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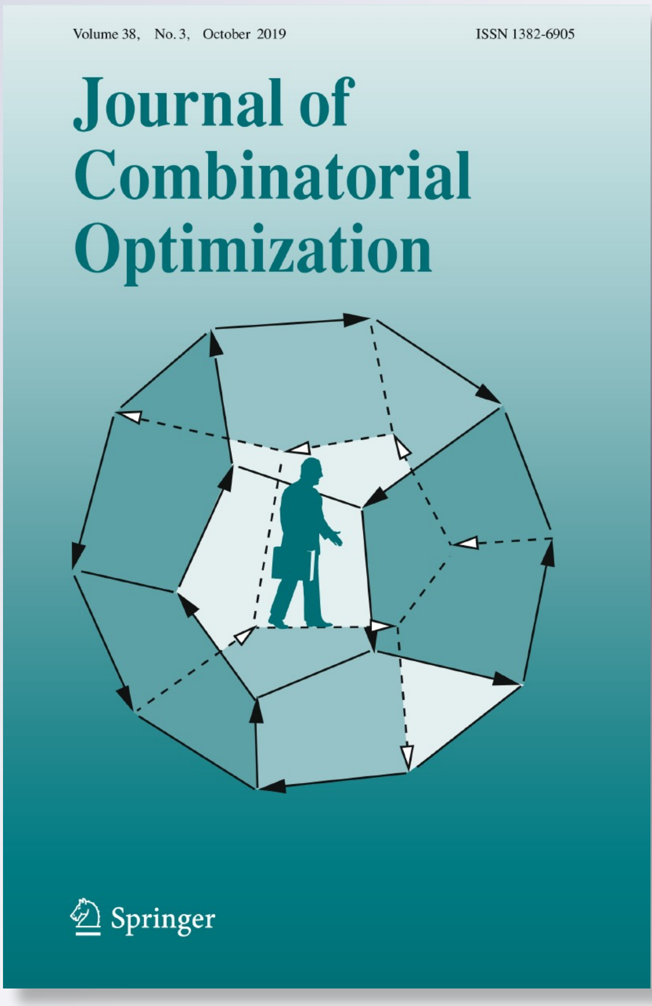

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Characterizations of matroids with an element lying in a restricted number of circuits

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Abstract

A matroid M with a distinguished element $e_0 \in E(M)$ is a rooted matroid with e_0 being the root. We present a characterization of all connected binary rooted matroids whose root lies in at most three circuits, and a characterization of all connected binary rooted matroids whose root lies in all but at most three circuits. While there exist infinitely many such matroids, the number of serial reductions of such matroids is finite. In particular, we find two finite families of binary matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ and prove the following. (i) For some $e_0 \in E(M)$, M has at most three circuits containing e_0 if and only if the serial reduction of M is isomorphic to a member in $\mathcal{M}_1$. (ii) If for some $e_0 \in E(M)$, M has at most three circuits not containing e_0 if and only if the serial reduction of M is isomorphic to a member in $\mathcal{M}_2$. These characterizations will be applied to show that every connected binary matroid M with at least four circuits has a 1-hamiltonian circuit graph.

Keywords Excluded minor characterizations · Matroid circuit graph · Hamiltonian · 1-hamiltonian

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1 The problem

Matroids and graphs considered in this paper are finite. We follow the notations and terminology in Bondy and Murty (2008) for graphs and Oxley (2011) for matroids except otherwise defined. As in Bondy and Murty (2008), $\kappa(G)$, $\delta(G)$ denote the connectivity and minimum degree of a graph G. For a matroid M, let $\mathcal{C}(M)$ and r_M denote the collection of circuits and the rank function of M, respectively. Following (Oxley 2011), a matroid M is connected if for any pair of distinct elements $e, e' \in E(M)$, there exists a circuit $C \in \mathcal{C}(M)$ with $e, e' \in C$. Throughout this paper, for any edge subset $X \subseteq E(G)$ of a graph G, X denotes an edge subset as well as the subgraph $G[X]$ induced by the edge subset X. Following matroid terminology, if G is a graph and $M = M(G)$ is the cycle matroid of M, any edge subset Z (as well as the subgraph $G[Z]$ induced by Z) will be called a **circuit** if $Z \in C(M(G))$. Let $h > 0$ be an integer. If $Z \in \mathcal{C}(M)$ with $|Z| = h$, we often call Z an h-circuit of M.

The distribution of circuits in a graph or a matroid has been studied by quite a few researchers. Murty (1971a) initially characterized all connected binary matroids with exactly one circuit length. Lemos et al. (2011) extended Murty's result by successfully characterizing all connected binary matroids with at most two circuit lengths. It is indicated in Lemos et al. (2011) that it is difficult to characterize the matroids having a particular circuit-spectrum set even when the set is small and the matroids belong to an interesting class. Cordovil et al. (2009), and Junior and Lemos (2001) constructed all matroids M whose circuit lengths are at most 5, and constructed all 3-connected binary matroids M whose circuit lengths are in $\{3, 4, 5, 6, 7\}$. Bollobás (1978) presented a characterization of all graphs with minimum degree at least 3 that do not have edge disjoint circuits. He indicated that this characterization can be applied to imply a slight extension of an earlier result of Erdös and Pósa (1965). The corresponding characterization of regular matroids without disjoint circuits is obtained in Fan et al. (2010). In this paper, we consider the problem of determining all binary matroids with an element lying in at most 3 circuits, as well as all binary matroids with an element lying in all but at most three circuits. The main results of this paper, to be stated in the next section after some of the terms are defined, are characterizations of such matroids.

Li and Liu (2007, 2008, 2010) initiated the investigation of graphical properties of matroid circuit graphs. Let M be a matroid, and let $k > 0$ be an integer. The **circuit graph** $G(M)$ of M has vertex set $V(G(M)) = \mathcal{C}(M)$. Two vertices $Z, Z' \in \mathcal{C}(M)$ are adjacent in $G(M)$ if and only if $|Z \cap Z'| \geq 1$. As an application of our main results, we prove that the circuit graph of a connected binary matroid with at least 4 circuits is 1-hamiltonian.

In the next section, we introduce rooted matroids and present characterizations of binary rooted matroids in which the root is in certain restricted number of circuits. An application of the characterizations to 1-hamiltonian circuit graphs will be presented in the last section.

2 Binary matroids with an element in restricted number of circuits

The main purpose of this section is to characterize all connected binary rooted matroids whose root is lying in at most three circuits, and all connected binary rooted matroids whose root is lying in all but at most three circuits.

A matroid M with a distinguished element $e_0 \in E(M)$ is a **rooted matroid** with e_0 being the root. We often use $M(e_0)$ to emphasize the root e_0. Two rooted matroids $M(e_0)$ and $N(f_0)$ are isomorphic if e_0 corresponds to f_0 under the matroid isomorphism. When f_0 is not emphasized, we often just say that M or $M(e_0)$ is isomorphic to N. Given a matroid $M(e_0)$, define $\mathcal{C}_{M,e_0} = \{C \in \mathcal{C}(M) : e_0 \in C\}$,

$$\begin{aligned}\mathcal{F}_1 &= \{M = M(e_0) : |\mathcal{C}_{M,e_0}| \le 3\}, \text{ and } \mathcal{F}_2 \\ &= \{M = M(e_0) : |\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \le 3\},\end{aligned} \tag{1}$$

Throughout this section, for fixed $i \in \{1, 2\}$, if M is such a matroid that for any $e_0 \in E(M)$, $M(e_0)$ is in $\mathcal{F}_1$, then we simply say that $M \in \mathcal{F}_i$ without indicating the root.

Excluded minor characterizations will be developed in this section. Let $\mathcal{F}$ be a collection of matroids. Define $EX(\mathcal{F})$ to be the family of matroids such that $M \in EX(\mathcal{F})$ if and only if M does not have a minor isomorphic to a member in $\mathcal{F}$. When $\mathcal{F} = \{N_1, N_2, \ldots, N_k\}$ is a finite collection, we also use $EX(N_1, N_2, \ldots, N_k)$ for $EX(\{N_1, N_2, \ldots, N_k\})$. Following (Oxley 2011), F_7 and F_7^* are the two binary vector matroids $F_7 = M_2[I_3|D]$ and $F_7^* = M_2[D^T|I_4]$, where

$$[I_3|D] = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \text{ and } [D^T|I_4] = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

Let M and N be matroids. If for some element $f \in E(M)$, f lies in a 2-circuit of M and $M - f = N$, then M is a **single element parallel extension** of N and N is a **single parallel deletion** of M. If M is obtained from N by taking a finite number of single element parallel extensions, then M is a **parallel extension** of N. If for some element $f \in E(M)$, f lies in a 2-cocircuit of M and $M/f = N$, then M is a **single element serial extension** of N and N is a **serial contraction** of M. If M is obtained from N by taking a finite number of single element serial extensions, then M is a **serial extension** of N. A subset $X \subseteq E(M)$ is a **serial class** if every pair of elements in X form a cocircuit of M such that X is a maximal subset of $E(M)$ with this property.

Proposition 2.1 (Li and Liu, Lemma 6 of Li and Liu (2008)) *Suppose that $e, e' \in E(M)$ and $\{e, e'\} \in \mathcal{C}(M^*)$.*

(i) *For any element $e_0 \neq e'$, $|\mathcal{C}_{M,e_0}| \le 3$ if and only if $|\mathcal{C}_{M/e',e_0}| \le 3$; and $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \le 3$ if and only if $|\mathcal{C}(M/e')| - |\mathcal{C}_{M/e',e_0}| \le 3$.*

(ii) *Consequently, if M is a serial extension of a matroid N, and if $e_0 \in E(N)$, then $|\mathcal{C}_{M,e_0}| \le 3$ if and only if $|\mathcal{C}_{N,e_0}| \le 3$; and $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \le 3$ if and only if $|\mathcal{C}(N)| - |\mathcal{C}_{N,e_0}| \le 3$.*

2.1 Rooted matroid minors

Let $M(e_0)$ be a rooted matroid. A **rooted minor** of $M(e_0)$ is a rooted matroid $N = N(e_0)$ such that for some disjoint subsets $S, T \subseteq E(M - e_0)$, $N = M/S - T$. Proposition 2.1 can be slightly extended to Lemma 2.2 below, showing that the properties of satisfying $|\mathcal{C}_{M,e_0}| \le 3$ and of satisfying $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \le 3$ are in fact closed under taking rooted minors.

Lemma 2.2 *Let $M = M(e_0)$ be a matroid rooted at e_0.*

(i) *If $|\mathcal{C}_{M,e_0}| \le 3$, the for any $x \in E(M) - e_0$, $|\mathcal{C}_{M-x,e_0}| \le 3$.*
(ii) *If $|\mathcal{C}_{M,e_0}| \le 3$, the for any $x \in E(M) - e_0$, $|\mathcal{C}_{M/x,e_0}| \le 3$.*
(iii) *If $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \le 3$, the for any $x \in E(M) - e_0$, $|\mathcal{C}(M-x)| - |\mathcal{C}_{M-x,e_0}| \le 3$.*
(iv) *If $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \le 3$, the for any $x \in E(M) - e_0$, $|\mathcal{C}(M/x)| - |\mathcal{C}_{M/x,e_0}| \le 3$.*

Proof Let $M = M(e_0) \in \mathcal{F}_1$, and let $x \in E(M) - e_0$. By definition, $|\mathcal{C}_{M,e_0}| \le 3$. As $\mathcal{C}(M - x) \subseteq \mathcal{C}(M)$, we have $\mathcal{C}_{M-x,e_0} \subseteq \mathcal{C}_{M,e_0}$. Moreover, for any $C \in \mathcal{C}(M - x) - \mathcal{C}_{M-x,e_0}$, as $\mathcal{C}(M - x) \subseteq \mathcal{C}(M)$ and $e_0 \notin C$, we have $C \in \mathcal{C}(M) - \mathcal{C}_{M,e_0}$, implying that $\mathcal{C}(M - x) - \mathcal{C}_{M-x,e_0} \subseteq \mathcal{C}(M) - \mathcal{C}_{M,e_0}$. Therefore, we have both $|\mathcal{C}_{M-x,e_0}| \le |\mathcal{C}_{M,e_0}| \le 3$ and $|\mathcal{C}(M - x) - \mathcal{C}_{M-x,e_0}| \le |\mathcal{C}(M) - \mathcal{C}_{M,e_0}| \le 3$, and so (i) and (iii) must hold.

We now prove (ii). As $\mathcal{C}(M/x)$ consists of the minimal members of $\{C - x : C \in \mathcal{C}(M)\}$, for each $C' \in \mathcal{C}_{M/x,e_0}$, there exists a circuit $C \in \mathcal{C}_{M,e_0}$ with $C' = C - x$. Thus the mapping $f(C') = C$ is injective. This implies that $|\mathcal{C}_{M/x,e_0}| \le |\mathcal{C}_{M,e_0}| \le 3$, and so (ii) holds. Similarly, for each $C' \in \mathcal{C}(M/x) - \mathcal{C}_{M/x,e_0}$, there exists a $C \in \mathcal{C}(M) - \mathcal{C}_{M,e_0}$ with $C' = C - x$. As the mapping from C' to C is injective, it follows that $|\mathcal{C}(M/x) - \mathcal{C}_{M/x,e_0}| \le |\mathcal{C}(M) - \mathcal{C}_{M,e_0}| \le 3$, implying (iv). □

The following theorem of Brylawski and Seymour will be needed in our arguments.

