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On *s*-hamiltonian line graphs of claw-free graphs

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ABSTRACT

For an integer $s \geq 0$, a graph G is s-hamiltonian if for any vertex subset $S \subseteq V(G)$ with $|S| \leq s$, $G - S$ is hamiltonian, and G is s-hamiltonian connected if for any vertex subset $S \subseteq V(G)$ with $|S| \leq s$, $G - S$ is hamiltonian connected. Thomassen in 1984 conjectured that every 4-connected line graph is hamiltonian (see Thomassen, 1986), and Kučzel and Xiong in 2004 conjectured that every 4-connected line graph is hamiltonian connected (see Ryjáček and Vrána, 2011). In Broersma and Veldman (1987), Broersma and Veldman raised the characterization problem of s-hamiltonian line graphs. In Lai and Shao (2013), it is conjectured that for $s \geq 2$, a line graph $L(G)$ is s-hamiltonian if and only if $L(G)$ is $(s+2)$-connected. In this paper we prove the following.

(i) For an integer $s \geq 2$, the line graph $L(G)$ of a claw-free graph G is s-hamiltonian if and only if $L(G)$ is $(s+2)$-connected.

(ii) The line graph $L(G)$ of a claw-free graph G is 1-hamiltonian connected if and only if $L(G)$ is 4-connected.

1. Introduction

Graphs considered here are finite and loopless. Unless otherwise noted, we follow [1] for notation and terms. As in [1], $\kappa(G)$ and $\kappa'(G)$ denote the connectivity and the edge-connectivity of a graph G, respectively. A graph is **nontrivial** if it contains edges. An edge cut X is **essential** if $G - X$ has at least two nontrivial components. For an integer $k > 0$, a graph G is **essentially** k**-edge-connected** if G does not have an essential edge cut X with $|X| < k$. For a connected graph G, let $ess'(G) = \max\{k : G \text{ is essentially } k\text{-edge-connected}\}$, and for an integer $i \geq 0$, let $D_i(G) = \{u \in V(G) : d_G(u) = i\}$ and $d_i(G) = |D_i(G)|$. Throughout this paper, for an integer $n \geq 2$, C_n denotes a cycle on n vertices (called an n-cycle), nK_2 denotes the loopless graph on two vertices with n edges, W_n denotes the graph obtained from an n-cycle by adding a new vertex and connecting it to every vertex of the n-cycle, and K_5^- denotes the graph obtained from K_5 by deleting an edge. If $S \subseteq V(G)$ or $S \subseteq E(G)$, $G[S]$ is the subgraph induced in G by S. We use $H \subseteq G$ to denote the fact that H is a subgraph of G. For $H \subseteq G$, $x \in V(G)$, $A \subseteq V(G)$, $X \subseteq E(G)$, and $Y \subseteq E(G) - E(H)$, define $N_H(x) = N_G(x) \cap V(H)$, $d_H(x) = |N_H(x)|$, $G - A = G[V(G) - A]$, $G - X = G[E(G) - X]$, and $H + Y = G[E(H) \cup Y]$. When $A = \{v\}$ and $X = \{e\}$, we use $G - v$ for $G - \{v\}$ and $G - e$ for $G - \{e\}$. Different from the notation in [1], for vertex-disjoint subgraphs H_1 and H_2 in G, we define $H_1 + H_2 = G[V(H_1) \cup V(H_2)]$.

A graph G is **claw-free** if it does not contain $K_{1,3}$ as an induced subgraph. The **line graph** of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are

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adjacent. It is straight forward to see that for a graph G with $|E(G)| \geq 3$, $L(G)$ is k-connected if and only if G is essentially k-edge-connected. The following are several fascinating conjectures in the literature.

Conjecture 1.1. *(i) (Thomassen [18]) Every 4-connected line graph is hamiltonian.*
(ii) (Matthews and Sumner [15]) Every 4-connected claw-free graph is hamiltonian.
(iii) (Kučzel and Xiong [11]) Every 4-connected line graph is hamiltonian connected.
(iv) (Ryjáček and Vrána [16]) Every 4-connected claw-free graph is hamiltonian connected.

Ryjacek and Vrána in [16] indicated that the statements in Conjecture 1.1 are mutually equivalent. There have been many studies on these conjectures in the literature. Among them are the following.

Theorem 1.2 (*Zhan [19]*)**.** *Every 7-connected line graph is hamiltonian connected.*

Theorem 1.3 (*Kriesell [10]*)**.** *Every 4-connected line graph of a claw-free graph is hamiltonian connected.*

For an integer $s \geq 0$, a graph G is s-**hamiltonian** (or s-**hamiltonian connected**, respectively) if for any vertex subset $S \subseteq V(G)$ with $|S| \leq s$, $G - S$ is hamiltonian (or hamiltonian connected, respectively). In [2], Broersma and Veldman proposed an open problem: for a given positive integer k determine the value s for which the statement "for a k-triangular graph G, the line graph $L(G)$ of G is s-hamiltonian if and only $L(G)$ is $(s + 2)$-connected" is valid. Broersma and Veldman in [2] proved that the statement holds for all values s with $0 \leq s \leq k$, and conjectured that it holds if $s \leq 2k$. Chen et al. in [7] proved this conjecture for all values s with $0 \leq s \leq \max\{2k, 6k - 16\}$. In [13], an attempt to characterize s-hamiltonian line graphs is made and the following is proved.

Theorem 1.4 (*[13]*)**.** *For $s \geq 5$, a line graph is s-hamiltonian if and only if it is $(s + 2)$-connected.*

An open problem was raised in [13] that whether a line graph $L(G)$ is s-hamiltonian if and only if $L(G)$ is $(s+2)$-connected for $s \in \{2, 3, 4\}$. The case when $s = 2$ implies Conjecture 1.1(i). Motivated by Conjecture 1.1 as well as the results in [7] and [13], we propose the following conjectures.

Conjecture 1.5. *Let s be an integer.*
(i) For $s \geq 2$, a line graph is s-hamiltonian if and only if it is $(s + 2)$-connected.
(ii) For $s \geq 2$, a claw-free graph is s-hamiltonian if and only if it is $(s + 2)$-connected.
(iii) For $s \geq 1$, a line graph is s-hamiltonian connected if and only if it is $(s + 3)$-connected.
(iv) For $s \geq 1$, a claw-free graph is s-hamiltonian connected if and only if it is $(s + 3)$-connected.

The main result in this paper is presented below, as an effort to support Conjecture 1.5(i) and (iii).

Theorem 1.6. *Let G be a claw-free graph.*
(i) For an integer $s \geq 2$, $L(G)$ is s-hamiltonian if and only if $\kappa(L(G)) \geq s + 2$.
(ii) $L(G)$ is 1-hamiltonian connected if and only if $\kappa(L(G)) \geq 4$.

In Section 2, we introduce Catlin's reduction method and the related results. In Section 3 we introduce a property of graphs which will be used in our arguments to prove the main results. The proof of Theorem 1.6 is given in Section 4.

2. Preliminaries

We view a trail of G as a vertex-edge alternating sequence $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$ such that all the e_i's are distinct and for each $i = 1, 2, \ldots, k$, e_i is incident to both v_{i-1} and v_i. The vertices in $v_1, v_2, \ldots, v_{k-1}$ are **internal vertices** of the trail. For edges $e', e'' \in E(G)$, an (e', e'')-trail of G is a trail T of G whose first edge is e' and whose last edge is e''. A **dominating** (e', e'')-**trail** of G is an (e', e'')-trail T of G such that every edge of G is incident to an internal vertex of T, and a **spanning** (e', e'')-**trail** of G is a dominating (e', e'')-trail T of G such that $V(T) = V(G)$. Harary and Nash–Williams [8] first showed the relationship between eulerian subgraphs in G and hamiltonicity in $L(G)$. Theorem 2.1(ii) is observed in [14].

Theorem 2.1. *Let G be a graph with $|E(G)| \geq 3$. Each of the following holds.*
(i) (Harary and Nash–Williams [8]) $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.
(ii) [14] $L(G)$ is hamiltonian connected if and only if for any pair of edges $e', e'' \in E(G)$, G has a dominating (e', e'')-trail.

We say that an edge $e \in E(G)$ is **subdivided** when it is replaced by a path of length 2 whose internal vertex, denoted by $v(e)$, has degree 2 in the resulting graph. The process of taking an edge e and replacing it by the path of length 2 is called **subdividing** e. For a graph G and edges $e', e'' \in E(G)$, let $G(e')$ denote the graph obtained from G by subdividing e', and let $G(e', e'')$ denote the graph obtained from G by subdividing both e' and e''. Then $V(G(e', e'')) - V(G) = \{v(e'), v(e'')\}$.

Lemma 2.2 (*Lemma 1.4 of [12]*)**.** *For a graph G and edges $e', e'' \in E(G)$, if $G(e', e'')$ has a spanning $(v(e'), v(e''))$-trail, then G has a spanning (e', e'')-trail.*

Let $X \subseteq E(G)$ be an edge subset of G. The **contraction** G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. If H is a subgraph of G, we write G/H for $G/E(H)$. If v_H is the vertex in G/H onto which H is contracted, then H is called the **preimage** of v, and denoted by $PI(v)$. Let $O(G)$ denote the set of odd degree vertices of G. A graph G is **eulerian** if $O(G) = \emptyset$ and G is connected. A graph G is **supereulerian** if G has a spanning eulerian subgraph. In [4] Catlin defined collapsible graphs. Given an even subset R of $V(G)$, a subgraph Γ of G is called an R-**subgraph** if $O(\Gamma) = R$ and $G - E(\Gamma)$ is connected. A graph G is **collapsible** if for any even subset R of $V(G)$, G has an R-subgraph. In particular, K_1 is collapsible. Catlin [4] showed that for any graph G, one can obtain the **reduction** G' of G by contracting all maximal collapsible subgraphs of G. A graph G' is **reduced** if G' has no nontrivial collapsible subgraphs. A vertex in G' is **c-nontrivial** (or **c-trivial**) if $|V(PI(x))| \geq 2$ (or $|V(PI(x))| = 1$). By definition, every collapsible graph is supereulerian. We summarize some results on Catlin's reduction method and other related facts below. Theorem 2.3(v) is a straightforward application of the definition of collapsible graphs.

Theorem 2.3. *Let G be a graph and let H be a collapsible subgraph of G. Let v_H denote the vertex onto which H is contracted in G/H. Each of the following holds.*

(i) (Catlin, Theorem 3 of [4]) G is collapsible if and only if G/H is collapsible. In particular, G is collapsible if and only if the reduction of G is K_1.

(ii) (Catlin, implied by definition and Theorem 3 of [4]) C_2, C_3 are collapsible, and when $n \geq 4$, for any $e_1, e_2 \in E(W_n)$, $(W_n - e_1)(e_2)$ is collapsible.

