Supereulerian digraphs with forbidden induced subdigraphs containing short dipaths

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August 11, 2017

Abstract

A digraph D is supereulerian if D has a spanning eulerian subdigraph. We investigate forbidden induced subdigraph conditions for a strong digraph to be supereulerian. Let P_k denote the dipath on k vertices. For $k \in \{2,3,4\}$, we determine the smallest integer h_k such that if a strong strict digraph D containing a subdigraph H isomorphic to P_k always satisfies $|A(D[V(H)])| \geq h_k$, then D is supereulerian. For $k \geq 5$, we show that $k^2 - 4k + 8 \leq h_k \leq k(k-1)$.

Key words. Strong arc connectivity, eulerian digraphs, supereulerian digraphs, forbidden induced subdigraphs

1 Introduction

We consider finite graphs and digraphs. Undefined terms and notations will follow [7] and [4]. We use (u, v) to represent an arc oriented from a vertex u

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to a vertex v. As in [7], a digraph D is **strict** if D has no loops and if for any pair of distinct vertices $u, v \in V(D)$, there is at most one arc in D oriented from u to v. Throughout out this paper, we only consider strict digraphs. We use $D \cong D'$ to mean that the two digraphs D and D' are isomorphic. For an integer $n > 0$, we use K_n^* to denote the complete digraph on n vertices. Hence for every pair of distinct vertices $u, v \in V(K_n^*)$, there is exactly one arc (u, v) in $A(K_n^*)$. For a digraph D, the underlying graph of D, denoted by $G(D)$, is obtained from D by erasing the orientations of all arcs of D. A digraph D is **weakly connected** if $G(D)$ is connected.

Following [4], for a digraph D with $X, Y \subseteq V(D)$, define

$$(X, Y)_D = \{(x, y) \in A(D) \; : \; x \in X, y \in Y\}.$$

When $Y = V(D) - X$, we define

$$\partial_D^+(X) = (X, V(D) - X)_D \text{ and } \partial_D^-(X) = (V(D) - X, X)_D.$$

For a vertex $v \in V(D)$, $d_D^+(v) = |\partial_D^+(\{v\})|$ and $d_D^-(v) = |\partial_D^-(\{v\})|$ are the **out-degree** and the **in-degree** of v in D, respectively. Finally, we define the following notations: $\delta^+(D) = \min\{d_D^+(v) \; : \; v \in V(D)\}$ and $\delta^-(D) = \min\{d_D^-(v) \; : \; v \in V(D)\}$. Let $N_D^+(v) = \{u \in V(D) - v : (v, u) \in A(D)\}$ and $N_D^-(v) = \{u \in V(D) - v : (u, v) \in A(D)\}$ denote the **out-neighbourhood** and **in neighbourhood** of v in D, respectively. We call the vertices in $N_D^+(v)$, $N_D^-(v)$ the **out-neighbours**, **in-neighbours** of v. When the digraph D is understood from the context, we often omit the subscript D. For an integer $k > 0$, let P_k denote a path (or a dipath) on k vertices.

Boesch, Suffel, and Tindell [6] in 1977 proposed the supereulerian problem, which seeks to characterize graphs that have spanning Eulerian subgraphs. Pulleyblank [16] later in 1979 proved that determining whether a graph is supereulerian is NP-complete. Since then, there have been lots of researches on this topic. For the literature of supereulerian graphs, see Catlin's first survey [8] on the topic and its updates [9] and [15].

It is natural to study supereulerian digraphs. A digraph D is **eulerian** if D is weakly connected and for every $v \in V(D)$, $d_D^+(v) = d_D^-(v)$; and is **supereulerian** if D contains a spanning eulerian subdigraph. The main problem is to determine supereulerian digraphs. Some earlier studies were done by Gutin [11, 12], and recent developments can be found in [1, 2, 3, 5, 13, 14], among others.