Theorem 2.3 (Brylawski 1972 and Seymour 1977) *Let N be a connected minor of a connected matroid M. For any $f \in E(M) - E(N)$, one of $M - f$ and M/f is connected and contains N as a minor.*

Lemma 2.4 *Let M, N be a connected matroids such that N is a minor of M, and let $e_0 \in E(M) - E(N)$. Each of the following holds.*

(i) *Either $|E(M)| = |E(N)| + 1$, or M has a connected proper minor L with $e_0 \in E(L)$ such that L contains N as a minor.*
(ii) *$M(e_0)$ contains a connected rooted minor $L(e_0)$ such that $L(e_0) - e_0 = N$.*

Proof As (ii) follows from (i), we argue by induction on $|E(M)|$ to prove (i). By assumption, $|E(M)| \ge |E(N) \cup e_0| = |E(N)| + 1$. If $|E(M)| = |E(N)| + 1$, then $L = M$. Assume that $|E(M)| > |E(N)| + 1$ and the lemma holds for smaller values of $|E(M)|$. Pick $f \in E(M) - (E(N) \cup e_0)$. By Theorem 2.3, either $M - f$ or M/f is connected, contains e_0 as an element and N as a minor. Thus by induction, either $M - f$ or M/f has a connected minor L with $e_0 \in E(L)$ such that L contains N as a minor. □

We need a few more notational conventions.

Notation 2.5 *For an integer $r > 0$, let $V(r, 2)$ denote the r-dimensional vector space over the 2-element field $GF(2)$. Suppose that $M = M_2[I_r|D]$ is a binary matroid with $E(M) = \{e_1, e_2, \ldots, e_m\}$ such that, for $1 \le i \le m$, e_i is the label of the ith column vector v_i of $[I_r|D]$. Then $B = \{e_1, e_2, \ldots, e_r\}$ is a basis of M and $\{v_1, v_2, \ldots, v_r\}$ is the standard basis of $V(r, 2)$. For any nonzero vector $v = (x_1, x_2, \ldots, x_r) \in V(r, 2) - \{0\}$,*

$$S(v) = \{i : x_i \ne 0\} \text{ and } B(v) = \{e_i : 1 \le i \le r \text{ and } x_i \ne 0\}. \tag{3}$$

Thus $B(v)$ is the unique minimum subset of B such that the vectors $\{v\} \cup \{v_i : e_i \in B(v)\}$ is a linearly dependent set in $\{v_1, v_2, \ldots, v_r, v\}$ that contains v.

Using the notation in Definition 2.5, we have the following observations. Observation 2.6 follows immediately from the definition of a vector matroid and from (3).

Observation 2.6 *Let $M = M_2[I_r|D]$ denote a binary matroid.*

- (i) *M is simple if and only if $[I_r|D]$ does not have an all zero column and does not have two identical columns. Consequently, if M is simple, then for any $j \ge r+1$, $|S(v_j)| \ge 2$.*
- (ii) *For vectors $w_1, w_2 \in V(r, 2)$, $B(w_1) = B(w_2)$ if and only if $w_1 = w_2$.*

Observation 2.7 *Let $M = M_2[I_r|D]$ be a simple binary matroid, let $v_{i_1}, v_{i_2}, \ldots, v_{i_t}$ be distinct column vectors of D, and suppose that $\{e_{i_1}, e_{i_2}, \ldots, e_{i_t}\} \in \mathcal{I}(M)$. Let $v = v_{i_1} + v_{i_2} + \cdots + v_{i_t}$. Then the following are equivalent.*

- (i) *$B(v) \cup \{e_{i_1}, e_{i_2}, \ldots, e_{i_t}\}$ is a circuit of M.*
- (ii) *For any partition of the set $\{i_1, i_2, \ldots, i_t\}$ into two disjoint nonempty sets J_1 and J_2, we have $S(\sum_{i \in J_1} v_i) \cap S(\sum_{j \in J_2} v_j) \ne \emptyset$.*

Proof Let $X = B(v) \cup \{e_{i_1}, e_{i_2}, \ldots, e_{i_t}\}$ and $J = \{i_1, i_2, \ldots, i_t\}$. Since M is binary and since $v \ne 0$, it follows by (3) that X is a disjoint union of circuits, and so there exist disjoint circuits $C_1, C_2, \ldots, C_s$ such that $X = \cup_{i=1}^{s} C_i$.

Assume (i) holds. Then $s = 1$. To show (ii), we argue by contradiction and assume that J can be partitioned into two disjoint nonempty sets J_1 and J_2 satisfying $S(\sum_{i \in J_1} v_i) \cap S(\sum_{j \in J_2} v_j) = \emptyset$. Let $w_1 = \sum_{i \in J_1} v_i$ and $w_2 = \sum_{j \in J_2} v_j$. Since $\{e_{i_1}, e_{i_2}, \ldots, e_{i_t}\} \in \mathcal{I}(M)$, we have $w_1 \ne 0$ and $w_2 \ne 0$. By (3), each of $B(w_1) \cup \{e_i : i \in J_1\}$ and $B(w_2) \cup \{e_i : i \in J_2\}$ is a disjoint union of circuits of M contained in X, contrary to the assumption that $s = 1$. Hence (i) implies (ii).

We shall show that (ii) implies $s = 1$. By contradiction, we assume that $s \ge 2$. Define $J'_1 = \{i : e_i \in C_1\}$ and $J'_2 = \{i : e_i \notin C_1\}$. Since B is a basis, we must have $J_1 = J'_1 - \{1, 2, \ldots, r\} \ne \emptyset$. With a similar argument, we also have $J_2 =$

$J_2' - \{1, 2, \ldots, r\} \neq \emptyset$. Since $C_1 \cap (\cup_{i=2}^{s} C_i) = \emptyset$, we have $J_2 = J - J_1$. Define $w_1 = \sum_{i \in J_1} v_i$ and $w_2 = \sum_{i \in J_2} v_i$. By (3), $B(w_1) = \{e_i : i \in J_1' \cap \{1, 2, \ldots, r\}\}$ and $B(w_2) = \{e_i : i \in J_2' \cap \{1, 2, \ldots, r\}\}$. Thus for any $1 \leq j \leq r$, if $j \in S(w_1)$, then $e_j \in B(w_1) \subset C_1$; and if $j \in S(w_2)$, then $e_j \in B(w_2) \subset X - C_1$. It follows that $S(w_1) \cap S(w_2) = \emptyset$, contrary to (ii). This shows that (ii) implies (i). □

Corollary 2.8 *Suppose that $M = M_2[I_r|D]$ is connected and simple such that D is an r by $m - r$ matrix with $m - r \geq 3$. If there exist distinct $h, k, \ell \in \{r+1, r+2, \ldots, m\}$ satisfying*

$$S(v_\ell) \cap S(v_h) \neq \emptyset,\ S(v_\ell) \cap S(v_k) \neq \emptyset,\ \textit{and}\ S(v_h) \cap S(v_k) = \emptyset, \tag{4}$$

then either $B(v_\ell + v_h + v_k) \cup \{e_\ell, e_h, e_k\} \in \mathcal{C}(M)$ (if $v_\ell + v_h + v_k \neq 0$), or $\{e_h, e_k, e_\ell\} \in \mathcal{C}(M)$ (if $v_\ell + v_h + v_k = 0$).

Proof Since M is simple, e_h, e_k, e_ℓ are mutually distinct non-zero vectors, and so if $v_\ell + v_h + v_k = 0$, then $\{e_h, e_k, e_\ell\} \in \mathcal{C}(M)$. Hence we assume that $\{e_h, e_k, e_\ell\} \notin \mathcal{C}(M)$. Again as M is simple, M contains no circuit of length at most 2, and so $\{e_h, e_k, e_\ell\} \in \mathcal{I}(M)$. For any partition of $\{e_h, e_k, e_\ell\}$ into two nonempty pats J_1 and J_2, (4) implies that $S(\sum_{i \in J_1} v_i) \cap S(\sum_{i \in J_2} v_i)$ contains either $S(v_\ell) \cap S(v_h)$ or $S(v_\ell) \cap S(v_k)$. Hence by Observation 2.7, Corollary 2.8 holds. □

As in Oxley (2011), for a basis B of M, for any $e \in E(M) - B$, we let $C_M(e, B)$ denote the fundamental circuit of e with respect to B. For the given basis $B = \{e_1, e_2, \ldots, e_r\}$, define a graph $H = H_B$ with $V(H)$ being the fundamental circuits of $e_{r+1}, \ldots, e_m$, with respect to B, such that two vertices of H are adjacent if and only if the corresponding fundamental circuits have a non-empty intersection. This graph H facilitates our arguments.

Observation 2.9 *A binary matroid $M = M_2[I_r|D]$ is connected if and only if M does not have any coloop and H_B is connected for any B. Or in another words, each of the following holds.*

(i) *For any $i \in \{1, 2, \ldots, r\}$, there must be a $j \in \{r+1, \ldots, m\}$ such that if $v_j = (x_1, x_2, \ldots, x_r)$, then $x_i = 1$.*
(ii) *If there exist distinct $i, j \in \{r+1, \ldots, m\}$ satisfying $S(v_i) \cap S(v_j) = \emptyset$, then there must be a $t_1, t_2, \ldots, t_k \in \{r+1, \ldots, m\} - \{i, j\}$, such that both $S(v_i) \cap S(v_{t_1}) \neq \emptyset$, $S(v_{t_1}) \cap S(v_{t_2}) \neq \emptyset, \ldots, S(v_{t_{k-1}}) \cap S(v_{t_k}) \neq \emptyset$, and $S(v_j) \cap S(v_{t_k}) \neq \emptyset$.*

Proof For sufficiency, we assume the validity of (i)-(ii) to show that M has only one component. Let $H = H_B$ denote the graph defined right before this observation. Condition (ii) indicates that H is connected. Let E_1 denote the component that contains the fundamental circuit of e_{r+1} with respect to the basis B. If $E_1 = E(M)$, then M is connected. Assume to the contrary, that there exists an element $e_t \in E(M) - E_1$.

If $t \in \{r+1, r+2, \ldots, m\}$, then as H is connected, there exists a sequence of fundamental circuits $C^1, C^2, \ldots, C^\ell$ with respect to B such that $C^1 = C_M(e_{r+1}, B)$ and $C^\ell = C_M(e_t, B)$, and such that $C^i \cap C^{i+1} \neq \emptyset$, for each $i = 1, 2, \ldots \ell - 1$. It follows that for each $i = 1, 2, \ldots \ell - 1$, elements in $C^i \cup C^{i+1}$ are in the same component of M. Thus the elements in C^ℓ, in particular e_t, must be in E_1, contrary to the assumption that $e_t \in E(M) - E_1$.

Hence we may assume that $t \in \{1, 2, \ldots, r\}$. By (i), there must be an index $j \in \{r+1, \ldots, m\}$ such that if $v_j = (x_1, x_2, \ldots, x_r)$, then $x_t = 1$. This implies that $e_t \in C_M(e_j, B)$. By the connectedness of H, we once again conclude that e_t must be in E_1, contrary to the assumption that $e_t \in E(M) - E_1$.

For necessity, by definition, M does not have any coloop. We use contradiction to show H_B is connected. Assume M is the minimum connected matroid such that H_B is disconnected for some B. Then H_B has two components, say H_1 and H_2. Similarly arguing as above, $M(H_1)$ and $M(H_2)$ are connected. Also $E(M(H_1)) \cap E(M(H_2)) = \emptyset$ and $E(M(H_1)) \cup E(M(H_2)) = E(M)$. The contradiction justifies this necessity. □

Observation 2.10 *In a binary matroid $M = M_2[I_r|D]$, we denote $D = (d_{ij})$ with $1 \le i \le r$ and $r+1 \le j \le m$; and let $w_i = (d_{i(r+1)}, d_{i(r+2)}, \ldots, d_{im})$ be the ith row of D. Each of the following holds.*

(i) *If for some $i \in \{1, 2, \ldots, r\}$, there is an $i' \in \{r+1, \ldots, m\}$ such that if $d_{ij} = 1$ if and only if $j = i'$, then $\{e_i, e_{i'}\} \in \mathcal{C}(M^*)$.*

(ii) *If there exist distinct $i, j \in \{1, 2, \ldots, r\}$ satisfying $w_i = w_j$, then then $\{e_i, e_j\} \in \mathcal{C}(M^*)$.*

(iii) *If there exist distinct $i, j, k \in \{1, 2, \ldots, m\}$ such that e_i, e_j, e_k belong to the same serial class of M, then $M/e_i = M_2[I_{r-1}|D_1]$, where D_1 is obtained from D by deleting the ith row of D, is also a simple matroid.*

(iv) *If there exist distinct $i, j \in \{1, 2, \ldots, m\}$ such that e_i, e_j belong to the same serial class of M, then $M/e_i = M_2[I_{r-1}|D_1]$, where D_1 is obtained from D by deleting the ith row of D, is also a connected matroid.*

Proof The justification of Observation 2.10 (i) and (ii) follow immediately from the fact that the dual of $M = M_2[I_r|D]$ is $M^* = M_2[D^T|I_{m-r}]$, in which every pair of identical columns form a cocircuit of M. The simpleness and the connectedness of $M/e_i = M_2[I_{r-1}|D_1]$ follow from Observation 2.6, and from Observation 2.9, respectively. □

Definition 2.11 For an integer $h > 0$, we have the following definitions.