(iii) (Theorem 2.3 (iii) of [14]) If G is collapsible, then for any pair of vertices $u, v \in V(G)$, G has a spanning (u, v)-trail.

(iv) (Theorem 2.3 (iv) of [14]) For vertices $u, v \in V(G/H) - \{v_H\}$, if G/H has a spanning (u, v)-trail, then G has a spanning (u, v)-trail.

(v) Let $e', e'' \in E(G) - E(H)$. Then G has a spanning (e', e'')-trail if and only if G/H has a spanning (e', e'')-trail.

(vi) (Theorem 3.3 of [14]) Let G be a 3-edge-connected graph. If every 3-edge-cut X has at least one edge in a 2-cycle or 3-cycle of G, then, for any two edges $e', e'' \in E(G)$, $G(e', e'')$ is collapsible.

Let $\tau(G)$ denote the maximum number of edge-disjoint spanning trees of G. Let $F(G)$ be the minimum number of additional edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. The following theorem summarizes results related to $F(G)$ and supereulerianicity.

Theorem 2.4. *Let G be a connected graph and let G' be the reduction of G. Then each of the following holds.*

(i) (Jaeger [9]) If $F(G) = 0$, then G is collapsible.

(ii) (Catlin [4]) If $F(G) \leq 1$, then $G' \in \{K_1, K_2\}$. Therefore, G is supereulerian if and only if $G' \neq K_2$.

(iii) (Catlin et al. [5]) If $F(G) \leq 2$, then $G' \in \{K_1, K_2, K_{2,t}\}$ for some integer $t \geq 1$. Therefore, G is supereulerian if and only if $G' \notin \{K_2, K_{2,t}\}$ for some odd integer t.

(iv) (Catlin [3]) $F(G') = 2|V(G')| - |E(G')| - 2$. Therefore, if $F(G') \geq 3$, then $3d_1(G') + 2d_2(G') + d_3(G') \geq 10$.

(v) (Theorem 1.1 of [6]) Let $k \geq 1$ be an integer. Then $\kappa'(G) \geq 2k$ if and only if for any edge subset $X \subseteq E(G)$ with $|X| \leq k$, $\tau(G - X) \geq k$.

Lemma 2.5. *Assume that $K = v_1v_2v_3v_1$ is a triangle in a connected graph G with $d_G(v_1) = 3$. Also assume that $N_G(v_1) = \{v_2, v_3, x\}$ and $e \in \{v_1v_2, v_2v_3\}$. Let w be the new vertex in G/K to which K is contracted, and let $u(\neq w) \in V(G/K)$. Let T be a spanning (u, w)-trail in G/K. Then each of the following holds.*

(i) For $e = v_1v_2$, $G(e)$ has a dominating $(u, v(e))$-trail T_1 such that $V(G(e)) - V(T_1) \subseteq \{v_1\}$.

(ii) For $e = v_2v_3$, if $xv_1 \notin E(T)$, then $G(e)$ has a spanning $(u, v(e))$-trail T_2.

Proof. Since $u \neq w$, we have $O(T) = \{u, w\}$. Let H be the subgraph induced by $E(T)$ in G. Then H may not be connected, $O(H) \subseteq \{u, v_1, v_2, v_3\}$, and $d_H(u)$ is odd. Since $d_G(v_1) = 3$ and $v_1v_2, v_1v_3 \notin E(T)$, $d_H(v_1) \in \{0, 1\}$.

Assume $d_H(v_1) = 0$. Then $xv_1 \notin E(H)$ and $d_H(v_2) + d_H(v_3) = d_T(w)$ is odd. Thus either $d_H(v_2)$ or $d_H(v_3)$ is odd. So $T_1 = \begin{cases} T + \{v_2v_3, v_3v_1, v_1v(v_1v_2)\}, & \text{if } d_H(v_2) \text{ is odd} \\ T + \{v_2v_3, v_2v(v_1v_2)\}, & \text{if } d_H(v_3) \text{ is odd} \end{cases}$ is a dominating $(u, v(e))$-trail of $G(e)$ with $V(G(e)) - V(T_1) \subseteq \{v_1\}$ if $e = v_1v_2$, and $T_2 = \begin{cases} T + \{v_2v_1, v_1v_3, v_3v(v_2v_3)\}, & \text{if } d_H(v_2) \text{ is odd} \\ T + \{v_2v_1, v_1v_3, v_2v(v_2v_3)\}, & \text{if } d_H(v_3) \text{ is odd} \end{cases}$ is a spanning $(u, v(e))$-trail in $G(e)$ if $e = v_2v_3$.

Assume $d_H(v_1) = 1$. Then $xv_1 \in E(H)$, $d_H(v_2) + d_H(v_3) = d_T(w) - 1$ is even, and $e = v_1v_2$. Thus both $d_H(v_2)$ and $d_H(v_3)$ are even or odd. If $d_H(v_2)$ and $d_H(v_3)$ are even, then $T_1 = T + \{v_1v_3, v_2v_3, v_2v(v_1v_2)\}$ is a spanning $(u, v(v_1v_2))$-trail in $G(v_1v_2)$. If both $d_H(v_2)$ and $d_H(v_3)$ are odd, $O(H) \subseteq \{u, v_1, v_2, v_3\}$, therefore H has at most two components. If v_1 and v_3 are in the same component of H, then $T_1 = T + \{v_2v_3, v_1v(v_1v_2)\}$ is a spanning $(u, v(v_1v_2))$-trail in $G(v_1v_2)$. If v_1 and v_3 are not in the same component of H, then $T_1 = T + \{v_1v_3, v_2v(v_1v_2)\}$ is a spanning $(u, v(v_1v_2))$-trail in $G(v_1v_2)$. ■

Lemma 2.6. *Let G be a 3-edge-connected, essentially 4-edge-connected graph. Let $v_1v_2v_3v_1$ be a triangle in G. If $d_G(v_i) = 3$ for $i = 1, 2, 3$, then $G = K_4$.*

Proof. Since G is essentially 4-edge-connected and $d_G(v_i) = 3$, we have $|N_G(v_i) \cap N_G(v_j)| \geq 2$ for some $\{i, j\} \subseteq \{1, 2, 3\}$. Without loss of generality, we assume that $x \in N_G(v_1) \cap N_G(v_2) - \{v_3\}$. Consider $N_G(v_3)$ and assume that $N_G(v_3) = \{v_1, v_2, y\}$. Then $\{xv_1, xv_2, yv_3\}$ is a 3-edge-cut in G. Since G is 3-edge-connected and essentially 4-edge-connected, we have $x = y$, and so $G = K_4$. ■

Lemma 2.7. *Let $s \geq 3$ be an integer and G be a graph with $\kappa'(G) \geq 3$ and $ess'(G) \geq s + 2$. If $v \in D_3(G)$, then $\kappa'(G - v) \geq 3$ and $ess'(G - v) \geq s + 1$.*

Proof. Let $N_G(v) = \{u_1, u_2, u_3\}$. Let X be an edge cut of $G-v$ and let H_1, H_2 be components of $(G-v)-X$. If $u_1, u_2, u_3 \in V(H_i)$ for some $i \in \{1, 2\}$, then $|X| \geq 3$. If $u_1 \in V(H_1)$ and $u_2, u_3 \in V(H_2)$, then $|X| \geq s \geq 3$ since $X \cup \{vu_2, vu_3\}$ is an essential edge cut in G, and so $\kappa'(G - v) \geq 3$. Let Y be an essential edge cut of $G - v$ and let H_1, H_2 be components of $(G - v) - Y$. If $u_1, u_2, u_3 \in V(H_i)$ for some $i \in \{1, 2\}$, then $|Y| \geq s + 2$. If $u_1 \in V(H_1)$ and $u_2, u_3 \in V(H_2)$, then $Y \cup \{vu_1\}$ is an essential edge cut of G, implying that $|Y| \geq s + 1$ and so $ess'(G - v) \geq s + 1$. ■

3. Graphs with property $\mathcal{K}(s)$

Throughout this section, we assume that $s \geq 2$ is an integer. We shall introduce a property of graphs which will play an important role in our arguments.

Definition 3.1. Let $\mathcal{K}$ denote the graph family such that a (connected) graph G is in $\mathcal{K}$ if and only if G satisfies each of the following.

(KS1) For any $w \in D_3(G)$, the subgraph induced by $N_G(w)$ contains at least one edge.

(KS2) Let $w \in N_G(x_1) \cap N_G(x_2)$, where $x_1, x_2 \in D_3(G)$ and $x_1x_2 \notin E(G)$. If $N_G(w) = \{x_1, x_2, v\}$, then either $vx_1 \notin E(G)$ or $vx_2 \notin E(G)$.

(KS3) Let $w_1, w_2 \in N_G(x_1) \cap N_G(x_2)$, where $x_1, x_2 \in D_3(G)$ and $x_1x_2 \notin E(G)$. If $w_1w_2 \in E(G)$, then $N_G(w_1) \cup N_G(w_2) \subseteq N_G(x_1) \cup N_G(x_2) \cup \{x_1, x_2\}$.

By definition, every claw-free graph satisfies (KS1) and (KS3). For an integer $s \geq 2$, a graph G is said to have **Property** $\mathcal{K}(s)$ if G is in $\mathcal{K} - \{K_4, W_4, W_5\}$ and satisfies both $\kappa'(G) \geq 3$ and $ess'(G) \geq s + 2$.

Lemma 3.2. *If the graph G has Property $\mathcal{K}(s)$, then there is a set $\triangle(G)$ of edge-disjoint triangles in G such that $D_3(G) \subseteq V(\triangle(G))$ and $D_3(G) \cap V(K) \neq \emptyset$ for each $K \in \triangle(G)$.*

Proof. By (KS1), each vertex with degree 3 is in a triangle. We choose a set $\triangle(G)$ of triangles in G such that

(i) $D_3(G) \subseteq V(\triangle(G))$ and $D_3(G) \cap V(K) \neq \emptyset$ for each $K \in \triangle(G)$;

(ii) subject to (i), the size of $T = \{e \in E(G) : e \in E(K) \cap E(L), \text{ where } K, L \in \triangle(G)\}$ is as small as possible.

To prove this lemma, it suffices to prove that $T = \emptyset$. By contradiction, we assume that $T \neq \emptyset$. Then there are two triangles $K = w_1u_1u_2w_1$ and $L = w_2u_1u_2w_2$ in $\triangle(G)$.

If $d_G(w_1) \geq 4$, then either $d_G(u_1) = 3$ or $d_G(u_2) = 3$ since $D_3(G) \cap V(K) \neq \emptyset$. Without loss of generality, we assume that $d_G(u_1) = 3$. By Lemma 2.6, we have either $d_G(u_2) \geq 4$ or $d_G(w_2) \geq 4$. If one of $d_G(u_2)$ and $d_G(w_2)$ equals three, we set $\triangle'(G) = \triangle(G) - \{K\}$. Then (i) is satisfied but (ii) is violated, a contradiction. So both $d_G(u_2) \geq 4$ and $d_G(w_2) \geq 4$. Let $\triangle'(G) = \triangle(G) - \{K\}$. Then (ii) is violated, a contradiction. So $d_G(w_1) = 3$. Similarly, $d_G(w_2) = 3$.