Forbidden induced subgraph conditions have been a widely investigated topic. Given a graph K, a graph G is said to be **K-free** if G does not have an induced subgraph isomorphic to K. This is equivalent to saying that if G has a subgraph H isomorphic to K, then $|E(G[V(H)])| \geq |E(H)| + 1$. Sufficient $K_{1,3}$-free conditions for hamiltonicity have been intensively studied, as seen in [10]. For a vertex w of G, let

$$M_G^i(w) = G[\{x \in V(G) : 1 \leq d_G(w, x) \leq i\}].$$

For $w \in V(G)$, let $N_2(w)$ be the subgraph induced by the set of edges uv, such that either u or v is adjacent to w. A vertex w of G is N^i-locally connected (N_2-locally connected, respectively) if $M_G^i(w)$ ($N_2(w)$, respectively) is connected. If every vertex of G is N^i-locally connected (N_2-locally connected, respectively), then G is **N^i-locally connected** (**N_2-locally connected**, respectively). Recently, Saito and Xiong proved the following.

Theorem 1.1 *(Saito and Xiong, [17]) Let H be a connected graph of order at least three, P_k be an undirected path on k vertices, and G be a connected N^3-locally connected graph. Each of the following holds.*
(i) Every 2-edge connected H-free graph is supereulerian if and only if H is $K_{1,2}$.
(ii) Every N^2-locally connected H-free graph is supereulerian if and only if H is either $K_{1,2}$ or $K_{1,3}$
(iii) If G is P_5-free, then G is supereulerian, if G is P_6-free, then G is supereulerian or the Petersen graph.

These motivates the current study on forbidden induced subdigraph sufficient conditions for supereulerian digraphs. Throughout the rest of the paper, for an integer $k \geq 2$, P_k denotes the dipath on k vertices. A subdigraph H of a digraph D is a P_k-subdigraph if H is isomorphic to P_k. If D does not have an induced P_k, then for any P_k-subdigraph H of D, we must have $|A(D[V(H)])| \geq k$.

Definition 1.1 *For integers $h \geq k \geq 2$, we define $\mathcal{F}(P_k, h)$ to be the family of all strict digraphs such that $D \in \mathcal{F}(P_k, h)$ if and only if D is strong and satisfies both of the following.*
(i) D contains at least one dipath P_k with $|A(D[V(P_k)])| = h$, and
(ii) for any dipath P_k in D, $|A(D[V(P_k)])| \geq h$.

If $D \in \mathcal{F}(P_k, h)$, then we also call D a $\mathcal{F}(P_k, h)$-digraph. It is known (for example, Corollary 3.1 of [2]) that every strong $\mathcal{F}(P_k, k^2 - k)$ digraph is supereulerian. Thus it is of interest to determine the smallest h_k such that every strong strict digraph in $\mathcal{F}(P_k, h_k)$ is supereulerian. A digraph D is **transitive** if for every triple of distinct vertices $x, y, z \in V(D)$ with $(x, y), (y, z) \in A(D)$, then $(x, z) \in A(D)$. Thus if D is a transitive digraph, then $D \in \mathcal{F}(P_3, 3)$, and so $\mathcal{F}(P_k, h)$ digraphs also represent some of the well studied families of digraphs.

The main purpose of this research is to investigate, for smaller values of k with $k \leq h \leq k(k-1)$, the behavior of graphs in $\mathcal{F}(P_k, h)$ and to determine the value of h_k. We show that $h_2 = 2$, $h_3 = 5$, $h_4 = 8$, and for $k \geq 5$, we show that $k^2 - 4k + 8 \leq h_k \leq k(k-1)$. Our results are presented in the subsequent sections.

2 Supereulerian digraphs in $\mathcal{F}(P_2, h) \cup \mathcal{F}(P_3, h)$

In this section, we investigate the supereulerianicity of digraphs in $\mathcal{F}(P_2, h)$ with $h = 2$ and in $\mathcal{F}(P_3, h)$ with $3 \leq h \leq 6$. We need a necessary condition for a digraph to be supereulerian. Let D be a digraph and $U \subset V(D)$. Let $t_0(U)$ be the smallest integer t such that $D[U]$ has a collection of arc disjoint ditrails $T_1, T_2, \cdots, T_t$ with $U = \cup_{i=1}^{t} V(T_i)$. For any subset $A \subseteq V(D) - U$, define $B =: V(D) - U - A$, and

$$h(U, A) =: min\{|\partial_D^+(A)|, |\partial_D^-(A)|\} + min\{|(U, B)_D|, |(B, U)_D|\} - t_0(U).$$

Then we have the following proposition.