(i) Let K_2^h be the loopless graph with 2 vertices and h parallel edges.

(ii) Let K_3P_3 be the loopless graph spanned by a 3-circuit $Z = u_1u_2u_3u_1$ such that $K_3P_3 - E(Z)$ is a path $u_1u_2u_3$. Thus the edge u_1u_3 is the only edge in K_3P_3 not lying in a 2-circuit. For any serial extension of $M(K_3P_3)$, let $[u_1u_3]$ denote the set of edges obtained by subdividing the edge $u_1u_3 \in E(K_3P_3)$.

(iii) Let $Z' = w_1w_2w_3w_4w_1$ denote a a 4-circuit. Define C_4M_2 to be the loopless multigraph spanned by Z' such that $C_4M_2 - E(Z')$ is a matching with edges $\{w_1w_2, w_3w_4\}$; and C_4P_4 to be the loopless graph spanned by Z' such that $C_4P_4 - E(Z')$ is a path $w_1w_2w_3w_4$. Thus the edge w_1w_4 is the only edge in C_4P_4 not lying in a 2-circuit. For any serial extension of $M(C_4P_4)$, let $[w_1w_4]$ denote the set of edges obtained by subdividing the edge $w_1w_4 \in E(C_4P_4)$.

(iv) Let L_5 denote the graph with $V(L_5) = \{u_1, u_2, u_3, z_1, z_2\}$ and $E(L_5) = \{u_1u_2, u_2u_3, u_3u_1, z_1u_1, z_1u_2, z_2u_2, z_2u_3\}$. For any serial extension of $M(L_5)$,

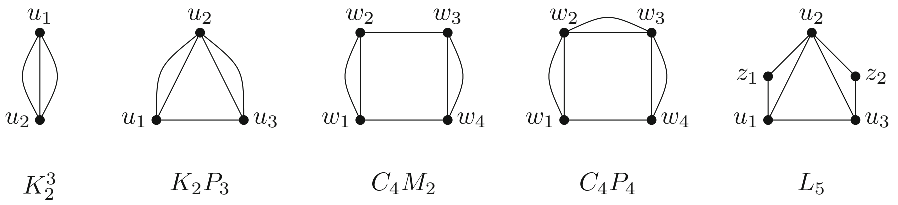

Fig. 1 Graphs in Definition 2.11

let $[u_1u_3]$ denote the set of edges obtained by subdividing the edge $u_1u_3 \in E(L_5)$ (Fig. 1).

By definition, both L_5 and C_4M_2 are serial extensions of K_3P_3. It is routine to verify the observations stated in Proposition 2.12 below.

Proposition 2.12 *We shall use the notation in Definition 2.11. For a given graph G, let $M = M(G)$ denote its cycle matroids.*

(i) *If $G \in \{K_2^2, K_2^3, K_2^4\}$, and e_0 is any edge in $E(G)$, or if $G = K_3P_3$ and $e_0 \in E(K_3P_3) - \{u_1u_3\}$, then $|\mathcal{C}_{M,e_0}| \leq 3$. If $G = K_3P_3$ and $e_0 = u_1u_3$, then $|\mathcal{C}_{M(K_3P_3),u_1u_3}| \geq 4$.*
(ii) *If $G \in \{K_2^2, K_2^3, K_2^4, K_3P_3, C_4M_2, K_4\}$, and e_0 is any edge in $E(G)$, or if $G = C_4P_4$ and $e_0 = w_1w_4$, then $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \leq 3$. If $G = C_4P_4$ and $e_0 \neq w_1w_4$, then $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \geq 4$.*
(iii) *If G is a member in $\{K_2^4, K_3P_3, C_4M_2\}$, and if G' is obtained from G by adding an edge joining two distinct vertices in G, then for any edge $e_0 \in E(G)$, $|\mathcal{C}_{M,e_0}| \geq 4$.*
(iv) *If G is a member in $\{K_2^4, K_3P_3, C_4M_2, C_4P_4, K_4\}$, and if G' is obtained from G by adding an edge joining two distinct vertices in G, then for any edge $e_0 \in E(G)$, $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \geq 4$.*
(v) *If $M \in \{M(K_4), F_7\}$, then for any $e \in E(M)$, $|\mathcal{C}_{M,e}| \geq 4$.*

In the next lemma, we will follow the language of Notation 2.5.

Lemma 2.13 *Let $r \geq 4$ be an integer and $M = M_2[I_r|D]$ be a connected simple binary matroid where D is an r by 3 matrix. Then M is isomorphic to $M(L_5)$ if each of the following holds.*

(i) *$S(v_{r+1}) \cap S(v_{r+3}) \neq \emptyset$ and $S(v_{r+2}) \cap S(v_{r+3}) = \emptyset$.*
(ii) *For any $\{e_i, e_j\} \in \mathcal{C}(M^*)$, M/e_i is not simple.*

Proof For $j = r+1, r+2, r+3$, denote $v_j = (x_1^j, x_2^j, \ldots, x_r^j)^T$. By (i), $S(v_{r+3}) \cap S(v_{r+2}) = \emptyset$, and so without loss of generality, we may assume that for some integers s, s_1, t, t_1 with $0 \leq s_1 \leq s < t \leq t_1 \leq r$, v_{r+1}, v_{r+2} and v_{r+3} satisfy the following:

$$x_1^{r+3} = x_2^{r+3} = \ldots x_s^{r+3} = 1 \text{ and } x_j^{r+3} = 0 \text{ if } j > s \text{ with } 2 \leq s \leq r-2,$$
$$x_t^{r+2} = x_{t+1}^{r+2} = \ldots x_r^{r+2} = 1 \text{ and } x_j^{r+2} = 0 \text{ if } j < t \text{ with } r-1 \leq t \leq r,$$
$$x_{s_1}^{r+1} = x_{s_1+1}^{r+1} = \ldots x_{t_1}^{r+1} = 1 \text{ and } x_j^{r+1} = 0 \text{ if } j < s_1 \text{ or } j > t_1$$

with $0 \leq s_1 \leq s < t \leq t_1 \leq r$.

Note that the assumed inequalities $2 \leq s \leq r-2$ and $r-1 \leq t \leq r$ follow from Observation 2.6, and the assumed inequalities $s_1 \leq s < t \leq t_1$ follow from Observation 2.9.

Claim 1 *We have these observations.*

(a) $0 \leq s_1 \leq s = 2$. *(By symmetry,* $t = r-1 \leq t_1 \leq r$.*)*
(b) $t = s+1$.
(c) $s = 2$, $t = 3$ *and* $r = 4$.

To justify Claim 1, we will use the fact $M^* = M_2[D^T|I_3]$ and Observation 2.10. If $s \geq 3$, then either $s_1 \geq 3$ and $\{e_1, e_2, e_{r+3}\}$ is contained in a serial class of M, or $s_1 \leq 2$ and $\{e_2, e_3\}$ is contained in a serial glass of M. In either case, by Observation 2.10, M/e_2 is simple, contrary to Lemma 2.13 (ii). Hence $s_1 \leq s \leq 2$. By Observation 2.6, $s = |S(v_{r+3})| \geq 2$ and so $s = 2$, and Claim 1(a) must hold.

If $t \geq s+2$, then $\{e_{s+1}, e_{r+1}\} \in \mathcal{C}^*(M)$. By Observation 2.9 and as $s_1 \leq s < t \leq t_1$, it follows by Observation 2.6 that M/e_2 is simple, contrary to Lemma 2.13 (ii). Hence Claim 1(b) must hold.

By Claim 1(a) and (b), we have $s = 2$, $t = 3$ and $r = 4$, and so (c) follows. This proves Claim 1.

As a consequence of of Claim 1(c), D must be one of the following matrices:

$$D \in \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \right\}. \tag{5}$$

It is routine to show that for any D in (5), $M = M_2[I_4|D]$ is always isomorphic to $M(L_5)$. □

By Observation 2.9, if $m = r+3$, the graph H_B is either a K_3 or a P_3. This gives us a bit more structural information of M. In the next lemma, we adopt the terms and notation in Definition 2.11.

Lemma 2.14 *Let M be a binary matroid with $r = r(M) > 0$ and $|E(M)| \geq 2$, and let $e \in E(M)$ be an arbitrary element. For any serial extension of $M(L_5)$, let $[u_1u_3]$ denote the set of edges obtained by subdividing the edge $u_1u_3 \in E(L_5)$. Each of the following holds.*

(i) *If M is loopless and coloopless with $|E(M)| \geq r(M)+5$, then $|\mathcal{C}(M) - \mathcal{C}_{M,e}| \geq 4$.*
(ii) *If M is connected and simple with $|E(M)| \geq r(M)+3$, then $|\mathcal{C}_{M,e}| \geq 4$ if and only if M is not isomorphic to a serial extension of $M(L_5)$ with $e \notin [u_1u_3]$.*
(iii) *If M is connected and simple with $|E(M)| \geq r(M)+4$, then $|\mathcal{C}(M) - \mathcal{C}_{M,e}| \geq 4$, unless M is a serial extension of $M(C_4P_4)$ and e is in the serial class obtained from subdividing the only edge in C_4P_4 that is not in a 2-circuit.*
(iv) *If M is connected and simple with $|E(M)| \geq r(M)+3$, then $|\mathcal{C}(M) - \mathcal{C}_{M,e}| \geq 3$, unless M is a serial extension of $M(K_3P_3)$ and e is in the serial class obtained from subdividing the only edge in K_3P_3 that not in a 2-circuit.*

Proof (i) Since M is coloopless, e is not a coloop and so there exists a basis $B \in \mathcal{B}(M)$ such that $e \notin B$. Let $e_1, e_2, e_3, e_4 \in E(M) - (B \cup e)$. Then the fundamental circuits $C_M(e_i, B)$, $1 \le i \le 4$, are all in $\mathcal{C}(M) - \mathcal{C}_{M,e}$, and so $|\mathcal{C}(M) - \mathcal{C}_{M,e}| \ge 4$. This proves (i).

In the proofs for (ii)–(iv), we assume that M is a binary connected simple matroid. Since M is connected, there exists a basis $B \in \mathcal{B}(M)$ such that $e \notin B$. Thus we may assume that for some r by $(m-r)$ binary matrix D, $M = M_2[I_r|D]$, $E(M) = \{e_1, e_2, \ldots, e_m\}$ such that e_i is the label of the ith column vector v_i of $[I_r|D]$ with $B = \{e_1, e_2, \ldots, e_r\}$ and $e \in \{e_{r+1}, \ldots, e_m\}$.

We are to argue by induction on $r = r(M)$ to prove (ii). Since M is simple and $|E(M)| \ge r+3$, we may assume that $r \ge 3$. If $r = 3$, then since M is simple, it follows by Observation 2.6 that $6 \le |E(M)| \le 7$, and so $M \in \{M(K_4), F_7\}$. Now by Proposition 2.12(v), for any $e \in E(M)$, $|\mathcal{C}_{M,e}| \ge 4$. Therefore, we assume that $r \ge 4$ and Lemma 2.14(ii) holds for smaller values of r.

Since L_5 is a serial extension of K_3P_3, it follows by Proposition 2.12 (i) that if M is isomorphic to a serial extension of $M(L_5)$ with $e \notin [u_1u_3]$, then $|\mathcal{C}_{M.e}| \le 3$. It remains to prove the sufficiency of (ii). In the proof for (ii), we may assume that $e = e_m$; and by Observation 2.9, there must be some j with $r+1 \le j \le m-1$ satisfying $S(v_m) \cap S(v_j) \ne \emptyset$. We may assume that $S(v_m) \cap S(v_{r+j}) \ne \emptyset$ for $1 \le j \le j_0 < m-r$. If $j_0 \ge 3$, then by Observation 2.7, $B(v_m) \cup \{e_m\}$, $B(v_m + v_{r+j}) \cup \{e_m, e_{r+j}\}$, $(1 \le j \le 3)$ are 4 distinct circuits of M containing e_m. Hence we assume that $j_0 \le 2$.