Notice that $G \neq K_4$. If $w_1w_2 \in E(G)$, by Lemma 2.6, $d_G(u_1) \geq 4$ and $d_G(u_2) \geq 4$. Let $\triangle'(G) = (\triangle(G) - \{K, L\}) \cup \{w_1w_2u_2w_1\}$. Then (ii) is violated. So $w_1w_2 \notin E(G)$. By (KS2), we have $d_G(u_1) \geq 4$ and $d_G(u_2) \geq 4$. By (KS3), $N_G(u_1) \cup N_G(u_2) \subseteq N_G(w_1) \cup N_G(w_2) \cup \{w_1, w_2\}$. Then there are two vertices x_1, x_2 such that $x_1w_1, x_1u_2, x_2u_1, x_2w_2 \in E(G)$. Thus $d_G(u_1) = d_G(u_2) = 4$. Since G is essentially 4-edge-connected, $d_G(x_1) \geq 4$ and $d_G(x_2) \geq 4$. Let $\triangle'(G) = (\triangle(G) - \{K\}) \cup \{x_1w_1u_2x_1\}$. Then (ii) is violated. This contradiction tells us that $T = \emptyset$. Hence $\triangle(G)$ is a set of edge-disjoint triangles in G. ■

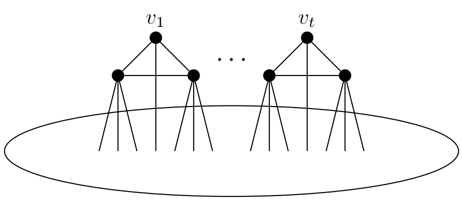

Fig. 1. $G_1^* = G_1/\triangle'(G)$.

Let $v \in D_3(G)$. By Lemma 3.2, there is a triangle containing v in $\triangle(G)$. We denote this triangle by $\triangle_v$. Thus, for $v, u \in D_3(G)$, we have either $E(\triangle_v) = E(\triangle_u)$ or $E(\triangle_v) \cap E(\triangle_u) = \emptyset$. Fix a given subset $X = \{e_1, e_2, \ldots, e_s\} \subseteq E(G)$. Define $\triangle'(G) = \bigcup_{v\in D_3(G), E(\triangle_v)\cap X=\emptyset}\{\triangle_v\}$ and $\triangle^*(G) = \triangle(G) - \triangle'(G)$. Then $\triangle(G) = \triangle'(G)$ if $X \cap E(\triangle(G)) = \emptyset$. Define $G_1 = G/\triangle(G)$, and we use G_1^* to denote $G/\triangle'(G)$. Thus if $X \cap E(\triangle(G)) = \emptyset$, then $G_1 = G_1^*$, and if $\triangle^*(G) = \{\triangle_{v_1}, \ldots, \triangle_{v_t}\}$, then $\{v_1, \ldots, v_t\} \subseteq D_3(G_1^*)$ and $E(\triangle_{v_i}) \cap X \neq \emptyset$ for $i = 1, \ldots, t$ (Fig. 1). We call G_1 a $\triangle$-**contraction** of G. By Theorem 2.4(v), for any $X \subseteq E(G_1)$ with $|X| \le 2$, $\tau(G_1 - X) = 2$, and so $F(G_1 - X) = 0$. Since $\kappa'(G) \ge 3$ and $ess'(G) \ge s + 2$, we have

$$\kappa'(G_1) \ge 4,\ ess'(G_1) \ge s+2,\ \kappa'(G_1^*) \ge 3,\ ess'(G_1^*) \ge s+2, \text{ and } D_i(G_1^*) \subseteq D_i(G) \text{ for } i \in \{3, \ldots, s+1\}. \quad (1)$$

Lemma 3.3. *Suppose that $s \in \{2, 3, 4\}$ and $N_G(x_1) \cap N_G(x_2) \cap N_G(x_3) = \emptyset$ for any $x_1, x_2, x_3 \in D_3(G)$ if $s \ge 3$. If G has Property $\mathcal{K}(s)$, then for any edge subset $X \subseteq E(G)$ with $|X| \le s$, $G - X$ has a dominating eulerian subgraph T such that $V(G) - V(T) \subseteq \bigcup_{i=3}^{s+1} D_i(G)$.*

Proof. Let $X = \{e_1, \ldots, e_s\}$. Let G_1 be a $\triangle$-reduction of G. By (1), $D_i(G_1^*) \subseteq D_i(G)$ for $i = 3, \ldots, s + 1$. Since a triangle is collapsible, to prove Lemma 3.3, it suffices to prove that

$$G_1^* - \{e_1, \ldots, e_s\} \text{ has a dominating eulerian subgraph } T \text{ such that } V(G_1^*) - V(T) \subseteq \bigcup_{i=3}^{s+1} D_i(G_1^*). \quad (2)$$

Claim 1. If $s = 2$, then $G_1^* - \{e_1, e_2\}$ has a dominating eulerian subgraph T such that $V(G_1^*) - V(T) \subseteq \{v\} \subseteq D_3(G_1^*)$. Furthermore, if $V(G_1^*) - V(T) = \{v\}$, then either e_1, e_2 are incident to v, or the reduction of $G_1^* - \{e_1, e_2\}$ is $K_{2,3}$.

Proof. Since G_1^* is 3-edge-connected, $G_1^* - \{e_1, e_2\}$ is connected. If G_1^* contains the triangle $\triangle_u$ with $V(\triangle_u) = \{u, w, v\}$, by Lemma 2.6, we have $\max\{d_{G_1^*}(v), d_{G_1^*}(w)\} \ge 4$. Without loss of generality, we assume that $d_{G_1^*}(w) \ge 4$. We add the new edge f_u parallel to the edge uw in G_1^*. Let $T = \{f_u : u \in D_3(G_1^*)\}$. Since G_1^* has at most two triangles that contain the vertices of degree 3, $|T| \le 2$. Let G_2 be the graph obtained from G_1^* by adding the edges in T. Then $\kappa'(G_2) \ge 4$. By Theorem 2.4(iv), $F(G_1^*) = F(G_2 - T) = 0$, and so $F(G_1^* - \{e_1, e_2\}) \le 2$. Let G' be the reduction of $G_1^* - \{e_1, e_2\}$. By Theorem 2.4(iii), $G' \in \{K_1, K_2, K_{2,t}\}$ for some odd integer $t \ge 1$.

If $G' = K_1$, then $G_1^* - \{e_1, e_2\}$ is collapsible. Hence $G_1^* - \{e_1, e_2\}$ has a spanning eulerian subgraph. If $G' = K_2$ with $V(G') = \{u_1, u_2\}$, then either $PI(u_1)$ or $PI(u_2)$ is trivial. Without loss of generality, we assume that $PI(u_1)$ is trivial. Since G_1^* is 3-edge-connected, e_1, e_2 are incident to u_1. Since $PI(u_2)$ is collapsible, $PI(u_2)$ has a spanning eulerian subgraph T. This subgraph T is a dominating eulerian subgraph of $G_1^* - \{e_1, e_2\}$ with $V(G_1^*) - V(T) = \{u_1\} \subseteq D_3(G_1^*)$. If $G' = K_{2,t}$, then $t \neq 1$ since G_1^* is 3-edge-connected, essentially 4-edge-connected. Notice that if $x \in D_2(G')$ is c-nontrivial, then both e_1, e_2 are incident to some vertices in $PI(x)$; if $x \in D_2(G')$ is c-trivial or $x \in D_3(G')$ is c-nontrivial, then either e_1 or e_2 is incident to some vertex in $PI(x)$. Thus $t \le 3$ and so $G' = K_{2,3}$. Claim 1 holds. ■

By Claim 1, we assume that $s \in \{3, 4\}$. Notice that $N_G(x_1) \cap N_G(x_2) \cap N_G(x_3) = \emptyset$ for $x_1, x_2, x_3 \in D_3(G)$. By (1), we have

$$\text{for } i \in \{3, \ldots, s+1\}, \text{ if } x \in D_i(G_1^*), \text{ then } x \in D_i(G) \text{ and } |N_{G_1^*}(x) \cap D_3(G_1^*)| \le 2. \quad (3)$$

Claim 2. If $s = 3$, then $G_1^* - \{e_1, e_2, e_3\}$ has a dominating eulerian subgraph T such that $V(G_1^*) - V(T) \subseteq D_3(G_1^*) \cup D_4(G_1^*)$ and $|V(G_1^*) - V(T)| \le 2$. Furthermore, if $V(G_1^*) - V(T) = \{x_1, x_2\}$, then $x_1, x_2 \in D_3(G_1^*)$, and if $V(G_1^*) - V(T) = \{x\}$ and $x \in D_4(G_1^*)$, then either e_1, e_2, e_3 are incident to x, or $G_1^* = G = K_5^-$ and $G_1^* - \{e_1, e_2, e_3\} = K_{2,3}$.

Proof. Assume that $G_1^* - \{e_1, e_2, e_3\}$ is not connected. Then e_1, e_2, e_3 are incident to a vertex v with $d_G(v) = 3$. As G_1^* is essentially 5-edge-connected, $d_G(x) \ge 4$ for $x \in N_G(v)$, and so $D_3(G_1^*) = \{v\}$. Let G_2 be the graph obtained from G_1^* by adding the edge e_1' that is parallel to the edge e_1. Then G_2 is 4-edge-connected. Thus $\tau(G_2 - \{e_1, e_1'\}) = \tau(G_1^* - e_1) \ge 2$. As $d_{G_1^*-e_1}(v) = 2$, $\tau(G_1^* - v) \ge 2$ and so $G_1^* - v$ is collapsible. Therefore, $G_1^* - v$ is supereulerian and $G_1^* - \{e_1, e_2, e_3\}$ has a dominating eulerian subgraph T_1 such that $V(G_1^*) - V(T_1) = \{v\} \subseteq D_3(G_1^*)$. Next we assume that $G_1^* - \{e_1, e_2, e_3\}$ is connected. Since $ess'(G) \ge 5$, $D_3(G_1^*)$ is an independent set. Thus $|D_3(G_1^*)| \le 3$.