Proposition 2.1 *(Hong, Lai and Liu, Proposition 2.1 of [13]) If D has a spanning eulerian subdigraph, then for any $U \subset V(D)$, we have $h(U, A) \geq 0$.*

Digraphs in $\mathcal{F}(P_3, h)$ with $3 \leq h \leq 4$ are not necessarily supereulerian, as can be seen in the example below.

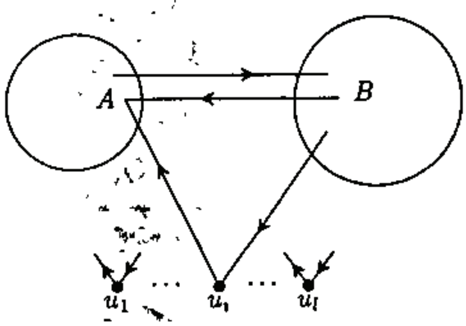

Figure 1 The digraph $D = D(\alpha, \beta, k, \ell)$.

Example 2.1 *Let $\alpha, \beta, k > 0$ be integers with $\alpha, \beta \geq k+1$, and let A and B be two disjoint set of vertices with $|A| = \alpha$ and $|B| = \beta$. Let $\ell \geq \alpha\beta + 1$ be an integer, and U be a set of vertices disjoint from $A \cup B$ with $|U| = \ell$. We construct a digraph $D = D(\alpha, \beta, k, \ell)$ such that $V(D) = A \cup B \cup U$ and the arcs of D are given as required in (D1) and (D2) below. (See Figure 1).*
(D1) $D[A \cup B] \cong K^*_{\alpha+\beta}$ is a complete digraph.
(D2) For every vertex $u \in U$, and for every $v \in A$, $(u, v) \in A(D)$ and for every $w \in B$, $(w, u) \in A(D)$. Thus for any $u \in U$, we have $N^+_D(u) = A$ and $N^-_D(u) = B$. No two vertices in U are adjacent.

Direct computation yields

$$h(U, A) = |\partial^+_D(A)| + |(U, B)_D| - t_0(U) = \alpha\beta - |U| < 0,$$

and so by Proposition 2.1, any $D = D(\alpha, \beta, k, \ell)$ is nonsupereulerian. By Definition 1.1, $D \in \mathcal{F}(P_3, 4)$. Thus Example 2.1 indicates that $\mathcal{F}(P_3, 4)$ contains infinitely many nonsupereulerian digraphs.

An arc (u, v) of D is **symmetric** in D if both $(u, v), (v, u) \in A(D)$. A digraph D is **symmetric** if $|V(D)| = 1$ or if $|A(D)| > 0$ and every arc of D is symmetric. A digraph $D \neq K_1$ is **symmetrically connected**, if for every $u, v \in V(D)$, D contains a symmetric (u, v)-dipath.

Theorem 2.1 *([2]) If D is symmetrically connected, then D is supereulerian.*

Observe that if $D \in \mathcal{F}(P_2, 2) \cup \mathcal{F}(P_3, 5)$, then D is symmetrically connected. Thus by Theorem 2.1, every digraph in $\mathcal{F}(P_2, 2) \cup \mathcal{F}(P_3, 5)$ is supereulerian. Hence we have our conclusions in this section.

Proposition 2.2 *Let D be a digraph.*
(i) Every digraph in $\mathcal{F}(P_2, 2) \cup \mathcal{F}(P_3, 5)$ is supereulerian.
(ii) Not every digraph in $\mathcal{F}(P_3, 3) \cup \mathcal{F}(P_3, 4)$ is supereulerian.