(ii-A) Suppose that $j_0 = 2$ and $m - r \ge 4$. Then for any j with $3 \le j < m-r$, $S(v_{r+j}) \cap S(v_m) = \emptyset$. By Observation 2.9, we may assume that $S(v_{r+3}) \cap S(v_m) = \emptyset$ and $S(v_{r+1}) \cap S(v_{r+3}) \ne \emptyset$. By Observation 2.7, $B(v_m) \cup \{e_m\}$, $B(v_m + v_{r+j}) \cup \{e_m, e_{r+j}\}$, $(1 \le j \le 2)$ are 3 distinct circuits of M containing e_m. By Corollary 2.8, either $B(v_m + v_{r+1} + v_{r+3}) \cup \{e_m, e_{r+1}, e_{r+3}\} \in \mathcal{C}_{M,e}$ or $\{e_m, e_{r+1}, e_{r+3}\} \in \mathcal{C}_{M,e}$. Thus in this case, $|\mathcal{C}_{M,e}| \ge 4$.

(ii-B) Suppose that $j_0 = 1$ and $m - r \ge 4$. Then $S(v_m) \cap S(v_{r+j}) = \emptyset$ for $j = 2, \ldots, m-r-1$. By Observation 2.9, we assume that $S(v_{r+1}) \cap S(v_{r+3}) \ne \emptyset$ and $S(v_{r+2}) \cap S(v_{r+3}) \ne \emptyset$. By Observation 2.7, $B(v_m) \cup \{e_m\}$, $B(v_m + v_{r+1}) \cup \{e_m, e_{r+1}\}$ are distinct circuits of M containing e_m. By Corollary 2.8, either $B(v_m + v_{r+1} + v_{r+3}) \cup \{e_m, e_{r+1}, e_{r+3}\} \in \mathcal{C}_{M,e}$ or $\{e_m, e_{r+1}, e_{r+3}\} \in \mathcal{C}_{M,e}$. To show that $|\mathcal{C}_{M,e}| \ge 4$, we need to find an additional circuit containing e_m.

If $S(v_{r+1}) \cap S(v_{r+2}) \ne \emptyset$, then by Corollary 2.8, either $B(v_m + v_{r+1} + v_{r+2}) \cup \{e_m, e_{r+1}, e_{r+2}\} \in \mathcal{C}_{M,e}$ or $\{e_m, e_{r+1}, e_{r+2}\} \in \mathcal{C}_{M,e}$. Hence $B(v_m) \cup \{e_m\}$, $B(v_m + v_{r+1}) \cup \{e_m, e_{r+1}\}$, either $B(v_m + v_{r+1} + v_{r+2}) \cup \{e_m, e_{r+1}, e_{r+2}\}$ or $\{e_m, e_{r+1}, e_{r+2}\}$, and either $B(v_m + v_{r+1} + v_{r+3}) \cup \{e_m, e_{r+1}, e_{r+3}\}$ or $\{e_m, e_{r+1}, e_{r+3}\}$ are in $\mathcal{C}_{M,e}$, and so $|\mathcal{C}_{M,e}| \ge 4$.

Assume that $S(v_{r+1}) \cap S(v_{r+2}) = \emptyset$ and $\{e_m, e_{r+1}, e_{r+3}\} \in \mathcal{C}_{M,e}$. Then $v_m + v_{r+1} + v_{r+3} = 0$. As $S(v_m) \cap S(v_{r+1}) \ne \emptyset$, $S(v_{r+1}) \cap S(v_{r+3}) \ne \emptyset$ and $S(v_m) \cap S(v_{r+3}) = \emptyset$, we must have $S(v_{r+1}) = S(v_m) \cup S(v_{r+3})$. It follows that $S(v_{r+1}) \cap S(v_{r+2}) \ne \emptyset$ as $S(v_{r+3}) \cap S(v_{r+2}) \subseteq S(v_{r+1}) \cap S(v_{r+2})$; and $v_m + v_{r+1} + v_{r+2} \ne 0$. By Corollary 2.8, $B(v_m) \cup \{e_m\}$, $B(v_m + v_{r+1}) \cup \{e_m, e_{r+1}\}$, $B(v_m + v_{r+1} + v_{r+2}) \cup \{e_m, e_{r+1}, e_{r+2}\}$ and $\{e_m, e_{r+1}, e_{r+3}\}$ are in $\mathcal{C}_{M,e}$. Thus $|\mathcal{C}_{M,e}| \ge 4$.

Assume that $S(v_{r+1}) \cap S(v_{r+2}) = \emptyset$ and $\{e_m, e_{r+1}, e_{r+3}\} \notin \mathcal{C}_{M,e}$. We are to apply Observe 2.7 to show that $B(v_{r+1}+v_{r+2}+v_{r+3}+v_m) \cup \{e_{r+1}, e_{r+2}, e_{r+3}, e_m\} \in \mathcal{C}_{M,e}$. Suppose we partition $\{r+1, r+2, r+3, m\}$ into two non-empty subsets J_1 and J_2 with $m \in J_1$. If $r+1 \in J_2$, then $S(\sum_{i \in J_1} v_i) \cap S(\sum_{i \in J_2} v_i)$ contains either $S(v_m) \cap S(v_{r+1})$; if $J_1 = \{r+1, m\}$, then $S(\sum_{i \in J_1} v_i) \cap S(\sum_{i \in J_2} v_i)$ contains either $S(v_{r+1}) \cap S(v_{r+3})$; if $\{r+1, m\} \subset J_1$ and $|\{r+2, r+3\} \cap J_1| = 1$, then $S(\sum_{i \in J_1} v_i) \cap S(\sum_{i \in J_2} v_i)$ contains either $S(v_{r+2}) \cap S(v_{r+3})$. In any case, $S(\sum_{i \in J_1} v_i) \cap S(\sum_{i \in J_2} v_i) \neq \emptyset$. It follows by Observation 2.7 that $B(v_m) \cup \{e_m\}$, $B(v_m + v_{r+1}) \cup \{e_m, e_{r+1}\}$, $B(v_m + v_{r+1} + v_{r+3}) \cup \{e_m, e_{r+1}, e_{r+3}\}$ and $B(v_{r+1}+v_{r+2}+v_{r+3}+v_m) \cup \{e_{r+1}, e_{r+2}, e_{r+3}, e_m\}$ are in $\mathcal{C}_{M,e}$. Thus $|\mathcal{C}_{M,e}| \geq 4$.

(ii-C) Suppose that $j_0 = 1$ and $m - r = 3$. Recall that $S(v_m) \cap S(v_{r+1}) \neq \emptyset$ and $S(v_m) \cap S(v_{r+2}) = \emptyset$. If M has a cocircuit $\{e_i, e_j\}$ such that M/e_i is simple, then by Observation 2.10, M/e_i is also a connected simple binary matroid with $r(M/e_i) < r(M)$ and $|E(M/e_i)| = r(M/e_i) + 3$. It follows by induction that $|\mathcal{C}_{M/e_i,e}| \geq 4$ if and only if M/e_i is not isomorphic to a serial extension of $M(L_5)$ with $e \in [u_1u_3]$. By Proposition 2.1, and since M is a serial extension of M/e_i, the conclusion of Lemma 2.14(ii) must hold. Hence we assume that for any $\{e_i, e_j\} \in \mathcal{C}(M^*)$, M/e_i is not simple. It follows by Lemma 2.13 that M is isomorphic to $M(L_5)$. This completes the proof for Lemma 2.14(ii).

To justify Lemma 2.14(iii) and (iv), we observe that

$$|\mathcal{C}(M(L_5))| \geq 4. \tag{6}$$

For a fixed element $e \in E(M)$, if $M - e$ is connected, then Lemma 2.14(iii) and (iv) follow by (6) and by applying Lemma 2.14(ii) to $M - e$. Therefore, we may assume that $M - e$ has connected components $M_1, M_2, \ldots, M_c$ with $c \geq 2$ such that

$$|E(M_1)| - r(M_1) \geq |E(M_2)| - r(M_2) \geq \cdots \geq |E(M_c)| - r(M_c).$$

Since M is connected, $r(M - e) = r(M)$. Thus $\sum_{i=1}^{c} |E(M_i)| = |E(M - e)| = |E(M)| - 1$ and $r(M - e) = r(M) = \sum_{i=1}^{c} r(M_i)$, and so $\sum_{i=1}^{c}(|E(M_i)| - r(M_i)) = |E(M)| - r(M) - 1$. Note that by matroid rank axioms, if for some i, $|E(M_i)| \geq r(M_i) + 1$, then $E(M_i) \in \mathcal{C}(M)$; and that by matroid circuit axioms, if for some i, $|E(M_i)| \geq r(M_i) + 2$, then $|\mathcal{C}(M_i)| \geq 3$. These, together with $|\mathcal{C}(M) - \mathcal{C}_{M,e}| = |\mathcal{C}(M - e)| = \sum_{i=1}^{c} |\mathcal{C}(M_i)|$, lead us to the following observations.

(iii-A) If $|E(M)| - r(M) \geq 5$, then $\sum_{i=1}^{c}(|E(M_i)| - r(M_i)) \geq 4$ and so $|\mathcal{C}(M) - \mathcal{C}_{M,e}| \geq 4$.

(iii-B) If $|E(M)| - r(M) = 4$, then as $\sum_{i=1}^{c}(|E(M_i)| - r(M_i)) = 3$, we conclude that either $|E(M_i)| - r(M_i) = 1$ for $i = 1, 2, 3$ and $|E(M_i)| - r(M_i) = 0$ for $i \geq 4$, whence M is isomorphic to a serial extension of $M(C_4P_4)$ with e being in the serial class obtained from subdividing the only edge in C_4P_4 that is not in a 2-circuit; or $|E(M_1|) - r(M_1) = 2$, $|E(M_2)| - r(M_2) = 1$, and $|E(M_i)| - r(M_i) = 0$ for $i \geq 3$, whence $|\mathcal{C}(M) - \mathcal{C}_{M,e}| \geq |\mathcal{C}(M_1)| + |\mathcal{C}(M_2)| \geq 3+1 = 4$; or $|E(M_1)| - r(M_1) = 3$, and $|E(M_i)| - r(M_i) = 0$ for $i \geq 3$, whence by applying Lemma 2.14(ii) to M_1 and by (6), $|\mathcal{C}(M) - \mathcal{C}_{M,e}| \geq |\mathcal{C}(M_1)| \geq 4$.

(iv) If $|E(M)| - r(M) = 3$, then as $\sum_{i=1}^{c}(|E(M_i)| - r(M_i)) = 2$, we conclude that either $|E(M_i)| - r(M_i) = 1$ for $i = 1, 2$ and $|E(M_i)| - r(M_i) = 0$ for $i \geq 3$, whence whence M is isomorphic to a serial extension of $M(K_3P_3)$ with e being in the serial class obtained from subdividing the only edge in K_3P_3 that is not in a 2-circuit; or $|E(M_1)| - r(M_1) = 2$, and $|E(M_i)| - r(M_i) = 0$ for $i \geq 2$, whence $|\mathcal{C}(M) - \mathcal{C}_{M,e}| \geq |\mathcal{C}(M_1)| \geq 3$. This proves the lemma.

□

2.2 Graphic matroids

We in this subsections study the graphic matroid memberships of $\mathcal{F}_1$ and $\mathcal{F}_2$. Let $G(e_0)$ be a graph with a distinguished edge $e_0 \in E(G)$, and let $M(e_0) = M(G(e_0))$ denote the cycle matroid of G rooted at e_0. Following (Oxley 2011), a matroid M is **planar** if for some planar graph G, $M = M(G)$ is the cycle matroid of G. The goal of this subsection is to determine all rooted planar matroids $M(e_0)$ such that $|\mathcal{C}_{M,e_0}| \leq 3$, as well as all rooted planar matroids $M(e_0)$ such that $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \leq 3$.

Definition 2.15 Let $M = M(e_0)$ be a connected rooted matroid with $r(M) \geq 1$.

(i) The **serial reduction** (a **rooted serial reduction**, respectively) of M is a matroid obtained from M by repeatedly taking serial contractions (serial contractions of elements in $M - e_0$, respectively) until the contraction either is isomorphic to $U_{1,2}$ or has no more 2-cocircuit left.

(ii) A rooted matroid $M(e_0)$ is a **rooted serial extension** of $N(f_0)$ if M is a serial extension of N and e_0 is in the serial class of M that contains f_0.

(iii) If $r(M) = 1$ or if $r(M) \geq 2$ and M contains no 2-cocircuits, then M is the serial reduction of itself. In this case, we said that M is **serially reduced**.

