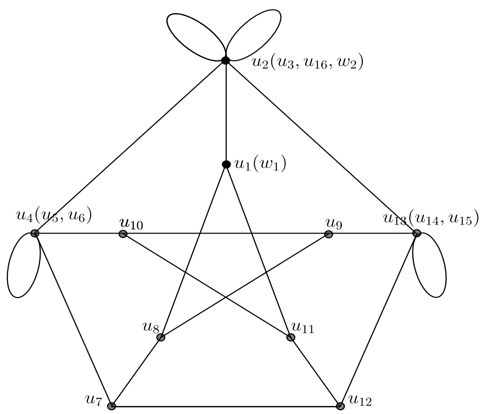

Fig. 6. $(G \circledast \Gamma)/(P_3 \cup 3C_3)$.

4. Hamiltonicity

In the following context, using the 2-generalized-cospectrally-rooted graphs shown in Example 1, we construct two families of generalized cospectral graphs in which one family have Hamiltonian cycles and the other one haven't.

Theorem 4.1. *Let G and H be the graphs depicted in Fig. 1 and Fig. 2, respectively. Let Γ be a graph with a Hamiltonian path $P_\Gamma = \{w_1w_2, w_2w_3, \cdots, w_{m-1}w_m\}$ where w_1 and w_m are the two end vertices of P_Γ. Denote by Γ_1 the 2-coalescence $G \circledast \Gamma$ about vertices u_1, u_2 and w_1, w_m (see for example Fig. 4); Γ_2 the 2-coalescence $H \circledast \Gamma$ about vertices v_1, v_2 and w_1, w_m, respectively. Then*

(i) Γ_1 *is generalized cospectral with* Γ_2.
(ii) Γ_1 *is non-Hamiltonian.*
(iii) Γ_2 *is Hamiltonian.*

Proof. (i) is a direct consequence of Example 1, Theorem 2.5 and Corollary 2.6.

We show statement (ii) by contradiction, let us assume that Γ_1 has a Hamiltonian cycle C. Since Γ_1 is the 2-coalescence $G \circledast \Gamma$ about vertices u_1, u_2 and w_1, w_m, C must traverse Γ from one of w_1, w_m along a Hamiltonian path to another. Denote by $\Gamma_1' = \Gamma/\{w_2w_3, \cdots, w_{m-1}w_m\}$, then Γ_1' also has a Hamiltonian cycle $C' = C/\{w_2w_3, \cdots, w_{m-1}w_m\}$. Let $\Delta_1 = \{u_2u_3, u_3u_{16}, u_{16}u_2\}$, $\Delta_2 = \{u_4u_5, u_5u_6, u_6u_4\}$ and $\Delta_3 = \{u_{13}u_{14}, u_{14}u_{15}, u_{15}u_{13}\}$ be the three triangles in G. The graph $GP(5,2) = \Gamma_1'/(\Delta_1 \cup \Delta_2 \cup \Delta_3)$, obtained from Γ_1' by contracting these three triangles, will be the Petersen graph (see for example Fig. 5 and Fig. 6 with the four loops deleted).

Since C' is a connected spanning subgraph of Γ_1', C' will be contracted to a connected spanning subgraph C'' of $GP(5,2)$. Moreover, every vertex of C' has even degree, and the same holds for C''. Hence C'' is a connected spanning subgraph of $GP(5,2)$, such that every vertex has an even degree. Thus C'' is a spanning eulerian subgraph of $GP(5,2)$. Since the degree of each vertex of C'' is even, and since $GP(5,2)$ is 3-regular, the degree of every vertex of C'' must be 2, and so C'' is a Hamiltonian cycle of the Petersen graph $GP(5,2)$. This contradicts the fact that Petersen graph $GP(5,2)$ is non-Hamiltonian.

Statement (iii) follows by identifying a Hamiltonian cycle of Γ_2 as follows: $v_1, w_2, w_3, \cdots, w_{m-1}, v_2, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_3, v_8, v_7, v_6, v_4, v_5, v_1$. $\square$

Declaration of competing interest

There is no competing interest.

Acknowledgements

The authors would like to express their gratitude to the anonymous referee for his/her careful reading of the manuscript and the valuable suggestions which led to a great improvement of the presentation of the paper.

The first author wants to thank Prof. Johannes Siemons for his helpful comments.

References

[1] Z.L. Blázsik, J. Cummings, W.H. Haemers, Cospectral regular graphs with and without a perfect matching, Discrete Math. 338 (2015) 199–201.

[2] A. Abiad, W.H. Haemers, Cospectral graphs and regular orthogonal matrices of level 2, Electron. J. Combin. 19 (3) (2012) P13.

[3] P.A. Catlin, H.J. Lai, Supereulerian graph and the Petersen graph, J. Combin. Theory Ser. B 66 (1996) 123–139.

[4] D.M. Cvetković, P. Rowlinson, S.K. Simić, An Introduction to the Theory of Graph Spectra, Cambridge University Press, Cambridge, 2010.

[5] E.R. van Dam, W.H. Haemers, Which graphs are determined by their spectrum, Linear Algebra Appl. 373 (2003) 241–272.

[6] E.R. van Dam, W.H. Haemers, Developments on spectral characterizations of graphs, Discrete Math. 309 (2009) 576–586.

[7] C.D. Godsil, G. Royle, Algebraic Combinatorics, Springer-Verlag, New York, 2001.

[8] W.H. Haemers, E. Spence, Enumeration of cospectral graphs, European J. Combin. 25 (2004) 199–211.
[9] C.R. Johnson, M. Newman, A note on cospectral graphs, J. Combin. Theory Ser. B 28 (1980) 96–103.
[10] A.J. Schwenk, Almost all trees are cospectral, in: New Directions in the Theory of Graphs, Academic Press, 1973, pp. 275–307.
[11] A.J. Schwenk, R.J. Wilson, On the eigenvalues of a graph, in: Selected Topics in Graph Theory, vol. 1, Academic Press, 1978, pp. 307–336.
[12] D.B. West, Introduction to Graph Theory, 2nd edition, Prentice-Hall, Englewood Cliffs, NJ, 2000.