3 Supereulerian digraphs in $\mathcal{F}(P_4, h)$

Throughout this section, $k > 0$ denotes an integer. In this section, we will study the supereulerianicity of digraphs in $\mathcal{F}(P_4, h)$, and determine the smallest value of h_4 such that every digraph in $\mathcal{F}(P_4, h_4)$ is supereulerian. We start with some terminology and definitions. For a digraph D and a subdigraph S of D, an (x, y)-dipath P is called an (S, S)-dipath if $V(P) \cap V(S) = \{x, y\}$. An (S, S)-dipath P is shortest if for some $x, y \in V(S)$, P is an (x, y)-dipath and P is shortest among all (x, y)-dipath in $D - A(S)$.

Definition 3.1 *Let D be a digraph. Suppose that S is an eulerian subdigraph of D with $|V(S)|$ maximized. A shortest (S, S)-dipath H with $|V(H)| = k + 2 \geq 3$ is called a* k-**handle** *of S in D.*

The following is a necessary condition for a digraph to be supereulerian.

Lemma 3.1 *(K.A. Alsatami et al, Lemma 2 of [3]) A digraph D is non-supereulerian if there exist vertex-disjoint subdigraphs $\{A, B_1, ..., B_m\}$ of D, for some integer $m > 0$, satisfying each of the following.*
(i) $N^-(B_i) \subseteq V(A), \forall i \in \{1, 2, ..., m\}$.
(ii) $|\partial^-(A)| \leq m - 1$.

In the rest of this section, we examine the supereulerian membership of digraphs in $\mathcal{F}(P_4, h)$, for each value h with $4 \leq h \leq 12 = |A(K_4^*)|$, and to determine the value of h_4 such that every digraph in $\mathcal{F}(P_4, h_4)$ is supereulerian.

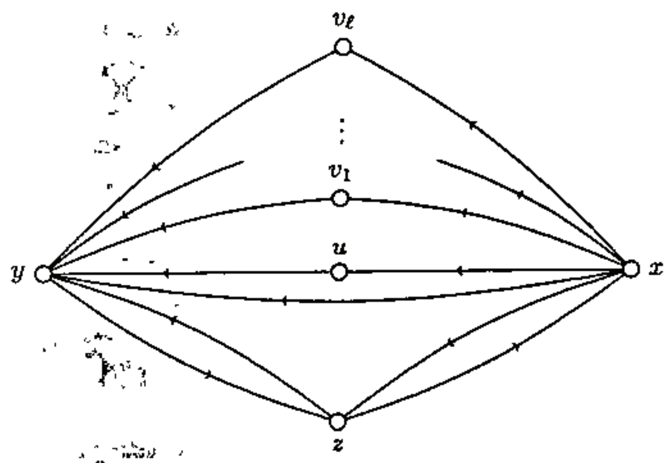

Figure 2. The digraph D_ℓ.

Proposition 3.1 *Let D be a digraph. There exists an infinite family of nonsupereulerian digraphs in $\mathcal{F}(P_4, 7)$. More precisely, there are infinitely many 1-handles in $\mathcal{F}(P_4, 7)$.*

Proof. Let $M = xzy$ be a symmetric dipath, $Q = xuy$ be a dipath and $H_i = xv_iy$, $1 \le i \le \ell$ be dipaths. Define $D_\ell = M \cup Q \cup \left(\bigcup_{i=1}^{\ell} H_i\right) \cup \{(x, y)\}$, as depicted in Figure 2. It is routine to verify that $D_\ell \in \mathcal{F}(P_4, 7)$. By Lemma 3 1 with $A = D[\{x\}]$, $B_1 = D[\{u\}]$ and $B_2 = D[\{v_1\}]$, we conclude that D_ℓ is nonsupereulerian. By Definition 3.1, D_ℓ is a 1-handle. ∎

We make the following observation for nonsupereulerian strong digraph

Observation 3.1 *Suppose that D is a nonsupereulerian strong digraph. Let S be a maximal eulerian subdigraph of D and let $H = xu_1...u_ky$ be a k-handle of S and $Q = xv_1...v_sy$ be a shortest (x, y)-dipath in S with $k + s$ is minimized. We have the following observations.*

(A) If for some i with $1 \le i \le k$, we have $\{(u_i, x), (y, u_i)\} \cap A(D) \neq \emptyset$, then either $S \cup \{(x, u_1), (u_1, u_2), ..., (u_{i-1}, u_i), (u_i, x)\}$ or $S \cup \{(y, u_i), (u_i, u_{i+1}), ..., (u_{k-1}, u_k), (u_k, y)\}$ would violate the maximality of S. Hence for any i with $1 \le i \le k$, we have $\{(u_i, x), (y, u_i)\} \cap A(D) = \emptyset$.