Theorem 2.16 *Let G be a planar graph with $\kappa(G) \geq 2$, and let $M = M(G)$. Each of the following holds.*

(i) *For some $e_0 \in E(G)$, $|\mathcal{C}_{M,e_0}| \leq 3$ if and only if the serial reduction of M is isomorphic to $M(H)$, where H is a member in $\{K_2^2, K_2^3, K_2^4\}$ and with e_0 being an arbitrary edge in $E(H)$, or $H = K_3P_3$, with e_0 being any edge of K_3P_3 lying in a 2-circuit.*

(ii) *If for some $e_0 \in E(G)$, $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \leq 3$ if and only if the serial reduction of M is isomorphic to $M(H)$, where H is a member in $\{K_2^2, K_2^3, K_2^4, K_3P_3, K_4\}$ with e_0 being an arbitrary edge in $E(H)$, or $H = C_4P_4$, with e_0 being the only edge not lying in a 2-circuit.*

Proof By Propositions 2.1 and 2.12 , it suffices to prove the necessity in (i) and (ii). Let M' denote the serial reduction of $M = M(G)$. As a serial contraction in the cycle matroid $M(G)$ amounts to contracting one edge in an edge cut of size 2, we have $M' = M(H)$ is also a cycle matroid of some planar graph H, where either $H = K_2^2$ or H is 3-edge-connected. If $H = K_2^2$, then done. Hence we assume that $H \neq K_2^2$. Hence $\kappa'(H) \geq 3$. Since serial contraction does not reduce connectivity, we assume that $\kappa(H) \geq 2$ as well.

(i) Suppose that for some $e_0 \in E(H)$, $|\mathcal{C}_{M',e_0}| \le 3$. By Lemma 2.2 and Proposition 2.12(v), we may assume that H does not have a K_4-minor. Let Z_0 be a shortest circuit in H with $e_0 \in Z_0$. Since Z_0 is shortest, every chord of Z_1 in H is parallel to an edge of Z_0. Let

$$s = |Z_0|, e_0 = v_s v_1 \text{ and } Z_0 - e_0 = v_1 v_2 ... v_s \text{ denote the } (v_1, v_s)\text{-path.}$$

If $3 \ge |V(H)| \ge |Z_0| \ge 2$, then by the assumption of $|\mathcal{C}_{M',e_0}| \le 3$ and by Proposition 2.12 (i) and (iii), either $H \in \{K_2^3, K_2^4\}$ with e_0 being any edge of H, or $H = K_3 P_3$ with e_0 being any edge of $K_3 P_3$ lying in a 2-circuit.

Now we assume that $|V(H)| \ge 4$.

Claim 1 $|V(H)| = s$. *We may assume that* $|V(H)| > s$. *Let* $V(H) - V(Z_0) = w_1, w_2, \ldots, w_t$. *Then* $t \ge 1$. *As* $\kappa'(H) \ge 3$ *and* $\kappa(H) \ge 2$, *for each* i *with* $1 \le i \le t$, *there exist three edge-disjoint paths* P_1^i, P_2^i *and* P_3^i, *internally vertex disjoint from* $V(Z_0)$, *joining* w_i *to at least two distinct vertices in* $V(Z_0)$. *Since* H *is* K_4*-minor-free,* $|\{z_1^i, z_2^i, z_3^i\}| \le 2$*; since* $\kappa(H) \ge 2$, *we can choose these path so that* $|\{z_1^i, z_2^i, z_3^i\}| \ge 2$. *Therefore, we may assume that* $z_2^i = z_3^i$. *Let* P_0 *be the* (z_1^1, z_2^1)*-path in* Z_0 *that contains* e_0. *Since* P_1^1, P_2^1 *and* P_3^1 *are edge-disjoint paths, it follows that for each* $j \in \{2, 3\}$, *there is a circuit* $Z^j \subseteq P_0 \cup P_1^1 \cup P_j^1$ *containing* e_0.

If $t \ge 2$, *then there exists a circuit* Z' *in* H, *containing* e_0 *and using at least one edge in* $P_1^2 \cup P_2^2 \cup P_3^2 - (Z^2 \cup Z^3)$. *It follows that* Z_0, Z', Z^2, Z^3 *are 4 circuits in* H *containing* e_0, *contrary to* $|\mathcal{C}_{M,e_0}| \le 3$. *Thus we must have* $t = 1$. *Since* $s + t = |V(H)| \ge 4$, *we must have* $s = 3$, *and so there exists a vertex* $z \in V(Z_0) - \{z_1^1, z_2^1\}$. *As* $\kappa'(H) \ge 3$, *there must be an edge* $e' \in E(H) - (Z_0 \cup Z^2 \cup Z^3)$ *incident with* z. *Since* $\kappa(H) \ge 2$, *there must be a circuit* Z'' *containing both* e_0 *and* e', *and so* Z_0, Z'', Z^2, Z^3 *are 4 circuits in* H *containing* e_0, *contrary to* $|\mathcal{C}_{M,e_0}| \le 3$.

Claim 2 $s \in \{2, 3\}$. *If* $s \ge 4$. *Since* $\delta(H) \ge 3$, *each* v_i, $1 \le i \le s-1$, *is incident with an edge* e_i *in* $E(H) - Z_0$. *Every* e_i *should be parallel to an edge of* Z_0, *and there are at least two such* e_i'*s, contrary to the assumption of* $|\mathcal{C}_{M',e_0}| \le 3$.

(ii) We argue by induction on $|E(H)|$ *to show that Theorem* 2.16(ii) *must hold. If* $|E(H)| = 2$, *then we must have* $H = K_2^2$. *We now assume that* $|E(H)| > 2$ *and Theorem* 2.16(ii) *holds for graphs with fewer edges. Pick an edge* $x \in E(G)$ *and* $x \neq e_0$. *Let* $M'' = M(H - x)$. *Since* $\kappa'(H) \ge 3$, *then* M'' *is connected. By induction,* $H - x \in \{K_2^2, K_2^3, K_2^4, K_3 P_3, K_4\}$ *with* e_0 *being an arbitrary edge in* $E(H - x)$, *or* $H - x = C_4 P_4$, *with* e_0 *being the only edge not lying in a 2-circuit. Since* $|\mathcal{C}(M(H))| - |\mathcal{C}_{M(H),e_0}| \le 3$, *by some routine checking,* H *has to be a member in* $\{K_2^2, K_2^3\}$. □

2.3 Binary matroids

Let $\{f, f'\}$ be a 2-circuit of a matroid L and let $M = L - f'$. We denote $L = M^{+,f}$ and call L the **parallel extension of** M **at** f.

The main purpose of this subsection is to characterize all rooted binary matroids $M(e_0)$ with $|\mathcal{C}_{M,e_0}| \le 3$, as well as all rooted binary matroids $M(e_0)$ with $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \le 3$.

Let G be a connected graph. If X, Y are subsets of $V(G)$, then following the notation of Bondy and Murty (2008), define

$$[X, Y] = \{xy \in E(G) : x \in X \text{ and } y \in Y\}, \text{ and } \partial_G(X) = [X, V(G) - X].$$

Thus $[X, Y]$ is a minimal edge cut if and only if $X \cap Y = \emptyset$ and both $G[X]$ and $G[Y]$ are connected subgraphs of G. Let $v \in V(G)$ be a vertex. Define $E_G(v) = [\{v\}, V(G) - \{v\}]$. Let $M = M(G)$ be the cycle matroid of G. If G is 2-connected, then every edge cut $[X, V(G) - X]$ with both $G[X]$ and $G - X$ being connected is a cocircuit of $M(G)$.

Throughout the rest of this section, we define

$$\mathbb{N} = \{F_7, M^*(K_5), M(K_5), (K_{3,3}), M^*(K_{3,3})\}.$$

By definition, every matroid in $\mathbb{N} \cup \{F_7^*\}$ is serially reduced, and contains K_4 as a minor. The next theorem is well known.

Theorem 2.17 (Kuratowski 1930 and Wagner 1937, see also Theorem 5.2.5 of Oxley 2011) *A binary matroid M is in $EX(\mathbb{N} \cup \{F_7^*\})$ if and only if $M = M(G)$ is a cycle matroid of a planar graph G.*

Lemma 2.18 *Let M be a connected matroid, N be a minor of M and $e_0 \in E(N)$. Each of the following holds.*

(i) *If $M \in \mathbb{N} \cup \{F_7^*\}$, then $|\mathcal{C}_{M,e_0}| \geq 4$.*
(ii) *If $M \in \mathbb{N}$, then $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \geq 4$.*
(iii) *If a rooted binary matroid $M(e_0)$ contains a rooted minor $N(e_0) \in \mathbb{N} \cup \{F_7^*\}$, then $M(e_0) \notin \mathcal{F}_1$; if a rooted binary matroid $M(e_0)$ contains a rooted minor $N(e_0) \in \mathbb{N}$, then $M(e_0) \notin \mathcal{F}_2$.*

Proof For any $M \in \mathbb{N} \cup \{F_7^*\}$, we have $|E(M)| - r(M) \geq 3$. Hence Lemma 2.14 implies both Lemma 2.18(i) and (ii). Lemma 2.18(iii) follows from Lemma 2.2. □

Lemma 2.19 *If M is a connected matroid and $\{f, f'\} \in \mathcal{C}(M^*)$, then M/f is also connected.*

Proof Let $G(M)$ denote the circuit graph of M. Then it is known that a coloopless matroid M is connected if and only if $G(M)$ is a connected graph. By a result of Li and Liu (2008) (see Lemma 3.3(ii) in Section 3), $G(M) = G(M/f)$ and so M/f is connected if and only if M is connected. □

Proposition 2.20 *Define $\mathbb{N}' = \{M(K_2^4), M(K_3P_3), M(K_4), F_7^*\}$. Let $r \geq 3$ be an integer and define $\mathcal{F}(r) = \{M : M$ is a connected simple binary matroid with $r(M) = r$ and $|E(M)| = r(M) + 3\}$. Define $A = [I_r|D]$, where D is an (0,1)-matrix of dimension r by 3. We shall adopt the notation in Notation 2.5 and so for $1 \leq i \leq m$, e_i is the label of the ith column vector v_i of $[I_r|D]$. For a fixed matroid $M \in \mathcal{F}(r)$, we have the following observations.*

(i) *$M = M_2[A]$ for some $(0,1)$-matrix D with $B = \{e_1, e_2, \ldots, e_r\}$ being a basis of M.*
(ii) *For any $N \in \mathcal{F}(r)$, N is serially reduced if and only if $r \le 4$ and D^T does not have a row vector with at most one nonzero entry and does not have two identical columns.*
(iii) *$M(K_4)$ and F_7^* are the only serially reduced matroids in $\cup_{r\ge3}\mathcal{F}(r)$.*
(iv) *If $S(v_i) \cap S(v_j) = \emptyset$ holds for some distinct $i, j \in \{r+1, r+2, r+3\}$, then either $\{e_{r+1}, e_{r+2}, e_{r+3}\} \in \mathcal{C}(M)$ or M is not serially reduced.*
(v) *Every matroid $M \in \cup_{r\ge3}\mathcal{F}(r)$ is a serial extension of a matroid in $\mathbb{N}'$.*
(vi) *For any $e \in E(M)$, if $M(e)$ is not a serial extension of $M(K_3P_3)(e_0)$ where e_0 is the only edge in K_3P_3 lying in a single element parallel class, then $|\mathcal{C}(M) - \mathcal{C}_{M,e}| = 3$.*
(vii) *Let $M \in \mathcal{F}(r)$ and M^+ be a single parallel extension of M. Then for any $e_0 \in E(M^+)$, $|\mathcal{C}(M^+) - \mathcal{C}_{M^+,e_0}| \ge 4$.*

To justify (ii), as $N = M_2[I_r|D]$, we have $N^* = M_2[D^T|I_3]$. Since N is connected, N^* is also connected and so N^* is loopless. It follows that N^* does not have a zero column. By definition, N is not serial educed if and only if N^* has a circuit of size 2, which amounts to that $[D^T|I_3]$ has two identical columns. As $[D^T|I_3]$ is a $(0,1)$-matrix of dimension 3 by $r+3$ without a zero column, we observe that $[D^T|I_3]$ does not have two identical columns only if $r \le 4$ and so (ii) must hold.

We apply (ii) to justify (iii), and assume that M is serially reduced and $|E(M)| = r+3$ with $r \in \{3,4\}$. By (ii), the matrix D does not have a row with only one nonzero entry, and does not have two identical rows. By Observation 2.9, we may assume without loss of generality that $1, 2 \in S(v_{r+1})$, and subject to $1, 2 \in S(v_{r+1})$, $|S(v_1)|$ is maximized. If $r = 3$, then

$$D \in \left\{ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right\} \tag{7}$$

and so it is routine to show that M is isomorphic to $M(K_4)$. If $r = 4$, then $|E(M)| = r+3 = 7$. Since $[D^T|I_3]$ is a 3 by 7 matrix without an all zero entry column, it follows by definition that $M^* = F_7$, and so $M = F_7^*$.

To justify (iv), we may assume that $S(v_{r+1}) \cap S(v_{r+3}) = \emptyset$. Thus by Observation 2.9, $S(v_{r+2}) \cap S(v_{r+1}) \ne \emptyset$ and $S(v_{r+2}) \cap S(v_{r+3}) \ne \emptyset$. If there exists an $i \in S(v_{r+1}) - S(v_{r+2})$, then the ith component of v_{r+1} is the only nonzero entry of the ith row of the matrix D. It follows by Observation 2.10 that $\{e_i, e_{3+i}\}$ is a 2-cocircuit of M. Similarly, if $S(v_{r+3}) - S(v_{r+2}) \ne \emptyset$ or if $S(v_{r+2}) - (S(v_{r+1}) \cup S(v_{r+3})) \ne \emptyset$, then M contains a 2-cocircuit and so M is not serially reduced. Thus we may assume that $S(v_{r+2}) = S(v_{r+1}) \cup S(v_{r+3})$, whence $v_{r+1} + v_{r+2} + v_{r+3} = 0$ and so $\{e_{r+1}, e_{r+2}, e_{r+3}\} \in \mathcal{C}(M)$. This proves (iv).