If $|D_3(G_1^*)| = 3$, then there are three triangles $\triangle_{v_1}, \triangle_{v_2}$ and $\triangle_{v_3}$ in G_1^* such that each triangle contains one of $\{e_1, e_2, e_3\}$. Let $V(\triangle_{v_i}) = \{v_i, u_i, w_i\}$ and $e_i \in E(\triangle_{v_i})$ for $i = 1, 2, 3$. By Lemma 2.7, $G_1^* - v_1$ is 3-edge-connected and essentially 4-edge-connected. By Claim 1, $(G_1^* - v_1) - \{e_2, e_3\}$ has a dominating eulerian subgraph T_4 such that $V(G_1^* - v_1) - V(T_4) \subseteq \{y_1\} \subseteq D_3(G_1^* - v_1)$. If $V(G_1^* - v_1) = V(T_4)$, then T_4 is a spanning eulerian subgraph of $(G_1^* - v_1) - \{e_2, e_3\}$ and $T_5 = \begin{cases} T_4 & \text{if } e_1 \notin E(T_4) \\ T_4 - \{u_1w_1\} + \{v_1u_1, v_1w_1\} & \text{if } e_1 \in E(T_4) \end{cases}$ is a dominating eulerian subgraph of $G_1^* - \{e_1, e_2, e_3\}$ with $V(G_1^*) - V(T_5) \subseteq \{v\} \subseteq D_3(G_1^*)$. So we assume $V(G_1^* - v_1) - V(T_4) = \{y_1\}$. Thus $v_1y_1 \in E(G_1^*)$ (otherwise, $T_6 = \begin{cases} T_4 & \text{if } e_1 \notin E(T_4) \\ T_4 - \{u_1w_1\} + \{v_1u_1, v_1w_1\} & \text{if } e_1 \in E(T_4) \end{cases}$ is a dominating eulerian subgraph of $G_1^* - \{e_1, e_2, e_3\}$ with $V(G_1^*) - V(T_6) \subseteq \{v_1, y_1\} \subseteq D_3(G_1^*)$).

If the reduction Q of $G_1^* - v_1 - \{e_2, e_3\}$ is $K_{2,3}$ with $D_2(Q) = \{a_1, a_2, a_3\}$, then $y_1 \in \{a_1, a_2, a_3\}$. Without loss of generality, we assume that $y_1 = a_3$. Since $ess'(G_1^*) \ge 5$, $N_{G_1^*}(v_1) \cap V(PI(a_i)) \neq \emptyset (i = 1, 2)$. Thus $G_1^* - \{e_1, e_2, e_3\}$ is supereulerian. So we assume that the reduction of $G_1^* - v_1 - \{e_2, e_3\}$ is not $K_{2,3}$. By Claim 1, e_2, e_3 are incident to y_1, and so $d_{G_1^*}(y_1) = 4$. Similarly, using the above discussion on $\triangle_{v_2}$ and $\triangle_{v_3}$, there are two vertices y_2, y_3 such that $\{e_1, e_3\} \subseteq E_{G_1^*}(y_2)$ and $\{e_1, e_2\} \subseteq E_{G_1^*}(y_3)$, and $d_{G_1^*}(y_2) = d_{G_1^*}(v_3) = 4$. Then $E(y_1y_2y_3y_1) = \{e_1, e_2, e_3\}$, contrary to the fact that e_1, e_2, e_3 are on the different triangles in G_1^*. So $|D_3(G_1^*)| \le 2$.

Let G_3 be the graph obtained from G_1^* by adding the new edge v_1v_2 if $D_3(G_1^*) = \{v_1, v_2\}$, or the edge parallel to vu if $D_3(G_1^*) = \{v\}$ and $u \in N_{G_1^*}(v)$. Then G_3 is 4-edge-connected. Thus $F(G_1^* - \{e_1, e_2, e_3\}) \le 2$. Let G' be the reduction of $G_1^* - \{e_1, e_2, e_3\}$. By Theorem 2.4(iii), $G' \in \{K_1, K_2, K_{2,t}\}$, where $t \ge 1$ is an odd integer. Notice that if $x \in D_2(G')$ is c-nontrivial, $|E_{G_1^*}(PI(x)) \cap \{e_1, e_2, e_3\}| \ge 2$, and if $x \in D_2(G')$ is c-trivial, $|E_{G_1^*}(x) \cap \{e_1, e_2, e_3\}| \ge 1$. So $t \le 3$. Since $\kappa'(G_1^*) \ge 3$ and $ess'(G_1^*) \ge 4$, $t \ge 3$. So $G' = K_{2,3}$ if $G' = K_{2,t}$.

If $G' = K_1$, then $G_1^* - \{e_1, e_2, e_3\}$ is collapsible. Hence $G_1^* - \{e_1, e_2, e_3\}$ has a spanning eulerian subgraph. If $G' = K_2$ with $V(G') = \{z_1, z_2\}$, then either $PI(z_1)$ or $PI(z_2)$ is trivial. Without loss of generality, we assume that $PI(z_1)$ is trivial. Since G_1^* is 3-edge-connected, $|E_{G_1^*}(z_1) \cap \{e_1, e_2, e_3\}| \ge 2$. Since $PI(z_2)$ is collapsible, $PI(z_2)$ has a spanning eulerian subgraph T. This subgraph T is a dominating eulerian subgraph of $G_1^* - \{e_1, e_2, e_3\}$ with $V(G_1^*) - V(T) = \{z_1\} \subseteq D_3(G_1^*) \cup D_4(G_1^*)$. In addition, if $z_1 \in D_4(D_1^*)$, then e_1, e_2, e_3 are incident to z_1. If $G' = K_{2,3}$, as G is essentially 5-edge-connected, $G = G_1^* = K_5^-$ and $G_1^* - \{e_1, e_2, e_3\} = K_{2,3}$. Thus $G_1^* - \{e_1, e_2, e_3\}$ has a dominating eulerian subgraph T with $V(G_1^* - \{e_1, e_2, e_3\}) - V(T) = \{x\}$, where $x \in D_4(G_1^*)$. ■

We will finish the proof of Lemma 3.3 by proving the following claim.

Claim 3. If $s = 4$, then $G_1^* - \{e_1, e_2, e_3, e_4\}$ has a dominating eulerian subgraph T such that $V(G_1^*) - V(T) \subseteq \bigcup_{i=3}^{5} D_i(G_1^*)$.

Proof. If $G_1^* - \{e_1, e_2, e_3, e_4\}$ is not connected, then we assume that H_1, H_2 are the components of $G_1^* - \{e_1, e_2, e_3, e_4\}$. As $\kappa'(G_1^*) \ge 3$ and $ess'(G_1^*) \ge 6$, we have either H_1 or H_2 is trivial. Assume that $V(H_1) = \{v\}$. Then $d_{G_1^*}(v) \in \{3, 4\}$, $N_{G_1^*}(v) \subseteq \{e_1, e_2, e_3, e_4\}$, and $\kappa'(H_2) \ge 2$ and $ess'(H_2) \ge 4$. We assume that $e_1, e_2, e_3 \in E_{G_1^*}(v)$. As $d_{G_1^*}(x) \ge 4$ for any $x \in N_{G_1^*}(v)$, G_1^* contains at most two vertices of degree three. Thus $\tau(G_1^* - e_4) \ge 2$. As $d_{G_1^*-e_4}(v) = 3$, $F(H_2) = F((G_1^* - e_4) - v) \le 1$. By Theorem 2.4(ii), H_2 is collapsible. So G_1^* has a dominating eulerian subgraph T_1 with $V(G_1^*) - V(T_1) = \{v\} \subseteq D_3(G_1^*) \cup D_4(G_1^*)$. Next we assume that $G_1^* - \{e_1, e_2, e_3, e_4\}$ is connected.

Since $ess'(G_1^*) \ge 6$, $D_3(G_1^*) \cup D_4(D_1^*)$ is independent. Let G' be the reduction of $G_1^* - \{e_1, e_2, e_3, e_4\}$. If $G' = K_1$, then $G_1^* - \{e_1, e_2, e_3, e_4\}$ is collapsible. Hence $G_1^* - \{e_1, e_2, e_3, e_4\}$ has a spanning eulerian subgraph. If $G' = K_2$ with $V(G') = \{a_1, a_2\}$, then either $PI(a_1)$ is trivial or $PI(a_2)$ is trivial. Without loss of generality, we assume that $PI(a_1)$ is trivial. As $PI(a_2)$ is collapsible, $PI(a_2)$ has a spanning eulerian subgraph T_1. This T_1 is a dominating eulerian subgraph in $G_1^* - \{e_1, e_2, e_3, e_4\}$ with $V(G_1^*) - V(T_1) = \{a_1\} \subseteq \bigcup_{i=3}^{5} D_i(G_1^*)$. So

$$\text{if } G' \in \{K_1, K_2\}, \text{ then Claim 3 is true.} \tag{4}$$

Assume that $D_3(G_1^*) = \emptyset$. Then $G_1^* = G_1$. Since G_1 is 4-edge-connected, $F(G_1^* - \{e_1, e_2, e_3, e_4\}) \le 2$. By Theorem 2.4(iii) and (4), $G' = K_{2,p}$, where $p \ge 1$ is an odd integer. As $\kappa'(G_1^*) \ge 4$ and $ess'(G_1^*) \ge 6$, $G' \ne K_{1,2}$ and $G' \ne K_{2,p}$ $(p \ge 5)$. Thus $G' = K_{2,3}$. Hence $G_1 = K_5$ and $G_1^* - \{e_1, e_2, e_3, e_4\} = K_{2,3}$, and so $G_1^* - \{e_1, e_2, e_3, e_4\}$ has a dominating eulerian subgraph T_2 such that $V(G_1^*) - V(T_2) = \{x\} \subseteq D_4(G_1^*)$.