(B) If for some i with $1 \le i \le k-1$, we have $\{(x, u_{i+1}), (u_i, y)\} \cap A(D) \neq \emptyset$, then $H' = xu_{i+1}...u_ky$ or $H'' = xu_1...u_iy$ is a shorter (S, S)-dipath,

contrary to the fact that H is a shortest (S, S)-dipath, stated in Definition 3.1. Hence for any $1 \leq i \leq k-1$, we have $\{(x, u_{i+1}), (u_i, y)\} \cap A(D) = \emptyset$.

(C) By Definition 3.1, H is a shortest (S, S)-dipath The minimality of $k+s$ implies that for every i, j with $1 \leq i \leq s, 1 \leq j \leq k$, we have $\{(v_i, u_j), (u_j, v_i)\} \cap A(D) = \emptyset$; and for every j with $i+2 \leq j$, we have $(u_i, u_j) \notin A(D)$.

Theorem 3.1 *Each of the following holds.*
(i) Every digraph D in $\mathcal{F}(P_4, 8)$ is supereulerian.
(ii) $h_4 = 8$.

Proof. As (ii) follows from (i) and Proposition 3.1, it suffices to prove (i). Assume that $D \in \mathcal{F}(P_4, 8)$. By contradiction, we assume that D is a nonsupereulerian digraph. Let S be a maximal eulerian subdigraph of D and let $H = xu_1...u_ky$ be a k-handle of S and, and for some integer $s \geq 1$, let $Q = xv_1...v_sy$ be a shortest (x, y)-dipath in S such that

$$k + s \text{ is minimized }. \tag{1}$$

We consider three cases.

Case 1. $k \geq 3$.

In this case, $P' = xu_1u_2u_3$ is a P_4 in D. By Observation 3.1 ((A) and (B)), we conclude that $\{(u_1, x), (x, u_2), (u_2, x), (x, u_3), (u_3, x)\} \cap A(D) = \emptyset$. It follows that $|A(D[V(P')])| < 8$, contrary to the assumption that $D \in \mathcal{F}(P_4, 8)$.

Case 2. $k = 2$.

In this case, $P'' = xu_1u_2y$ is a P_4 in D. By Observation 3.1 ((A) and (B)), we conclude that $\{(u_1, x), (x, u_2), (u_2, x), (y, u_2), (y, u_1), (u_1, y)\} \cap A(D) = \emptyset$. It follows that $|A(D[V(P'')])| < 8$, contrary to the assumption that $D \in \mathcal{F}(P_4, 8)$.

Case 3. $k = 1$.

Claim 1. $(y, x) \notin A(D)$.

By contradiction, we assume that $(y, x) \in A(D)$. If $(y, x) \notin A(S)$, then $S \cup A(H) \cup \{(y, x)\}$ is violation to the maximality of S. Hence $(y, x) \in A(S)$.

Since $P^{(3)} = u_1yxv_1$ is a P_4 in D, by Observation 3.1 ((A) and (C)), we conclude that $\{(u_1,x),(y,u_1),(u_1,v_1),(v_1,u_1)\} \cap A(D) = \emptyset$. Since $D \in \mathcal{F}(P_4,8)$, we have $|A(D[V(P^{(3)})])| \geq 8$, and so $\{(y,v_1),(v_1,x)\} \subset A(D)$.