We are to justify (v). Let $M \in \cup_{r\ge3}\mathcal{F}(r)$. By (i), $M = M_2[I_r|D]$. Let M' denote the serial reduction of M. We argue by induction on $r(M)$ to $M' \in \mathbb{N}'$. By (iii), M is not serially reduced, and so there must be a 2-cocircuit $\{f, f'\} \in \mathcal{C}(M^*)$. If $r = 3$, then M/f is a connected matroid with $r(M/f) = 2$ and $|E(M/f)| - r(M/f) = 3$, which must be the cycle matroid of a graph H with $|V(H)| = 3$. It follows that either

$M' = M(K_3P_3) \in \mathbb{N}'$; or H is spanned by a K_3 with 6 edges and a vertex of degree 2, whence $M' = M(K_2^4) \in \mathbb{N}'$. Hence we assume that $r \geq 4$. If there exists a 2-cocircuit $\{f, f'\} \in \mathcal{C}(M^*)$ such that M/f is simple, then as by Lemma 2.19, M/f is connected, we have $M/f \in \cup_{r\geq 3}\mathcal{F}(r)$. Thus by induction, the serial reduction of M/f, (and so M'), must be in $\mathbb{N}'$. Therefore, we assume that

$$r \geq 4 \text{ and, if } \{f, f'\} \in \mathcal{C}(M^*), \text{ then } M/f \text{ is not simple.} \tag{8}$$

Then M/f has two parallel elements f', f'' and $(M/f) - f'$ is simple and connected. Also $|E((M/f) - f')| - r((M/f) - f') = 2$. Then $(M/f) - f'$ is a simple connected matroid of corank 2. Hence $(M/f) - f'$ is a serial extension of $M(K_2^3)$ without parallel elements. Therefore M is a serial extension of $M(K_2^4)$ or $M(K_3P_3)$.

To justify (vi), we apply Lemma 2.14(iv) to obtain that $|\mathcal{C}(M) - \mathcal{C}_{M,e}| \geq 3$. To see that $|\mathcal{C}(M) - \mathcal{C}_{M,e}| < 4$, we again assume that $B \in \mathcal{B}(M - e) \subset \mathcal{B}(M)$ and so $e \in \{e_{r+1}, e_{r+2}, e_{r+3}\}$. We further assume that $e = e_{r+3}$. For each $C \in \mathcal{C}(M) - \mathcal{C}_{M,e_{r+3}}$, $C - B \neq \emptyset$ and so either $\{e_{r+1}, e_{r+2}\} \cap C = \{e_{r+1}\}$, or $\{e_{r+1}, e_{r+2}\} \cap C = \{e_{r+2}\}$ or $\{e_{r+1}, e_{r+2}\} \cap C = \{e_{r+1}, e_{r+2}\}$. Accordingly, $C \in \{B(v_{r+1}) \cup \{e_{r+1}\}, B(v_{r+2}) \cup \{e_{r+2}\}, B(v_{r+1} + v_{r+2}) \cup \{e_{r+1}, e_{r+2}\}\}$. This proves (vi).

To prove (vii), let $e \in E(M^+) - E(M)$. Then there exists an $e' \in E(M)$ such that $\{e, e'\} \in \mathcal{C}(M^+)$. If $\{e_0, e_0'\} \in \mathcal{C}(M^+)$, then by Lemma 2.14(iv), there are three circuits in $\mathcal{C}(M^+ - \{e_0, e_0'\})$. These, together with a circuit using in $\mathcal{C}(M^+ - e_0)$ using e_0', implies $|\mathcal{C}(M^+) - \mathcal{C}_{M^+,e_0}| \geq 4$. If $\{e_0, e_0'\} \notin \mathcal{C}(M^+)$, then we may assume that $e' \in E(M - e_0)$ and $\{e, e'\} \in \mathcal{C}(M^+)$. Thus by Lemma 2.14(iv), there are three circuits in $\mathcal{C}(M - \{e_0\}) = \mathcal{C}(M^+ - \{e_0, e\})$. These, together with $\{e, e'\} \in \mathcal{C}(M - e_0)$, implies $|\mathcal{C}(M) - \mathcal{C}_{M,e_0}| \geq 4$. This proves (vii).

Definition 2.21 Suppose that N is a minor of M such that N is serially reduced. A minor L of M is a **maximum serial extension of N in M** if N is a serial reduction of L with $|E(L)|$ maximized. We similarly define maximum rooted serial extensions in a rooted matroid.

Define $\mathcal{M}_1 = \{M(K_2^2), M(K_2^3), M(K_2^4), M(K_3P_3)\}$ and $\mathcal{M}_2 = \{M(K_2^2), M(K_2^3), M(K_2^4), M(K_3P_3), M(K_4), M(C_4P_4), F_7^*\}$.

Theorem 2.22 *Let M be a binary matroid. Each of the following holds.*

- (i) *There exists an $e_0 \in E(M)$ satisfying $|\mathcal{C}_{M,e_0}| \leq 3$ if and only if the rooted serial reduction of $M(e_0)$ is isomorphic either to a member in $\mathcal{M}_1 - \{M(K_3P_3)\}$ with $e_0 \in E(M)$; or to $M(K_3P_3)$ with e_0 being any edge of K_3P_3 lying in a 2-circuit.*
- (ii) *There exists an $e_0 \in E(M)$ satisfying $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \leq 3$ if and only if the rooted serial reduction of $M(e_0)$ is isomorphic either to a member in $\mathcal{M}_2 - \{M(C_4P_4)\}$ with $e_0 \in E(M)$; or to $M(C_4P_4)$ with e_0 being the only edge not lying in a 2-circuit.*

Proof The sufficiencies of both (i) and (ii) follow from Proposition 2.1(ii), Proposition 2.12 and Proposition 2.20(vi). It remains to show the necessities.

Assume that M is a binary matroid with $|\mathcal{C}_{M,e_0}| \leq 3$. By Lemma 2.18, $M(e_0) \in EX(\mathbb{N} \cup \{F_7^*\})$, and so by Theorem 2.17, M is isomorphic to the cycle matroid $M(G)$

for a planar graph G. By Theorem 2.16, the rooted serial reduction of $M(e_0)$ is isomorphic either to a member in $\{M(K_2^2), M(K_2^3), M(K_2^4)\}$ with $e_0 \in E(M)$; or to $M(K_3P_3)$ with e_0 being any edge of K_3P_3 lying in a 2-circuit. This proves the necessity of (i).

Assume that M is a binary matroid with $|\mathcal{C}(M)| - |\mathcal{C}_{M,e_0}| \leq 3$. By Lemma 2.18, $M(e_0) \in EX(\mathbb{N})$. Suppose that M contains a minor isomorphic to F_7^*. If $M \in \cup_{r\geq 3}\mathcal{F}(r)$, then by Proposition 2.20(v), M is a serial extension of a matroid in $\mathbb{N}' = \{M(K_2^4), M(K_3P_3), M(K_4), F_7^*\}$. Thus we conclude that if F_7^* is a minor of M, then M is a serial extension of F_7^*. Now assume that $M(e_0) \in EX(\mathbb{N} \cup F_7^*)$. Then by Theorem 2.17, M is isomorphic to the cycle matroid $M(G)$ for a planar graph G. By Theorem 2.16, the rooted serial reduction of $M(e_0)$ is isomorphic either to a member in $\{M(K_2^2), M(K_2^3), M(K_2^4), M(K_3P_3), M(K_4), F_7^*\}$ with $e_0 \in E(M)$; or to $M(C_4P_4)$ with e_0 being the only edge not lying in a 2-circuit. This proves the necessity of (ii). □

3 Application to 1-Hamiltonian circuit graphs of matroids

There have been many studies on the properties of graphs arising from matroids. In Tutte (1965), Tutte defined a graph $C(M)$ of a matorid M. The vertices of $C(M)$ are the circuits of M, where the two vertices in $C(M)$ are adjacent if and only if they are distinct circuits of the same connected line. Tutte (1965) showed that a matroid M is connected if and only if $C(M)$ is a connected graph. Maurer (1973a, b) defined the base graph of a matroid. The vertices are the bases of M and two vertices are adjacent if and only if the symmetric difference of these two bases is of cardinality 2. The graphical properties of the base graph of a matroid are discussed in Maurer (1973a, b). Alspach and Liu (1989) studied the properties of paths and circuits in base graphs of matroids. The connectivity of the base graph of matroids is investigated by Liu (1988, 1990). The graphical properties of the matroid base graphs have also been investigated by many other researchers, as seen in Harary and Plantholt (1989), Holzman and Harary (1972), Li and Liu (2004), Liu and Zhang (2005), among others.

Li and Liu (2007, 2008, 2010) initiated the investigation of graphical properties of matroid circuits graphs. Let M be a matroid, and let $k > 0$ be an integer. The **circuit graph** $G(M)$ of M has vertex set $V(G(M)) = \mathcal{C}(M)$. Two vertices $Z, Z' \in \mathcal{C}(M)$ are adjacent in $G(M)$ if and only if $|Z \cap Z'| \geq 1$. For notational convenience, for a circuit $Z \in \mathcal{C}(M)$, we shall use Z to denote both a vertex in $G(M)$ and a circuit (also as a subset of $E(M)$) of M.

In their studies Li and Liu (2007, 2008, 2010), they proved that $G(M)$ possesses quite good graphical connectivity properties. A recent study on the connectivity of certain spanning subgraphs of $G(M)$ is done in Xu et al. (2012).

Theorem 3.1 *Let M be a connected matroid with $|\mathcal{C}(M)| \geq 3$ and rank $r(M)$, and let $G = G(M)$ be the circuit graph of M. Each of the following holds.*

(i) (Li and Liu 2010) $\kappa(G) \geq 2(|E(M)| - r(M) - 1)$.
(ii) (Li and Liu 2007) *G is edge-pancyclic. That is, for any edge $e \in E(G)$ and for any integer ℓ with $3 \leq \ell \leq |V(G)|$, G contains a circuit C_ℓ containing e with length ℓ.*

(iii) (Li and Liu 2008) *For any edge* $e \in E(G)$, *G has two Hamilton circuits* Z' *and* Z'' *such that* Z' *contains e and* Z'' *does not contain e.*
(iv) (Liu and Li 2008) *For any distinct vertices* $u, v \in V(G)$, *and for any integer* ℓ *with* $2 \le \ell \le |V(G)| - 1$, *G has an* (u, v)*-path of length* ℓ. *That is, G is pan-connected. Consequently, G is hamiltonian with* $\kappa(G) \ge 3$.

For an integer $s \ge 0$, a graph G is ***s*-hamiltonian** if for any subset $S \subset V(G)$ with $|S| \le s$, $G - S$ is hamiltonian. Motivated by Theorem 3.1, the main purpose of this section is to investigate the conditions to warrant the circuit graph of a binary matroid to be 1-hamiltonian.

Throughout this section, M denotes a matroid with $|\mathcal{C}(M)| \ge 4$, and $G = G(M)$ denotes the circuit graph of M. The main goal of this section is to prove that the circuit graph of every connected binary matroid M is 1-hamiltonian. The first subsection below is devoted to developing some useful tools for the arguments; and the main result will be proved in the second subsection.

3.1 Lemmas

In this section, we will develop some lemmas to be utilized in the arguments of the next subsection, in which the main result of this section will be proved. For two sets X and Y, define the **symmetric difference of** X **and** Y as

$$X \triangle Y = (X \cup Y) - (X \cap Y).$$

Lemma 3.2 *Let M be a loopless matroid with* $|E(M)| \ge 2$.

(i) *(Strong circuit elimination, Page 15 of Oxley (2011)) Let* $C_1, C_2 \in \mathcal{C}(M)$ *be distinct circuits. If* $e \in C_1 \cap C_2$ *and* $f \in C_1 - C_2$, *then there exists* $C_3 \in \mathcal{C}(M)$ *such that* $f \in C_3 \subseteq (C_1 \cup C_2) - e$.
(ii) *If* $|E| \le 3$, *then* $M \in \{U_{1,3}, U_{2,3}\}$ *and so* $|\mathcal{C}(M)| \le 3$.
(iii) *Suppose that* $|E| = 4$. *Then* $|\mathcal{C}(M)| \ge 4$ *if and only if* $M \in \{U_{1,4}, U_{2,4}\}$.

Proof It suffices to assume to prove (ii) and (iii). Let $r = r(M)$. As M is connected and $|E| \ge 2$, M contains at least one circuit and so $1 \le r \le \max\{1, |E| - 1\}$.