Next we assume that there is a triangle $\triangle_v$ containing e_1 in G_1^* such that $d_{G_1^*}(v) = 3$. Let $V(\triangle_v) = \{v, u_2, u_3\}$ and $N_{G_1^*}(v) = \{u_1, u_2, u_3\}$. Then $d_{G_1^*}(u_i) \ge 5$ $(i = 1, 2, 3)$. By Lemma 2.7, $G_1^* - v$ is 3-edge-connected, essentially 5-edge-connected. Since $ess'(G_1^*) \ge 6$, we have $G_1^* - v \ne K_5^-$. By Claim 2, $(G_1^* - v) - \{e_2, e_3, e_4\}$ has a dominating eulerian subgraph T_3 with $V(G_1^* - v) - V(T_3) \subseteq D_3(G_1^* - v) \cup D_4(G_1^* - v)$ and $|V(G_1^* - v) - V(T_3)| \le 2$. If $(V(G_1^* - v) - V(T_3)) \cap \{u_1, u_2, u_3\} = \emptyset$, then $T_4 = \begin{cases} T_3 - \{u_2u_3\} + \{vu_2, vu_3\}, & \text{if } e_1 = u_2u_3 \in E(T_3) \\ T_3, & \text{otherwise} \end{cases}$ is a dominating eulerian subgraph of $G_1^* - \{e_1, e_2, e_3, e_4\}$ such that $V(G_1^*) - V(T_4) \subseteq D_3(G_1^*) \cup D_4(G_1^*)$. So we may assume that $u_i \in (V(G_1^* - v) - V(T_3)) \cap \{u_1, u_2, u_3\}$ for some $i \in \{1, 2, 3\}$. As $d_{G_1^*-v}(u_i) \ge 4$, by Claim 2, $V(G_1^* - v) - V(T_4) = \{u_i\} \subseteq D_4(G_1^* - v)$ and e_2, e_3, e_4 are incident to u_i. Thus $D_3(G_1^*) = \{v\}$. Since $\kappa'(G_1^*) \ge 3$ and $ess'(G_1^*) \ge 6$, $G^* - v$ is 4-edge-connected. Thus $F((G_1^* - v) - \{e_2, e_3, e_4\}) \le 1$ and so $F(G_1^* - \{e_1, e_2, e_3, e_4\}) \le 1$. By Theorem 2.4(ii), $G' \in \{K_1, K_2\}$. By (4), Claim 3 is true. ■

Lemma 3.4. *Let $s \ge 2$ be an integer and G be a graph having Property $\mathcal{K}(s)$. Then for any three edges e, e_1, e_2, $G - e$ has a dominating (e_1, e_2)-trail T such that $V(G) - V(T) \subseteq D_3(G)$.*

Proof. By contradiction, we assume that G is a counterexample to Lemma 3.4 with $|V(G)|$ minimized. Then there exist three edges $e, e_1, e_2 \in E(G)$ such that

$$G - e \text{ does not have a dominating } (e_1, e_2)\text{-trail } T \text{ such that } V(G) - V(T) \subseteq D_3(G). \tag{5}$$

Thus $G \notin \{K_4, W_4, W_5\}$. Let $X = \{e, e_1, e_2\}$. Since G satisfies Property $\mathcal{K}(s)$, let G_1 be a $\triangle$-reduction of G. By (1), we have $\kappa'(G_1) \ge 4$, $\kappa'(G_1^*) \ge 3$ and $ess'(G_1^*) \ge 4$. Notice that a triangle is collapsible. By Theorem 2.3(iii), (iv), and by (5),

$$(G_1^* - e)(e_1, e_2) \text{ has no a dominating } (v(e_1), v(e_2))\text{-trail } T \text{ with } V((G_1^* - e)(e_1, e_2)) - V(T) \subseteq D_3(G_1^*). \tag{6}$$

Therefore, $(G_1^*-e)(e_1,e_2)$ is not collapsible. Since G is 3-edge-connected and essentially 4-edge-connected, $G_1^*(e_1,e_2)$ is 2-edge-connected and essentially 4-edge-connected, and $(G_1^*-e)(e_1,e_2)$ is 2-edge-connected and essentially 3-edge-connected. Let G' be the reduction of $(G_1^*-e)(e_1,e_2)$. Then $G'\notin\{K_1,K_2\}$.

Claim 1. (i) Each vertex in $D_2(G')$ is c-trivial. Therefore, $D_2(G')\subseteq\{v(e_1),v(e_2),v,u\}$, where $e=uv$.
(ii) If $x\in D_3(G')$ is c-nontrivial, then e is incident to a vertex in $PI(x)$.
(iii) $F((G_1^*-e)(e_1,e_2))\ge 3$, and $2d_2(G')+d_3(G')\ge 10$.

Proof. If $x\in D_2(G')$ is c-nontrivial, then e is incident to a vertex in $PI(x)$. Without loss of generality, we assume that $v\in PI(x)$. Since $G_1^*(e_1,e_2)$ is essentially 4-edge-connected, $V((G_1^*-e)(e_1,e_2))-V(PI(x))=\{u\}$ and $d_{G_1^*}(u)=3$. Thus $G'=2K_2$, a contradiction. Thus any vertex in $D_2(G')$ is trivial, and so $D_2(G')\subseteq\{v(e_1),v(e_2),v,u\}$. Since G_1^* is essentially 4-edge-connected, (ii) holds.

Assume that $F((G_1^*-e)(e_1,e_2))\le 2$. By Theorem 2.4(iii), $G'\in\{K_{2,2},K_{2,3},K_{2,4}\}$. If $G'=K_{2,2}$, then $G'=v(e_1)uv(e_2)vv(e_1)$. Thus $G_0=G_1^*=3K_2$, contrary to the hypothesis that G is a simple graph. If $G'=K_{2,4}$, then $v(e_1),v(e_2)\in D_2(G')$ and G' has a spanning $(v(e_1),v(e_2))$-trail. Thus $(G_1^*-e)(e_1,e_2)$ has a spanning $(v(e_1),v(e_2))$-trail, contrary to (6). So $G'=K_{2,3}$. If $D_2(G')=\{v(e_1),v(e_2),v\}$, then G' has a spanning $(v(e_1),v(e_2))$-trail. Hence, $(G_1^*-e)(e_1,e_2)$ has a spanning $(v(e_1),v(e_2))$-trail, contrary to (6). If $D_2(G')=\{v(e_1),u,v\}$ with $D_3(G')=\{a,b\}$, then $v(e_2)\in PI(a)\cup PI(b)$. Without loss of generality, we assume that $v(e_2)\in PI(a)$. Then the edge cut between $V(PI(a))$ and $V(G_1^*)-V(PI(a))$ is an essential 3-edge cut in G_1^*, a contradiction. So $F((G_1^*-e)(e_1,e_2))\ge 3$. By Theorem 2.4(iv) and the fact that $(G_1^*-e)(e_1,e_2)$ is 2-edge-connected, $2d_2(G')+d_3(G')\ge 10$. ■

Claim 2. $|D_3(G_1^*)|\ge 2$.

Proof. By contradiction, we assume that $|D_3(G_1^*)|\le 1$. If there is a triangle $xyzx$ in G_1^* with $d_{G_1^*}(x)=3$, by Lemma 2.6, we have either $d_{G_1^*}(y)\ge 4$ or $d_{G_1^*}(z)\ge 4$. Let G_2 be the graph obtained from G_1^* by adding the edge parallel to xz if $D_3(G_1^*)=\{x\}$ with $V(\triangle_x)=\{x,y,z\}$ and $d_{G_1^*}(y)\ge 4$, or $G_2=G_1^*$ if $D_3(G_1^*)=\emptyset$. Then G_2 is 4-edge-connected. Thus $\tau(G_1^*-e_1)\ge 2$ and so $F((G_1^*-e_1)(e_2,e_3))\le 2$, contrary to Claim 1(iii). Claim 2 holds. ■

Claim 3. $|D_3(G_1^*)|=3$.

Proof. Assume that G_1^* contains exactly two triangles $\triangle_{v_1}$ and $\triangle_{v_2}$ with $V(\triangle_{v_i})=\{v_i,u_i,w_i\}(i=1,2)$. Then $\{v_1,v_2\}\subseteq D_3(G_1^*)$ and $\tau(G_1^*)\ge 2$. For $i=1,2$, by Lemma 2.6, either $d_{G_1^*}(w_i)\ge 4$ or $d_{G_1^*}(u_i)\ge 4$. Without loss of generality, we assume that $d_{G_1^*}(w_i)\ge 4$.

Claim 3.1. If $E(\triangle_{v_1})$ contains e_1 only, then $e_1=u_1w_1$, and $\{u_1,w_1\}\cap D_3(G_1^*)=\emptyset$.

Proof. By contradiction, we assume that $e_1=v_1u_1$. Let $G_{11}^*=G_1^*/E(\triangle_{v_1})$ and let z_1 be the vertex in G_{11}^* to which $\triangle_{v_1}$ is contracted. Let G_2 be the graph obtained from G_{11}^* by adding the new edge f parallel to v_2u_2. Then G_2 is 4-edge-connected. Thus $\tau(G_2-\{f,e\})=\tau(G_{11}^*-e)\ge 2$ and so $F((G_{11}^*-e)(e_2))\le 1$. Since $(G_{11}^*-e)(e_2)$ is 2-edge-connected, by Theorem 2.4(ii), $(G_{11}^*-e)(e_2)$ is collapsible. Thus $(G_{11}^*-e)(e_2)$ has a spanning $(v(e_2),z_1)$-trail. By Lemma 2.5(i), $(G_1^*-e)(e_1,e_2)$ has a dominating eulerian trail T such that $V((G_1^*-e)(e_1,e_2))-V(T)\subseteq\{v_1\}\subseteq D_3(G_1^*)$, contrary to (6). So $e_1=u_1w_1$. If $u_1\in D_3(G_1^*)$, then $\triangle_{u_1}=\triangle_{v_1}$. Using the above discussion on u_1, $(G_1^*-e)(e_1,e_2)$ has a dominating eulerian trail T such that $V((G_1^*-e)(e_1,e_2))-V(T)\subseteq\{u_1\}\subseteq D_3(G_1^*)$, contrary to (6). So $\{u_1,w_1\}\cap D_3(G_1^*)=\emptyset$. Claim 3.1 holds. ■

Claim 3.2. $e,e_1,e_2\in E(\triangle_{v_1})\cup E(\triangle_{v_2})$.

Proof. Assume that $e\notin E(\triangle_{v_1})\cup E(\triangle_{v_2})$. Then for $i=1,2$, $|E(\triangle_{v_i})\cap\{e_1,e_2\}|=1$. By Claim 3.1, $\{u_1,w_1,u_2,w_2\}\cap D_3(G_1^*)=\emptyset$. Let G_3 be the graph obtained from G_1^* by adding the edge v_1v_2. Then G_3 is 4-edge-connected. Thus $\tau(G_1^*-e)\ge 2$ and so $F((G_1^*-e)(e_1,e_2))\le 2$, contrary to Claim 1(iii). So $e\in E(\triangle_{v_1})\cup E(\triangle_{v_2})$.

Assume that $e_1\notin E(\triangle_{v_1})\cup E(\triangle_{v_2})$. Also we assume that the triangles $\triangle_{v_1},\triangle_{v_2}$ contain e and e_2, respectively. By Claim 3.1, $e_2=u_2w_2$ and $d_{G_1^*}(u_2)\ge 4$ and $d_{G_1^*}(w_2)\ge 4$. Let $v',u',w'\in V(G')$ whose preimages contain v_1,u_1,w_1, respectively. By Claim 1(i), $d_2(G')\le 4$. If $d_2(G')=4$, then $D_2(G')=\{v(e_1),v(e_2),v_1,u_1\}$, where $e=v_1u_1$. Thus $d_{G_1^*}(u_1)=3$. By Claim 1(ii), each vertex in $D_3(G')$ is c-trivial. Thus $D_3(G')\subseteq\{v_2\}$, and so $2d_2(G')+d_3(G')\le 9$, contrary to Claim 1(iii). If $d_2(G')=3$, Then $D_2(G')=\{v_1,v(e_1),v(e_2)\}$. Thus $D_3(G')\subseteq\{v_2,u',w'\}$, and so $2d_2(G')+d_3(G')\le 9$. If $d_2(G')\le 2$, then $D_3(G')\subseteq\{v_2,v',u',w'\}$, and so $2d_2(G')+d_3(G')\le 8$, contrary to Claim 1(iii). So Claim 3.2 holds. ■

We use the following two cases to finish the proof of Claim 3.