Suppose first that $s = 1$. If $\{(y,v_1),(v_1,x)\} \cap A(S) = \emptyset$, then $S \cup \{(y,v_1),(v_1,x),(x,u_1),(u_1,y)\}$ is a violation to the maximality of S. Hence $\{(y,v_1),(v_1,x)\} \cap A(S) \neq \emptyset$, whence $S - \{(x,v_1),(v_1,y)\} + \{(x,u_1),(u_1,y)\}$ is a violation to the maximality of S. In either case, a contradiction obtains and so we must have $s \geq 2$. Note that $P^{(4)} = u_1yv_1v_2$ is a P_4 in D. By Observation 3.1((A) and (C)), we conclude that $\{(y,u_1),(u_1,v_1),(v_1,u_1),(u_1,v_2),(v_2,u_1)\} \cap A(D) = \emptyset$. It follows that $|A(D[V(P^{(4)})])| < 8$, contrary to the assumption that $D \in \mathcal{F}(P_4,8)$. This justifies Claim 1.

Claim 2. For any i with $1 \leq i \leq s$, $(y,v_i) \notin A(D)$.

By contradiction, we assume that $(y,v_i) \in A(D)$ for some i with $1 \leq i \leq s$. Then $P^{(5)} = xu_1yv_i$ is a P_4 in D. Since $D \in \mathcal{F}(P_4,8)$, we must have $|A(D[V(P^{(5)})])| \geq 8$. By Observation 3.1(($A$) and ($C$)) and Claim 1, $\{(u_1,x),(y,u_1),(u_1,v_i),(v_i,u_1),(y,x)\} \cap A(D) = \emptyset$. It follows that $|A(D[V(P^{(5)})])| < 8$, contrary to the assumption that $D \in \mathcal{F}(P_4,8)$. This justifies Claim 2.

By Claims 1 and 2, we have,

$$\{(y,v_1),(y,v_2),\cdots,(y,v_s),(y,x)\} \cap A(D) = \emptyset. \tag{2}$$

Claim 3. For any $z \in V(D)$, $(y,z) \notin A(D)$.

By contradiction, we assume that for some $z \in V(D)$, we have $(y,z) \in A(D)$. Then $P^{(6)} = xu_1yz$ is a P_4 in D. Since $D \in \mathcal{F}(P_4,8)$, we have $|A(D[V(P^{(6)})])| \geq 8$.

By Observation 3.1 (A) and by (2), we have $\{(u_1,x),(y,u_1),(y,x)\} \cap A(D) = \emptyset$. It follows that $|\{(z,u_1),(u_1,z)\} \cap A(D)| \geq 1$.

Suppose first that $\{(z,u_1),(u_1,z)\} \subseteq A(D)$. If $z \notin V(S)$, then $S \cup \{(y,z),(z,u_1),(u_1,y)\}$ violates the maximality of S. If $z \in V(S)$, then $S \cup \{(z,u_1),(u_1,z)\}$ violates the maximality of S. We have $|\{(z,u_1),(u_1,z)\} \cap A(D)| = 1$. Since $D \in \mathcal{F}(P_4,8)$, therefore, we have $|A(D[V(P^{(6)})])| \geq 8$. Thus $\{(z,y),(z,x),(x,z),(x,y),(x,u_1),(u_1,y),(y,z)\} \subseteq A(D)$. Since

$|\{(z,u_1),(u_1,z)\}\cap A(D)| = 1$, we consider just two subcases.

Subcase 3.1. $(z,u_1)\in A(D)$ but $(u_1,z)\notin A(D)$.

If $z\notin V(S)$, then $S\cup\{(y,z),(z,u_1),(u_1,y)\}$ violates the maximality of S Hence we assume that $z\in V(S)$. The dipath $P^{(7)} = v_s yzu_1$ is a P_4 in D. By Observation 3.1((A) and (C)) and by (2), we have $\{(u_1,z),(v_s,u_1),(u_1,v_s),(y,u_1),(y,v_s)\}\cap A(D)=\emptyset$. It follows that $|A(D[V(P^{(7)})])| < 8$, contrary to the assumption that $D\in\mathcal{F}(P_4,8)$. This contradiction indicates that Subcase 3.1 does not occur.

Subcase 3.2. $(u_1,z)\in A(D)$ but $(z,u_1)\notin A(D)$.