Assume first that $|E| \le 3$. If $r = 1$, then $M = U_{1,3}$ and so $|\mathcal{C}(M)| = 3$. If $r = 2$, then $M = U_{2,3}$ and so $|\mathcal{C}(M)| = 1$. This justifies (ii).

To prove (iii), we first observe that if $M \in \{U_{1,4}, U_{2,4}\}$, then $|\mathcal{C}(M)| \ge 4$. Now we assume that $|\mathcal{C}(M)| \ge 4$. If $r = 1$, then $M = U_{1,4}$ and so $|\mathcal{C}(M)| = 6$. If $r = 3$, then $M = U_{3,4}$ and so $|\mathcal{C}(M)| = 1$. Hence we assume that $r = 2$. If M contains no circuit of size 2, then $M = U_{2,4}$ and so $|\mathcal{C}(M)| = 4$. Thus we assume that M has a 3-circuit C. Then M must be a single parallel extension of $U_{2,3}$ and so $|\mathcal{C}(M)| = 3$. $\square$

Lemma 3.3 (Li and Liu 2008) *Let M be a matroid,* $e \in E(M)$, $V_1 = \mathcal{C}(M - e)$ *and* $V_2 = \mathcal{C}(M) - \mathcal{C}(M - e)$. *Each of the following holds.*

(i) *The circuit graph of* $M - e$ *is a subgraph of G induced by* V_1, *and the subgraph of G induced by* V_2 *is a complete subgraph of G.*
(ii) *If* $\{e', e''\} \in \mathcal{C}(M^*)$, *then* $G(M) = G(M/e')$.

(iii) *Suppose that $e \in E(M)$ is an element such that $M - e$ is connected, If $|V_1| \geq 2$, then for any $Z_1Z_2 \in E(G)$, there exists a 4-circuit $\Gamma = Z_1Z_2Z_3Z_4Z_1$ in G such that $|E(\Gamma) \cap E(G_1)| \geq 1$, $|E(\Gamma) \cap E(G_2)| \geq 1$ and both $Z_1Z_2, Z_2Z_3, Z_3Z_1 \in E(G)$.*

We need a slightly stronger version of Lemma 3.3(iii) for binary matroids, as stated in Lemma 3.4 below.

Lemma 3.4 *Let M be a connected binary matroid, $G = G(M)$ be the circuit graph of M. For a fixed element $e \in E(M)$, let $V_1 = \mathcal{C}(M - e)$ and $V_2 = \mathcal{C}(M) - \mathcal{C}(M - e)$, and define $G_1 = G[V_1]$ and $G_2 = G[V_2]$. If $M - e$ is connected, and both $|V_1| \geq 3$ and $|V_2| \geq 4$, then for any $Z_0 \in V(G)$ and for any $Z_1Z_2 \in E(G - Z_0)$, there exists a 4-circuit $\Gamma = Z_1Z_2Z_3Z_4Z_1$ in $G - Z_0$ such that $|E(\Gamma) \cap E(G_1)| = 1$ and $|E(\Gamma) \cap E(G_2)| = 1$.*

Proof Let $Z_0 \in V_1$, and $Z_1Z_2 \in E(G - Z_0)$. We shall show that existence of the desired 4-circuit $\Gamma = Z_1Z_2Z_3Z_4Z_1$ in $G - Z_0$ according to the different situations of e.

Case 1 $e \in E - (Z_1 \cup Z_2)$.

Then $Z_1Z_2 \in E(G - Z_0)$, and so there exists an element $e_1 \in Z_1 \cap Z_2$. Since M is connected, both e_1 and e are contained in a circuit $Z_3 \in V_2$. Thus $Z_3 \neq Z_0$ and $Z_1Z_3, Z_2Z_3 \in E(G)$. Since $e \in Z_3 - (Z_1 \cup Z_2)$, both $Z_1 \neq Z_3$ and $Z_2 \neq Z_3$.

Assume first that $e \notin Z_0$. Since $Z_1 \neq Z_3$, there exists an $e_2 \in Z_1 - Z_3$. As $Z_1 \in V_1$, $e \neq e_2$. Since M is connected, M has a circuit Z_4 with $e_2, e \in Z_4$. Thus $e \in (Z_3 \cap Z_4) - (Z_1 \cup Z_2)$, $e_1 \in (Z_1 \cap Z_3) - Z_4$ and $e_2 \in (Z_1 \cap Z_4) - Z_3$, and so $\Gamma = Z_1Z_2Z_3Z_4Z_1$ is a 4-circuit of G with $E(\Gamma) \cap E(G_1) = \{Z_1Z_2\}$ and $E(\Gamma) \cap E(G_2) = \{Z_3Z_4\}$. As $Z_1, Z_2 \in V(G - Z_0)$ and as $Z_3, Z_4 \in V_2$, we conclude that $Z_0 \notin \{Z_1, Z_2, Z_3, Z_4\}$. Hence $\Gamma = Z_1Z_2Z_3Z_4Z_1$ is a desired 4-circuit of $G - Z_0$.

Next we assume that $e \in Z_0$. If there exists an element $e_3 \in Z_1 - (Z_0 \cup Z_3)$, then as M is connected, M has a circuit Z_4 with $e, e_3 \in Z_4$. As $e_3 \in Z_4$, $Z_4 \notin \{Z_0, Z_3\}$. Thus $\Gamma_1 = Z_1Z_2Z_3Z_4Z_1$ is a 4-circuit of $G - Z_0$ with $E(\Gamma) \cap E(G_1) = \{Z_1Z_2\}$ and $E(\Gamma) \cap E(G_2) = \{Z_3Z_4\}$. Therefore, we assume that $Z_1 \subseteq Z_0 \cup Z_3$. As Z_1 is not a proper subset of Z_0, we have $Z_1 \cap Z_3 \neq \emptyset$. Since M is binary, $Z_1 \triangle Z_3$ is a disjoint union of circuits different from Z_1 and Z_3. Since $e \in Z_3 - Z_1$, there must be a circuit $Z' \subseteq Z_1 \triangle Z_3$ such that $e \in Z'$. If $Z' \neq Z_0$, then set $Z'_4 = Z'$ and so in this case $Z_1Z_2Z_3Z'_4Z_1$ is a desired 4-circuit of $G - Z_0$. Thus we assume that $Z' = Z_0$. If Z_0 is a proper subset of $Z_1 \triangle Z_3$, then $Z_1 \triangle Z_3$ contains another circuit Z'', disjoint from Z_0 and intersecting with both Z_1 and Z_3. Hence there exists an element $e'_1 \in Z_1 - (Z_0 \cup Z_3)$. In this case, by the connectedness of M, there must be a circuit $Z''_4 \in \mathcal{C}(M)$ such that $e, e'_1 \in Z''_4$. It follows that $Z_1Z_2Z_3Z''_4Z_1$ is a desired 4-circuit of $G - Z_0$. Hence we conclude that if no desirable 4-circuit exists, then we must have $Z_1 \triangle Z_3 = Z_0$. By the symmetry between Z_1 and Z_2, we also have $Z_2 \triangle Z_3 = Z_0$, which leads to the contradiction that $Z_1 = Z_0 \triangle Z_3 = Z_2$. This contradiction indicates that we always can find a desirable 4-circuit satisfying the conclusion of the lemma.

Case 2 $e \in Z_1 - Z_2$ or $e \in Z_2 - Z_1$.

By symmetry, we assume that $e \in Z_2 - Z_1$, $e_1 \in Z_1 \cap Z_2$.

Assume first that $e \notin Z_0$. By Lemma 3.2(i), M has a circuit $Z_3 \subseteq Z_1 \cup Z_2 - \{e_1\}$ with $e \in Z_3$. Since $e \in Z_3$ and $Z_0 \in V_1$, we have $Z_3 \neq Z_0$. As Z_3 cannot be a proper subset of Z_2, there must be an element $e_2 \in Z_1 \cap Z_3$. Since $Z_1 \in V_1$, we note that $e_2 \neq e$.

If there exists an element $e_3 \in E(M) - (Z_0 \cup Z_1 \cup e)$, then by the assumption that $M - e$ is connected, there exists a circuit $Z_4 \in \mathcal{C}(M - e)$ with $e_2, e_3 \in Z_4$. In this case, $\Gamma = Z_1Z_2Z_3Z_4Z_1$ is a 4-circuit of G with $E(\Gamma) \cap E(G_1) = \{Z_1Z_4\}$ and $E(\Gamma) \cap E(G_2) = \{Z_2Z_3\}$. As $Z_1, Z_2 \in V(G - Z_0)$, $Z_3 \neq Z_0$ and $e_3 \in Z_4 - Z_0$, we conclude that $Z_0 \notin \{Z_1, Z_2, Z_3, Z_4\}$. It follows that in this case $\Gamma = Z_1Z_2Z_3Z_4Z_1$ is a desired 4-circuit of $G - Z_0$. Hence we may assume that $E(M) = Z_0 \cup Z_1 \cup e$. Since $Z_0 \neq Z_1$, $e \notin Z_0 \cup Z_1$ and since $M - e$ is also binary, $Z_0 \triangle Z_1$ is a disjoint union of circuits. Since $Z_3 \subset E(M) = Z_0 \cup Z_1 \cup e$, $Z_3 \neq e$ and $Z_3 \neq Z_0$, there must be an element $e_3' \in Z_3 - (Z_0 \cup e)$. Let Z_4' be a circuit in $Z_0 \triangle Z_1$ with $e_3' \in Z_4'$. In this case, $\Gamma' = Z_1Z_2Z_3Z_4'Z_1$ is a 4-circuit of G with $E(\Gamma') \cap E(G_1) = \{Z_1Z_4\}$ and $E(\Gamma') \cap E(G_2) = \{Z_2Z_3\}$. As $Z_1, Z_2 \in V(G - Z_0)$, $Z_3 \neq Z_0$ and $e_3' \in Z_4 - Z_0$, we conclude that $Z_0 \notin \{Z_1, Z_2, Z_3, Z_4'\}$, and so in this case $\Gamma' = Z_1Z_2Z_3Z_4'Z_1$ is a desired 4-circuit of $G - Z_0$.

Next we assume that $e \in Z_0$. Since $|V_2| \geq 4$, we may assume that Z_0, Z_1, Z_1', Z_1'' are different vertices in V_2. If there is an element $e_1' \in Z_1' - (Z_2 \cup \{e\})$, then set $Z_4 = Z_1'$ and, as M is connected, there exists a circuit $Z_3 \in \mathcal{C}(M - e)$ with $e_1, e_1' \in Z_3$. As $Z_1Z_4 \in E(G_2)$ and $Z_2Z_3 \in E(G_1)$, $Z_1Z_2Z_3Z_4Z_1$ is a desired 4-circuit of $G - Z_0$. Hence we may assume that $Z_1' \subseteq Z_2 \cup e$. By the symmetry between Z_1' and Z_1'', we may also assume that $Z_1'' \subseteq Z_2 \cup e$. This forces that $Z_2 = Z_1' \triangle Z_1''$. Let Z_3 be a circuit in $Z_1 \triangle Z_1'$. Then $Z_3 \cap Z_1' \neq \emptyset$ and $Z_3 \cap Z_2 \neq \emptyset$. Hence letting $Z_4 = Z_1'$, once again we have $Z_1Z_4 \in E(G_2)$ and $Z_2Z_3 \in E(G_1)$, and so $Z_1Z_2Z_3Z_4Z_1$ is a desired 4-circuit of $G - Z_0$. This proves Case 2.

Case 3 $e \in Z_1 \cap Z_2$, whence both Z_1 and Z_2 are vertices in G_2.

Assume first that $e \notin Z_0$. If $Z_0 = Z_1 \triangle Z_2$, then as $Z_1 \neq Z_2$, there must be an element $e_1 \in Z_1 - Z_2$ and an element $e_2 \in Z_2 - Z_1$. As $e_1, e_2 \in E(M - e)$ and as $M - e$ is connected, there exists a circuit $Z_3 \in \mathcal{C}(M - e)$ such that $e_1, e_2 \in Z_3$. Since Z_3 is not a proper subset of Z_0, we have $Z_3 \neq Z_0$. Since $|V_1| \geq 3$, there must be a $Z \in V_1 - \{Z_0, Z_3\}$. If $e_1 \in Z$, then $\Gamma_1 = Z_1Z_2Z_3ZZ_1$ is a 4-circuit of G with $E(\Gamma_1) \cap E(G_1) = \{Z_3Z\}$ and $E(\Gamma_1) \cap E(G_2) = \{Z_1Z_2\}$, and with $Z_0 \notin \{Z_1, Z_2, Z_3, Z\}$. Hence by symmetry, we may assume that $\{e_1, e_2\} \cap Z = \emptyset$. In this case, we pick $e_3 \in Z - Z_3$. As $M - e$ is connected and as $e_1, e_3 \in E(M - e)$, there must be a $Z_4 \in \mathcal{C}(M - e)$ with $e_1, e_3 \in Z_4$. It follows that $\Gamma_2 = Z_1Z_2Z_3Z_4Z_1$ is a 4-circuit of G with $E(\Gamma_2) \cap E(G_1) = \{Z_3Z_4\}$ and $E(\Gamma_2) \cap E(G_2) = \{Z_1Z_2\}$, and with $Z_0 \notin \{Z_1, Z_2, Z_3, Z_4\}$.