Case 1. $e_1,e_2\in E(\triangle_{v_1})$, and $e\in E(\triangle_{v_2})$.

Without loss of generality, we assume that $e_2=v_1w_1$. First we prove that $e_1=u_1w_1$. Otherwise, $e_1=v_1u_1$. As $\tau(G_1^*)\ge 2$, $F(G_1^*-e)\le 1$. By Theorem 2.4(ii), G_1^*-e is collapsible. Let T_1 be a spanning eulerian subgraph of G_1^*-e. Then

$|E(T_1) \cap E_{G_1^*}(v_1)| = 2$. Let $E_{G_1^*}(v_1) = \{v_1u_1, v_1w_1, f_1\}$. Then $T_2 = \begin{cases} T_1 - \{e_1\} + \{u_1v(e_1), v_1v(e_2)\}, & \text{if } e_1, f_1 \in E(T_1) \\ T_1 - \{e_1, e_2\} + \{v(e_1)u_1, v(e_2)w_1\}, & \text{if } e_1, e_2 \in E(T_1) \end{cases}$ is a dominating $(v(e_1), v(e_2))$-trail in $(G_1^* - e)(e_1, e_2)$ with $V((G_1^* - e)(e_1, e_2)) - V(T_2) \subseteq \{v_1\} \subseteq D_3(G_1^*)$, contrary to (6). So $e_1 = u_1w_1$.

Consider $G_4 = G_1^* - \{e, e_2\}$. Then $F(G_4) \leq 2$. Since $\kappa'(G_1^*) \geq 3$ and $ess'(G_1^*) \geq 4$, and since e, e_2 are in different triangles, G_4 is 2-edge-connected. Let G_4' be the reduction of G_4. By Theorem 2.4(iii), $G_4' \in \{K_1, K_{2,p}\}(p \geq 2)$. Notice that if $x \in D_2(G_4')$ is c-nontrivial, then both e, e_2 are incident to some vertices in $PI(x)$; if $x \in D_2(G_4')$ is c-trivial or $x \in D_3(G_4')$ is c-nontrivial, then either e or e_2 is incident to some vertex in $PI(x)$. So $p \leq 4$. Furthermore, $G_4' \neq K_{2,3}$ (otherwise, $G = W_4$, a contradiction). By Theorem 2.4(iii), G_4 is supereulerian. Let T_3 be a spanning eulerian subgraph of G_4. Then $T_4 = \begin{cases} T_3 - \{e_1\} + \{u_1v(e_1), w_1v(e_2)\}, & \text{if } e_1 \in E(T_3) \\ T_3 + \{v(e_1)w_1, v(e_2)w_1\}, & \text{if } e_1 \notin E(T_3) \end{cases}$ is a spanning $(v(e_1), v(e_2))$-trail in $(G_1^* - e)(e_1, e_2)$, contrary to (6).

Case 2. $e, e_1 \in E(\triangle_{v_1})$, $e_2 \in E(\triangle_{v_2})$,

We claim that $e_1 = w_1u_1$. Otherwise, assume that $e_1 = v_1w_1$. Let $G_5 = (G_1^* - e)(e_2)$. Then $\kappa'(G_5) \geq 2$ and $ess'(G_5) \geq 3$. Let G_5' be the reduction of G_5. Then each vertex $x \in D_2(G_5')$ is c-trivial. As $d_2(G_5) \leq 3$, $d_2(G_5') \leq 3$. Furthermore, if $d_2(G_5') = 3$, then $D_2(G_5') = \{v_1, u_1, v(e_2)\}$, where $e = v_1u_1$ and $d_{G_1^*}(u_1) = 3$. Since $\tau(G_1^*) \geq 2$, $F(G_5) \leq 2$. By Theorem 2.4(iii), $G_5' \in \{K_1, K_{2,3}\}$. If $G_5' = K_{2,3}$, then $G = K_4$, a contradiction. Thus $G_5' = K_1$. So G_5 has a spanning $(v_1, v(e_2))$-trail T_5. Thus $T_6 = \begin{cases} T_5 + v_1v(e_1), & \text{if } e_1 \notin E(T_5) \\ T_5 - \{e_1\} + \{w_1v(e_1)\}, & \text{if } e_1 \in E(T_5) \end{cases}$ is a dominating $(v(e_1), v(e_2))$-trail in $(G_1^* - e)(e_1, e_2)$ with $V((G_1^* - e)(e_1, e_2)) - V(T_6) \subseteq \{v_1\} \subseteq D_3(G_1^*)$, contrary to (6). So $e_1 = w_1u_1$. Using this discussion, we can get $d_{G_1^*}(u_1) \geq 4$ and $d_{G_1^*}(w_1) \geq 4$. By Claim 3.1, $e_2 = w_2u_2$ and $\{u_2, w_2\} \cap D_3(G_1^*) = \emptyset$. Thus $G_1^* + v_1v_2$ is 4-edge-connected, and so $F((G_1^* - e)(e_1, e_2)) \leq 2$, contrary to Claim 1(iii). We finish the proof of Claim 3. ∎

By Claim 3, we assume that three edges e, e_1 and e_2 belong to 3 distinct triangles $\triangle_v, \triangle_{v_1}$, and $\triangle_{v_2}$, respectively. Let $f = vx \in E_{G_1^*}(v) - E(\triangle_v), f_1 = v_1x_1 \in E_{G_1^*}(v_1) - E(\triangle_{v_1})$ and $f_2 = v_2x_2 \in E_{G_1^*}(v_2) - E(\triangle_{v_2})$. Let $V(\triangle_v) = \{v, u, w\}, V(\triangle_{v_1}) = \{v_1, u_1, w_1\}$, and $V(\triangle_{v_2}) = \{v_2, u_2, w_2\}$. Also we assume that z, z_1, z_2 are vertices in G_1 to which $\triangle_v, \triangle_{v_1}, \triangle_{v_2}$ are contracted, respectively. Let $G_2 = G_1^*/E(\triangle_{v_1}) \cup E(\triangle_{v_2})$. Then $\kappa'(G_2) \geq 3$ and $ess'(G_2) \geq 4$, and $\tau(G_2 - e) \geq 2$ and $\tau(G_2 - f_i) \geq 2$ $(i = 1, 2)$. Let $G_3 = G_2 - \{f_1, f_2\}$ and $G_4 = G_2 - \{e, f_1, f_2\}$. Then $F(G_3) \leq 1$ and $F(G_4) \leq 2$.

If G_3 has a cut edge e', then $f_1 \neq f_2$ and $\{e', f_1, f_2\}$ is a 3-edge-cut of G_2. Thus $v_1v_2 \notin E(G_1^*)$. As G_2 is essentially 4-edge-connected, e', f_1, f_2 are incident to a vertex y. Thus $d_{G_1^*}(y) = 3$. As $d_{G_2}(z_i) \geq 4 (i = 1, 2)$, $x_1 = x_2 = y$. Let $e' = yq$. Since G is claw-free, we have either $v_1q \in E(G)$ or $yv_2 \in E(G)$. Without loss of generality, we assume that $v_1q \in E(G)$. This implies that $\{q, y_1, u_1, w_1\} \subseteq N_{G_1^*}(v_1)$, contrary to the fact that $d_{G_1^*}(v_1) = 3$. So G_3 is 2-edge-connected.

As $F(G_3) \leq 1$, by Theorem 2.4(ii), G_3 is collapsible, so G_3 has a spanning (z_1, z_2)-trail T. By Lemma 2.5(ii) and (6), $e \in E(T)$. If $|E(\triangle_v) \cap E(T)| = 1$, then $T' = (T - \{e\}) + (E(\triangle_v) - \{e\})$ is a spanning (z_1, z_2)-trail in G_3. By Lemma 2.5(ii), $(G_1^* - e)(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail, a contradiction. So $|E(\triangle_v) \cap E(T)| \geq 2$. Furthermore, we have the following.

$$\text{if } e = vu \text{ and } d_{G_1^*}(u) = 3, \text{ then } |E(\triangle_v) \cap E(T)| = 3. \tag{7}$$

(Otherwise, then $|E(\triangle_v) \cap E(T)| = 2$. Since $d_{G_1^*}(u) = 3$, by symmetry, we may assume that $vu, uw \in E(T)$ and $vw \notin E(T)$. Then $T' = (T - \{vu, uw\}) \cup \{vw\}$ is a dominating (z_1, z_2)-trail in G_3. By Lemma 2.5(ii), $(G_1^* - e)(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T'' with $V((G_1^* - e)(e_1, e_2)) - V(T'') = \{u\} \subseteq D_3(G_1^*)$, a contradiction).

Consider $G_5 = G_1^*/E(\triangle_{v_2})$. Then $\kappa'(G_5) \geq 3$ and $ess'(G_5) \geq 4$, and $\tau(G_5) \geq 2$. Thus $(G_5 - e)(e_1)$ is 2-edge-connected and $F((G_5 - e)(e_1)) \leq 2$. Let G_5' be the reduction of G_5. Then $G_5' \in \{K_1, K_{2,p}\}(p \geq 2)$ and each vertex in $D_2(G_5')$ is c-trivial. As $d_2(G_5) \leq 3$, $p \leq 3$. If $G_5' = K_{2,3}$, then $G = G_5 = K_4$, a contradiction. So $G_5' = K_1$ and G_5 has a spanning $(v(e_1), z_2)$-trail. By Lemma 2.5(i) and (6), $e_2 = w_2u_2$. Using this discussion, we can get $d_{G_1^*}(u_2) \geq 4$ and $d_{G_1^*}(w_2) \geq 4$. Similarly, $e_1 = u_1w_1$, and $d_{G_1^*}(u_1) \geq 4$ and $d_{G_1^*}(w_1) \geq 4$.

Consider G_4. Let G_4' be the reduction of G_4. Since $F(G_4) \leq 2$, by Theorem 2.4(iii), Lemma 2.5(ii), and (6), $G_4' \in \{K_2, K_{2,p}\}(p \geq 1)$. Notice that G_2 is 3-edge-connected, essentially 4-edge-connected and $d_{G_1^*}(v_1) = d_{G_1^*}(v_2) = 3$. If $a \in D_1(G_4')$ is c-trivial, then $|E_{G_1^*}(a) \cap \{e, f_1, f_2\}| \geq 2$. If $a \in D_2(G_4')$ is c-trivial, then $|N_G(a) \cap \{e, f_1, f_2\}| \geq 1$, and if $a, b \in D_2(G_4')$ are c-trivial, then $ab \notin \{f_1, f_2\}$. If $a \in D_2(G_4')$ is c-nontrivial, then $|E_{G_1^*}(a) \cap \{e, f_1, f_2\}| \geq 2$. If $a \in D_3(G_4')$ is c-nontrivial, then $|E_{G_1^*}(a) \cap \{e, f_1, f_2\}| \geq 1$. Thus, if $G_4' = K_{2,p}$, then $p \leq 4$. So $G_4' \in \{K_2, K_{1,2}, K_{2,2}, K_{2,3}, K_{2,4}\}$.