Note that $P^{(8)} = u_1 z x v_1$. Since $D\in\mathcal{F}(P_4,8)$, we have

$$|A(D[V(P^{(8)})])| \geq 8.$$

By Observation 3.1((A) and (C)) and by (2), $\{(u_1,x),(u_1,v_1),(v_1,u_1),(z,u_1)\}\cap A(D)=\emptyset$, and so $(z,v_1)\in A(D)$.

Recall that Q is a shortest (x,y)-dipath in S as defined in (1). The dipath $P^{(9)} = u_1 z v_1 v_2$ is a P_4 in D, where $v_2 = y$ when $Q = xv_1 y$. Since $D\in\mathcal{F}(P_4,8)$, we have $|A(D[V(P^{(9)})])|\geq 8$. By Observation 3 1 (C) and by (2), $\{(u_1,v_1),(v_1,u_1),(v_2,u_1),(z,u_1)\}\cap A(D)=\emptyset$, where $(u_1,v_2)\notin A(D)$ when $v_2\neq y$ and $(y,v_1)\notin A(D)$ when $v_2 = y$ These imply that $|A(D[V(P^{(9)})])| < 8$, contrary to the assumption that $D\in\mathcal{F}(P_4,8)$. ∎

4 Lower bound of h_k

For a given integer $k > 1$, let h_k denote the smallest integer such that every strong strict digraph in $\mathcal{F}(P_k,h_k)$ is supereulerian. It is known that $k < h_k \leq k(k-1)$. In this final remark section, we would present a lower bound of h_k for $k\geq 5$, as stated below.

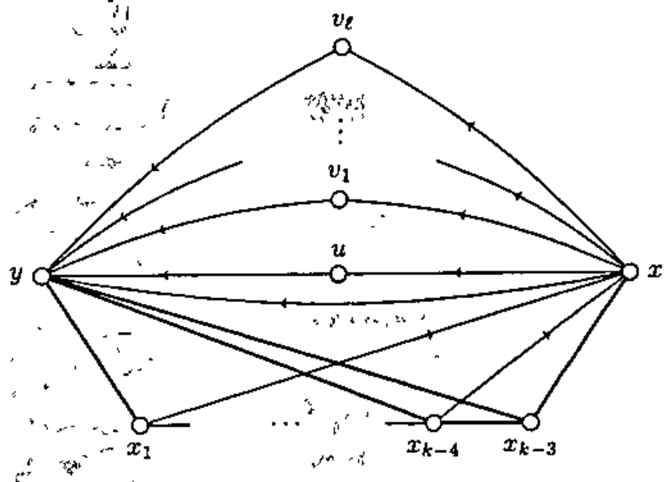

Figure 3. The digraph D_ℓ.

Proposition 4.1 *For $k \geq 5$, $h_k \geq k^2 - 4k + 8$.*

Proof. For each integer $k \geq 5$, we shall show that there exists an infinite family of nonsupereulerian strong strict digraphs in $\mathcal{F}(P_k, k^2 - 4k + 7)$.

Let M' be a complete digraph isomorphic to K_{k-1}^* with vertice set $\{x, u_1, u_2, \cdots, u_{k-3}, y\}$ and set $x_0 = y$. Let M be the digraph obtained from M' by deleting all the arcs (x_j, x) from M', where $0 \leq j \leq k-3$. Let $Q = xuy$ be a dipath and $H_i = xv_iy$, $1 \leq i \leq \ell$ be dipaths. Define $D_\ell = M \cup Q \cup \left(\bigcup_{i=1}^{\ell} H_i\right)$, as depicted in Figure 3. It is routine to verify that $D_\ell \in \mathcal{F}(P_k, k^2 - 4k + 7)$. By Lemma 3.1 with $A = D[\{x\}]$, $B_1 = D[\{u\}]$, $B_2 = D[\{v_1\}]$, we conclude that D_ℓ is nonsupereulerian. ∎

We conclude this section with the following conjecture.

Conjecture 4.1 *For every integer $k \geq 5$, $h_k = k^2 - 4k + 8$.*

Acknowledgment. The research of Juan Liu is supported by NSFC grant (NSFC 61363020).

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