Next, we assume that $e \notin Z_0$ and $Z_0 \neq Z_1 \triangle Z_2$. Since M is binary, $Z_1 \triangle Z_2$ contains a circuit Z_3' such that Z_3' contains an element $e_1' \in (Z_1 \triangle Z_2) - Z_0$. As $e_1' \in Z_1 \triangle Z_2$, we by symmetry may assume that $e_1' \in Z_1 - Z_2$. Since Z_3' cannot be a proper subset of Z_1, there must be an element $e_2' \in Z_3' \cap Z_2 - Z_1$. Since $|V_1| \geq 3$, there must be a $Z'' \in V_1 - \{Z_0, Z_3'\}$. If $e_1' \in Z''$, then $\Gamma_3 = Z_1Z_2Z_3'Z''Z_1$ is a 4-circuit of G with $E(\Gamma_3) \cap E(G_1) = \{Z_3'Z''\}$ and $E(\Gamma_3) \cap E(G_2) = \{Z_1Z_2\}$, and with $Z_0 \notin \{Z_1, Z_2, Z_3', Z''\}$. Hence by symmetry, we may assume that $\{e_1, e_2\} \cap Z'' = \emptyset$.

In this case, we pick $e_3' \in Z'' - Z_3'$. As $M - e$ is connected and as $e_1', e_3' \in E(M - e)$, there must be a $Z_4' \in \mathcal{C}(M - e)$ with $e_1', e_3' \in Z_4$. It follows that $\Gamma_4 = Z_1 Z_2 Z_3' Z_4' Z_1$ is a 4-circuit of G with $E(\Gamma_4) \cap E(G_1) = \{Z_3' Z_4'\}$ and $E(\Gamma_4) \cap E(G_2) = \{Z_1 Z_2\}$, and with $Z_0 \notin \{Z_1, Z_2, Z_3', Z_4'\}$.

As the arguments above show that if $e \notin Z_0$, then a desirable 4-circuit always exists, we assume throughout the rest of the proof of this lemma that $e \in Z_0$. Since $e \in Z_1 \cap Z_2$, $Z_1 \neq Z_2$, and as M is binary, $Z_1 \triangle Z_2$ contains a circuit $Z_3 \in V_1$. Since $|V_1| \geq 3$, there exists a circuit $Z' \in V_1 - \{Z_3\}$. Pick $e' \in Z' - Z_3 \subseteq E - \{e\}$. As $Z_3 \subseteq Z_1 \triangle Z_2$, there must be an element $e'' \in Z_3 \cap Z_1$. Since $e \notin Z_3$, $e'' \neq e$. By the connectedness of $M - e$, there exists a circuit $Z_4 \in \mathcal{C}(M - e)$ such that $e', e'' \in Z_4$. Since $Z_1 Z_2 \in E(G_2)$ and $Z_3 Z_4 \in E(G_1)$, it follows that $Z_1 Z_2 Z_3 Z_4 Z_1$ is a desirable 4-circuit. This completes the proof of this case as well as the lemma. □

An element $e \in E(M)$ of a connected matroid M is **essential** if $M - e$ is not connected. A matroid M is **critically connected** if M is connected and every $e \in E(M)$ is essential.

Theorem 3.5 (Murty 1974) *If M is critically connected with $r(M) \geq 2$, then M contains a cocircuit of 2 element.*

Lemma 3.6 *If $M \in \{K_2^2, K_2^3, K_2^4, K_3 P_2, C_4 P_3, K_4, F_7^*\}$, then either $G(M)$ has fewer than 4 vertices, or for any $z \in V(G(M))$ and any edge $f \in E(G(M) - z)$, $G(M) - z$ has a hamiltonian circuit containing f.*

Proof If $M \in \{K_2^2, K_2^3\}$, then $|V(G(M))| \leq 3$. As every pair of distinct circuits of F_7^* or of $M(K_4)$ must have nonempty intersection. both $G(F_7^*)$ and $G(M(K_4))$ are complete graphs with at least 6 vertices. By definition, if $G(M(K_2^4))$ is the graph obtained from K_6 by deleting perfect matching. Let e be the edge in $P_2 K_3$ not lying in a 2-circuit. Then circuits in $P_2 K_3$ containing e, as vertices in $G(M(P_2 K_3))$, induces a K_4, and so $G(M(P_2 K_3))$ is the graph obtained from K_6 by deleting an edge. Likewise, Let e' be the edge in $P_3 C_4$ no lying in a 2-circuit. Then circuits in $P_3 C_4$ containing e', as vertices in $G(M(P_3 C_4))$, induces a K_8, and so $G(M(P_3 C_4))$ is the graph obtained from K_{11} by deleting a 3-circuit. It is routine to show that each of these graphs has the indicated property. □

Lemma 3.7 *If M be a connected serially reduced binary matroid with $|E(M)| - r(M) \leq 2$. Then $M = U_{1,3}$.*

Proof Let B be a basis of M, let e_1, e_2 be the only two elements in $E(M) - B$, and Z_1, Z_2 be the fundamental circuit of e_1 and e_2 with respect to B, respectively. Then $Z_1 \triangle Z_2 = \{e_1, e_2\}$ is a circuit. Since M is connected, It follows that both $Z_1 = Z_2 \triangle \{e_1, e_2\} = B \cup e_1$ and $Z_2 = Z_1 \triangle \{e_1, e_2\} = B \cup e_2$. As M is serially reduced, M contains no 2-element cocircuits, and so for some element e_3, we have $B = \{e_3\}$. This shows that $M \cong U_{1,3}$. □

3.2 A result on 1-edge-hamiltonian circuit graphs

If for any vertex subset $S \subset V(G)$ with $|S| \leq 1$ and for any edge $e \in E(G-S)$, $G-S$ has a Hamilton circuit containing e, then G is said to be **1-edge-hamiltonian**. Recall that $M(e_0)$ is a matroid with e_0 being its root.

We prove a slightly stronger result than the statement we made in the beginning of this section, as follows.

Theorem 3.8 *Let $M = (E, \mathcal{I})$ be a connected binary matroid with $|\mathcal{C}(M)| \geq 4$, and let $G = G(M)$ be the circuit graph of M. Then G is 1-edge-hamiltonian.*

Proof By Theorem 3.1(ii), it suffices to show that

$$\text{for any } v \in V(G) \text{ and } e \in E(G-v),\ G-v \text{ has a Hamilton circuit containing } e. \tag{9}$$

We argue by induction on $|E|$ to prove (9). By Lemma 3.2, every matroid $M = (E, \mathcal{I})$ with $|E| \leq 3$ has $|\mathcal{C}(M)| < 4$. By Lemmas 3.2 and 3.6 , (9) holds for any connected binary matroid on 4 elements. Hence we assume that $|E| \geq 5$, and (9) holds for connected binary matroids with smaller number of elements.

If for some element $e_0 \in E(M)$, $M(e_0)$ is in $\mathcal{F}_1 \cup \mathcal{F}_2$, then by $|E| \geq 5$ and by Lemma 3.6, $G(M)$ is 1-edge-hamiltonian. Hence we assume that

$$\text{for any } e_0 \in E(M),\ M(e_0) \text{ is not in } \mathcal{F}_1 \cup \mathcal{F}_2. \tag{10}$$

If M has a 2-cocircuit $\{e', e''\}$, then by Lemma 3.3(ii), $G(M) = G(M/e')$, and so by induction, we may assume that

$$M \text{ is serially reduced.} \tag{11}$$

Suppose that M is critically connected. Then by Theorem 3.5, M has a 2-cocircuit $\{e', e''\}$. By Lemma 3.3(ii), $G(M) = G(M/e')$. By induction $G(M/e')$, and so $G(M)$, is 1-edge-hamiltonian. Therefore, we assume that M is not critically connected. By definition, there exists an element $e \in E(M)$ such that $M-e$ is connected. Define $V_1 = \mathcal{C}(M) - \mathcal{C}_{M,e}$, $V_2 = \mathcal{C}_{M,e}$, $G_1 = G[V_1]$ and $G_2 = G[V_2]$. If $|V_1| \geq 4$, then it follows by induction that

$$G_1 = G(M-e) \text{ is 1-edge-hamiltonian.} \tag{12}$$

By (1), if $|V_1| \leq 3$, then $M(e) \in \mathcal{F}_2$; and if $|V_2| \leq 3$, then $M(e) \in \mathcal{F}_1$. In either case, a contradiction to (10) is found. If $|E(M)| - r(M) \leq 2$, then by Lemma 3.7, $G(M) \in \{K_1, K_3\}$ and so we may assume $|E(M)| - r(M) \geq 3$. These, together with Lemma 3.3(iii) and Lemma 3.7, imply that

$$|V_1| \geq 4, |V_2| \geq 4, \kappa(G) \geq 4 \text{ and that } G_2 \text{is a complete graph.} \tag{13}$$

Let $Z_0 \in V(G) = \mathcal{C}(M)$, and an edge $f = Z'Z'' \in E(G - Z_0)$ be given. We shall show that $G - Z_0$ has a Hamilton circuit containing f. By (12), G_1 (if $e \in Z_0$) or $G_1 - Z_0$ (if $e \notin Z_0$) has a Hamilton circuit C.

Case 1. $e \notin Z' \cup Z''$.

Then $f = Z'Z'' \in E(G_1)$. By (13), $|V_1| \geq 4$ and so there must be a two vertices $Z_1, Z_2 \in V_1 - \{Z', Z''\}$ such that $Z_1Z_2 \in E(C - f)$. By Lemma 3.4, $G - Z_0$ has a 4-circuit $Z_1Z_2Z_3Z_4Z_1$ such that $Z_3Z_4 \in E(G_2)$. By Lemma 3.3(i) and by (13), $G_2 - Z_0$ (if $e \in Z_0$) or G_2 (if $e \notin Z_0$) is a complete graph on at least 3 vertices, and so $G_2 - Z_0$ contains a spanning (Z_3, Z_4)-path P. It follows that $E(C - Z_1Z_2) \cup E(P) \cup \{Z_2Z_3, Z_1Z_4\}$ induces a Hamilton circuit of $G - Z_0$ which contains $f = Z'Z''$.

Case 2. $e \in Z' - Z''$ or $e \in Z'' - Z'$.

By symmetry, we may assume that $e \in Z'' - Z'$, and so $Z' \in V_1$ and $Z'' \in V_2$. Let $Z_1 = Z'$ and $Z_2 = Z''$. By Lemma 3.4, $G - Z_0$ has a 4-circuit $Z_1Z_2Z_3Z_4Z_1$ such that $Z_2Z_3 \in E(G_2)$. If $Z_1Z_4 \in E(G_1)$. By (12), $G_1 - Z_0$ (if $e \notin Z_0$) or G_1 (if $e \in Z_0$) has a Hamilton circuit C_1 with $Z_1Z_4 \in E(C_1)$. As G_2 is a complete graph on at least 3 vertices, $G_2 - Z_0$ (if $e \in Z_0$) or G_2 (if $e \notin Z_0$) contains a spanning (Z_2, Z_3)-path P. It follows that $E(C - Z_1Z_4) \cup E(P) \cup \{Z_1Z_2, Z_3Z_4\}$ induces a Hamilton circuit of $G - Z_0$ which contains $f = Z'Z''$.

Case 3. $e \in Z' \cap Z''$.

Then $f = Z'Z'' \in E(G_2)$. By (13), $\kappa(G) \geq 4$, and so $G - \{Z_0, Z', Z''\}$ is connected. Therefore, there must be an edge $Z_1Z_1' \in E(G - \{Z_0, Z', Z''\})$ such that $Z_1 \in V_1$ and $Z_1' \in V_2$. Pick and edge $Z_1Z_2 \in E(C)$. By Lemma 3.4, $G - Z_0$ has a 4-circuit $Z_1Z_2Z_3Z_4Z_1$ with $Z_3, Z_4 \in V_2 - \{Z_0\}$. Assume that $Z_1' \neq Z_3$ ($Z_1' = Z_3 \neq Z_4$, respectively). By Lemma 3.3(i) and (13), G_2 is a complete graph on at least 4 vertices, and so G_2 (if $e \notin Z_0$) or $G_2 - Z_0$ (if $e \in Z_0$) has a spanning (Z_1', Z_3)-path ((Z_1', Z_4)-path, respectively) P with $f = Z'Z'' \in E(P)$. It follows that $E(C - Z_1Z_2) \cup E(P) \cup \{Z_1Z_1', Z_2Z_3\}$ (or $E(C - Z_1Z_2) \cup E(P) \cup \{Z_1Z_4, Z_2Z_3\}$, respectively) induces a Hamilton circuit of $G - Z_0$ which contains $f = Z'Z''$.

As in every cases, $G - Z_0$ always has a Hamilton circuit containing f, the theorem is now proved. □

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