Assume that $G_4' = K_2$ and $V(G_4') = \{b_1, b_2\}$. Then one of b_1, b_2, say b_1, is c-trivial. Thus $z_1, z_2 \in V(PI(b_2))$ and $PI(b_2)$ has a spanning (z_1, z_2)-trail. By Lemma 2.5(ii), $(G_1^* - e)(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T with $V((G_1^* - e)(e_1, e_2)) - V(T) = \{b_1\} \subseteq D_3(G_1^*)$, contrary to (6). So $G_4' \neq K_2$.

Assume that $G_4' = K_{1,2}$ and $V(G_4') = \{b_1, b_2, b_3\}$, where $d_{G_4'}(b_3) = 2$. Then b_1, b_2 are c-trivial vertices of G_4', $e = b_1b_2$, and $e, f_1 \in E_{G_1^*}(b_1)$ and $e, f_2 \in E_{G_1^*}(b_2)$. Thus $f_1 = b_1v_1$, $f_2 = b_2v_2$, $v \in \{b_1, b_2\}$, and $z_1, z_2 \in PI(b_2)$. Let $V(\triangle_v) = \{c, b_1, b_2\}$. Since $PI(b_2)$ is collapsible, $PI(b_2)$ has a spanning (c, z_2)-trail T_2. Let H be the subgraph in $G_1^*/E(\triangle_{v_2})$ induced by $E(T_2)$. Then $d_H(u_1) + d_H(w_1)$ is even. If $d_H(u_1)$ and $d_H(w_1)$ are even, then c and z_2 are in the same component of T_2. Also this component contains (at least) one of u_1 and w_1. Without loss of generality, we assume that w_1 is in this component. Let $T_3 = T_2 + \{cb_1, b_1v_1, v_1u_1, u_1v(e_1)\}$. If $d_H(u_1)$ and $d_H(w_1)$ are odd, by symmetry, we assume that c and w_1 are in a component of H and z_2 and u_1 are in a component of H. Let $T_3 = T_2 + \{cb_1, b_1v_1, v_1u_1, w_1v(e_1)\}$. Then T_3 is a dominating

$(v(e_1), z_2)$-trail of $(G_1^*/E(\triangle_{v_2})-e)(e_1)$ with $V((G_1^*/E(\triangle_{v_2})-e)(e_1))-V(T_3) = \{b_2\} \subseteq D_3(G_1^*)$. By Lemma 2.5(ii), $(G_1^*-e)(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T_3' with $V((G_1^* - e)(e_1, e_2)) - V(T_3') = \{b_2\} \subseteq D_3(G_1^*)$, contrary to (6). So $G_4' \neq K_{1,2}$.

Assume that $G_4' = K_{2,2}$ and $V(G_4') = \{b_1, b_2, b_3, b_4\}$. Then two of b_1, b_2, b_3, b_4 are c-trivial and they are not adjacent in G_4'. Without loss of generality, we assume that b_1 and b_3 are c-trivial. Then $e = b_1b_3$, and b_2 and b_4 are c-nontrivial. Thus f_1 and f_2 are edges joining vertices between $PI(b_2)$ and $PI(b_4)$. So $z_1, z_2 \in V(PI(b_2)) \cup V(PI(b_4))$. Since e is in $\triangle_v$, we may assume that $V(\triangle_v) - \{b_1, b_3\} = \{c_1\} \subseteq V(PI(b_2))$. Also we assume that $N_{G_4}(b_1) \cap V(PI(b_4)) = \{c_2\}$ and $N_{G_4}(b_3) \cap V(PI(b_4)) = \{c_3\}$. Consider G_3 and the spanning (z_1, z_2)-trail T. By (7), $b_1b_3, b_1c_1, b_3c_1 \in E(T)$. Thus $b_1c_2, b_3c_3 \notin E(T)$. It is impossible. So $G_4' \neq K_{2,2}$.

Assume that $G_4' = K_{2,3}$ and $V(G_4') = \{b_1, b_2, b_3, b_4, b_5\}$, where $d_{G_4'}(b_4) = d_{G_4'}(b_5) = 3$. Then b_1, b_2 are c-trivial vertices and b_3, b_4, b_5 are c-nontrivial vertices of G_4', and $e = b_1b_2$. Since e is in $\triangle_v$, we may assume that $V(\triangle_v) - \{b_1, b_2\} = \{c_1\} \subseteq V(PI(b_4))$. Also we assume that $N_{G_4}(b_1) \cap V(PI(b_5)) = \{c_2\}$ and $N_{G_4}(b_2) \cap V(PI(b_5)) = \{c_3\}$. Since b_3 is a c-nontrivial vertex, we assume that f_1 joins $PI(b_3)$ and $PI(b_4)$ and f_2 joins $PI(b_3)$ and $PI(b_5)$. Let c_4c_5 be the edge joining $PI(b_4)$ and $PI(b_3)$, where $c_4 \in V(PI(b_4))$ and $c_5 \in V(PI(b_3))$, and let c_6c_7 be the edge joining $PI(b_5)$ and $PI(b_3)$, where $c_6 \in V(PI(b_3))$ and $c_7 \in V(PI(b_5))$. Consider G_3 and the spanning (z_1, z_2)-trail T. By (7), $b_1b_2, b_1c_1, b_2c_1 \in E(T)$. Thus $b_1c_2, b_2c_3 \notin E(T)$. So we may assume that $z_1 \in V(PI(b_4))$ and $z_2 \in V(PI(b_5))$. Consider the subgraph Q_1 induced by $V(PI(b_5)) \cup \{b_1, b_2\}$ in G_2. Then Q_1 is collapsible. Let T_4 be a spanning (c_7, z_2)-trail in Q_1. Since $d_{Q_1}(b_1) = d_{Q_1}(b_2) = 2$, $e, b_1c_2, b_2c_3 \in E(T_4)$. Let T_5 be a spanning (z_1, c_4)-trail in $PI(b_4)$, T_6 be spanning (c_5, c_6)-trail in $PI(b_4)$. Then the subgraph induced by $(E(T_4) - \{e\}) \cup \{b_1c_1, c_1b_2\} \cup E(T_5) \cup E(T_6) \cup \{c_4c_5, c_6c_7\}$ is a spanning (z_1, z_2)-trail in G_4. By Lemma 2.5(ii), $(G_1^* - e)(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail, contrary to (6). So $G_4' \neq K_{2,3}$.

Therefore, $G_4' = K_{2,4}$. Let $V(G_4') = \{b_1, b_2, b_3, b_4, b_5, b_6\}$, where $d_{G_4'}(b_5) = d_{G_4'}(b_6) = 4$. Then b_1, b_2 are c-trivial vertices and b_3, b_4 are c-nontrivial vertices of G_4', and $e = b_1b_2$. Since e is in $\triangle_v$, we may assume that $V(\triangle_v) - \{b_1, b_2\} = \{c_1\} \subseteq V(PI(b_5))$. Also we assume that $N_{G_4}(b_1) \cap V(PI(b_6)) = \{c_2\}$ and $N_{G_4}(b_2) \cap V(PI(b_6)) = \{c_3\}$. Since b_3, b_4 are c-nontrivial vertices, f_1 and f_2 join $PI(b_3)$ and $PI(b_4)$, so $z_1, z_2 \in V(PI(b_3)) \cup V(PI(b_4))$. Let $c_4, c_6 \in V(PI(b_5))$, $c_5, c_9 \in V(PI(b_3))$, $c_7, c_{11} \in V(PI(b_4))$, and $c_8, c_{10} \in V(PI(b_6))$ such that $c_4c_5, c_6c_7, c_8c_9, c_{10}c_{11} \in E(G_4)$. Consider G_3 and the spanning (z_1, z_2)-trail T. By (7), $b_1b_2, b_1c_1, b_2c_1 \in E(T)$. Thus $b_1c_2, b_2c_3 \notin E(T)$. So $z_1, z_2 \in V(PI(b_3))$ or $z_1, z_2 \in V(PI(b_4))$. Without loss of generality, we assume that $z_1, z_2 \in V(PI(b_3))$. Consider the subgraph Q_2 induced by $V(PI(b_6)) \cup \{b_1, b_2\}$ in G_2. Then Q_2 is collapsible. Thus there is a spanning (c_8, c_{10})-trail T_7 in Q_2. Since $d_{Q_2}(b_1) = d_{Q_2}(b_2) = 2$, $e, b_1c_2, b_2c_3 \in E(T_7)$. Let Q_3 be the graph obtained from $PI(b_3)$ by adding a new vertex c_{12} and the new edges $c_{12}z_1$ and $c_{12}z_2$. Then Q_3 is collapsible. Let T_8 be a spanning (c_5, c_9)-trail in Q_3. Then $c_{12}z_1, c_{12}z_2 \in E(T_8)$. Let $T_9 = T_8 - \{c_{12}\}$. Let T_{10} be the spanning (c_7, c_{11})-trail in $PI(b_4)$, T_{11} be the spanning (c_4, c_6)-trail in $PI(b_5)$. Then the subgraph induced by $E(T_9) \cup (E(T_7) - \{e\}) \cup \{b_1c_1, c_1b_2\} \cup E(T_{10}) \cup E(T_{11}) \cup \{c_4c_5, c_6c_7, c_8c_9, c_{10}c_{11}\}$ is a spanning (z_1, z_2)-trail in G_4. By Lemma 2.5(ii), $(G_1^*-e)(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail, contrary to (6). ■

4. Proof of Theorem 1.6

In this section we assume that s is a positive integer, and assume that G is connected with $ess'(G) \geq 4$. Following [17], we define the **core** of G, denoted by G_0, to be the graph obtained from G by deleting all the vertices of degree 1, and contracting the edge xy for each path xyz for each $y \in D_2(G)$. As shown in [17], we observe that G_0 is well-defined, and

$$G_0 \text{ is claw-free with } \kappa'(G_0) \geq 3,\ ess'(G_0) \geq 4 \text{ and } D_3(G_0) = D_3(G). \tag{8}$$

We need one more notation. Let $e = xy \in E(W_5)$ with $x, y \in D_3(W_5)$ and H be a graph and $e' = x'y' \in E(H)$. Define a new graph $H \oplus W_5$ to be a graph obtained from the disjoint union of $H - e$ and W_5 by identifying x and x' to form a new vertex, also called x, and by identifying y and y' to form a new vertex, also called y.

Lemma 4.1. *Suppose that $s \geq 2$ and that G is claw-free such that $\kappa(L(G)) \geq s + 2$. Let G_0 be the core of G and let $w_1, w_2, w_3 \in D_3(G_0)$ be vertices with $N_{G_0}(w_2) = \{w_1, w_3, v\}$. If $vw_1, vw_3 \in E(G_0)$, then each of the following holds.*
(i) $s = 2$.
(2) either $G = G_0 \in \{K_4, W_4, W_5\}$, or there exists a subgraph Γ of G with $\kappa'(\Gamma) \geq 3$ and $ess'(\Gamma) \geq 4$ such that $G_0 = \Gamma \oplus W_5$.

Proof. Since $(E_{G_0}(w_1) - \{w_1w_2\}) \cup \{w_2v, w_2w_3\}$ is an essential 4-edge cut of G_0, we must have $s = 2$. If $w_1w_3 \in E(G_0)$ or $d_{G_0}(v) = 3$, then by Lemma 2.6, we have $G = G_0 = K_4$. Thus we assume that $d_{G_0}(v) \geq 4$ and $w_1w_3 \notin E(G_0)$. Let $w_4 \in N_{G_0}(v) - \{w_1, w_2, w_3\}$. As G_0 is claw-free and by symmetry, we may assume that $w_4w_3 \in E(G_0)$.

If $d_{G_0}(v) = 4$, then $w_1w_4 \in E(G_0)$ (otherwise, let $z \in N_{G_0}(w_1) - \{v, w_2\}$. Then $\{zw_1, w_4v, w_4w_3\}$ is an essential 3-edge cut in G_0, a contradiction). As G_0 is claw-free and $d_{G_0}(w_1) = d_{G_0}(w_3) = 3$, $G_0 = W_4$. Since G is essentially 4-edge-connected, $G = G_0 = W_4$.

Assume that $d_{G_0}(v) \geq 5$. Let $w_5 \in N_{G_0}(v) - \{w_1, w_2, w_3, w_4\}$. Since G_0 is claw-free and since $w_1, w_3 \in D_3(G_0)$, we have $w_1w_5 \in E(G_0)$ and $d_{G_0}(v) = 5$. Since $G_0[\{v, w_2, w_4, w_5\}] \neq K_{1,3}$, we must have $w_4w_5 \in E(G_0)$. Let $X = N_{G_0}(w_4) \cup N_{G_0}(w_5) - \{v, w_1, w_2, \ldots, w_5\}$. If $X = \emptyset$, then $G_0 = W_5$, and so $G = G_0$. Assume that $X = \{v_1, \ldots, v_k\} \neq \emptyset$. As G_0 is claw-free, $G_0[\{v_1, \ldots, v_k, w_4, w_5\}] = K_{k+2}$, as depicted in Fig. 2. Since $\kappa'(G_0) \geq 3$, we have $k \geq 2$. Let $\Gamma = G_0 - \{w_1, w_2, w_3, v\}$. Then $G_0 = \Gamma \oplus W_5$. As $G_0[\{v_1, \ldots, v_k, w_4, w_5\}] = K_{k+2}$ and $k \geq 2$, we conclude that $\kappa'(\Gamma) \geq 3$ and $ess'(\Gamma) \geq 4$. ■

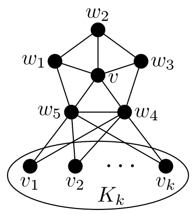

Fig. 2. $K_{k+2} \oplus W_5$ in Lemma 4.1.

Throughout the rest of the proofs, we will adopt the following notation and assumptions. Let $s \geq 2$ be an integer, G be a claw-free graph, $H = L(G)$ with $\kappa(L(G)) \geq s+2$ in the proof of Theorem 1.6(i) or $\kappa(L(G)) \geq 4$ in the proof of Theorem 1.6(ii). Since every complete graph of order at least $s+3$ is s-hamiltonian and 1-hamiltonian-connected, we will assume that $L(G)$ is not a complete graph, and so $ess'(G) = \kappa(L(G))$. Let G_0 be the core of G. As shown in [17], we have $\kappa'(G_0) \geq 3$ and $ess'(G_0) \geq \kappa(L(G))$. Thus if $ess'(G_0) \geq s+2$, then for $i = 3, \ldots, s+1$, we have $D_i(G) = D_i(G_0)$. As G is claw-free, G_0 is also claw-free.

Proof of Theorem 1.6. (i). It suffices to prove that if $\kappa(L(G)) \geq s+2$, then $L(G)$ is s-hamiltonian. By Theorem 1.4, we assume that $s \in \{2, 3, 4\}$. To prove H is s-hamiltonian, it suffices to prove that for any $X = \{e_1, \ldots, e_s\} \subset E(G_0)$,

$$G_0 - X \text{ has a dominating eulerian subgraph } T \text{ such that } V(G_0) - V(T) \subseteq \bigcup_{i=3}^{s+1} D_i(G_0). \tag{9}$$

If $G_0 \in \{K_4, W_4, W_5\}$, then $s = 2$ and $G = G_0$. Thus (9) holds, and so we may assume that $G_0 \notin \{K_4, W_4, W_5\}$.

If $s \geq 3$, then as G_0 is claw-free and essentially 5-edge-connected, for any $x_1, x_2, x_3 \in D_3(G_0)$, we have $N_{G_0}(x_1) \cap N_{G_0}(x_2) \cap N_{G_0}(x_3) = \emptyset$. Hence by Lemma 3.3, G_0 does not have Property $\mathcal{K}(s)$. Since G_0 is claw-free, G_0 must violate (KS2). Arguing by contradiction, we assume that

$$G \text{ is a counterexample to Theorem 1.6(i) with } |V(G_0)| \text{ minimized.} \tag{10}$$

Since G_0 violates (KS2), there exist $w_1, w_2, w_3 \in D_3(G_0)$ with $N_{G_0}(w_2) = \{w_1, w_3, v\}$ and $vw_1, vw_3 \in E(G_0)$. Since $G_0 \notin \{K_4, W_4, W_5\}$, by Lemma 4.1, $s = 2$ and $G_0 = \Gamma \oplus W_5$, for a claw-free graph Γ with $\kappa'(\Gamma) \geq 3$ and $ess'(\Gamma) \geq 4$. Assume that $V(W_5) = \{v, w_1, \ldots, w_5\}$ with $w_4w_5 \in E(\Gamma) \cap E(W_5)$, as depicted in Fig. 2.

If $e_1, e_2 \in E(\Gamma)$, then by (10), $\Gamma - \{e_1, e_2\}$ has a dominating eulerian subgraph T_1 such that $V(\Gamma) - V(T_1) \subseteq D_3(\Gamma)$. Thus $T_2 = T_1 + w_1w_2w_3w_4vw_5w_1$ is a dominating eulerian subgraph in $G_0 - \{e_1, e_2\}$ such that $V(G_0) - V(T_2) \subseteq D_3(G_0)$, a contradiction.

If $e_1 \in E(\Gamma)$ and $e_2 \in E(W_5) - E(\Gamma)$, then by (10), $\Gamma - \{e_1, w_4w_5\}$ has a dominating eulerian subgraph T_3 such that $V(\Gamma) - V(T_3) \subseteq D_3(\Gamma)$. By Theorem 2.3(iii), $W_5 - e_2$ is collapsible. Thus $W_5 - e_2$ has a spanning eulerian subgraph T_4. Therefore, $L_1 = G_0[E(T_3) \cup E(T_4)]$ is a dominating eulerian subgraph in $G_0 - \{e_1, e_2\}$ such that $V(G_0) - V(L_1) \subseteq D_3(G_0)$, a contradiction.

If $e_1, e_2 \in E(W_5) - E(\Gamma)$, then $W_5 - \{e_1, e_2\}$ has a dominating eulerian subgraph T_5 such that $V(W_5) - V(T_5) \subseteq D_3(G_0)$. By (10), $\Gamma - \{w_4w_5\}$ has a dominating eulerian subgraph T_6 such that $V(\Gamma) - V(T_6) \subseteq D_3(\Gamma)$. Thus $L_2 = G_0[E(T_5) \cup E(T_6)]$ is a dominating eulerian subgraph in $G_0 - \{e_1, e_2\}$ such that $V(G_0) - V(L_2) \subseteq D_3(G_0)$, a contradiction. These contradictions establish the theorem. ∎

Proof of Theorem 1.6. (ii). By Theorem 2.1(ii), it suffices to show that for any three edges $e, e_1, e_2 \in E(G)$, $G - e$ has a dominating (e_1, e_2)-trail. In view of this goal, for any $y \in D_2(G)$ with $N_G(y) = \{x_y, z_y\}$, we may assume that $x_y y \notin \{e, e_1, e_2\}$. With this, and letting G_0 be the core of G, it suffices to assume that $e, e_1, e_2 \in E(G_0)$, and to show $G_0 - e$ has a dominating (e_1, e_2)-trail T with $V(G_0) - V(T) \subseteq D_3(G_0)$. By contradiction, we assume that G is a counterexample to Theorem 1.6(ii) with $|V(G_0)|$ minimized. Thus by Lemma 2.2, there exist edges $e, e_1, e_2 \in E(G_0)$, with G_0^* denoting $(G_0 - e)(e_1, e_2)$, such that

$$G_0^* \text{ does not have a dominating } (v(e_1), v(e_2))\text{-trail } T \text{ such that } V(G_0^*) - V(T) \subseteq D_3(G_0). \tag{11}$$

By (11) and Theorem 2.3(iii), we assume that $G_0 \notin \{K_4, W_4, W_5\}$ and G_0^* is not collapsible. By Lemma 3.4, G_0 does not have Property $\mathcal{K}(s)$. As G_0 is claw-free, (KS2) is violated. Thus there exist $w_1, w_2, w_3 \in D_3(G_0)$ with $N_{G_0}(w_2) = \{w_1, w_3, v\}$ and $vw_1, vw_3 \in E(G_0)$. By Lemma 4.1, $G_0 = \Gamma \oplus W_5$, for a subgraph Γ of G_0 with $\kappa'(\Gamma) \geq 3$ and $ess'(\Gamma) \geq 4$. Assume that $V(W_5) = \{v, w_1, \ldots, w_5\}$ with $w_4w_5 \in E(\Gamma) \cap E(W_5)$, as depicted in Fig. 2.

If $\{e, e_1, e_2\} \cap E(W_5) = \emptyset$, then by the minimality of G_0, $(\Gamma - e)(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T_1 with $V((\Gamma - e)(e_1, e_2)) - V(T_1) \subseteq D_3(\Gamma)$. It follows from $G_0 = \Gamma \oplus W_5$ that (11) is violated. If $e, e_1, e_2 \in E(W_5)$, then by inspection, $(W_5 - e)(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T_2 that contains either w_4 or w_5. By Theorem 2.3(vi), Γ has a spanning eulerian trail T_3. Thus $T_4 = G_0^*[(E(T_2) - E(T_3)) \cup (E(T_3) - E(T_2))]$ is a dominating $(v(e_1), v(e_2))$-trail in G_0^* with $V(G_0^*) - V(T_4) \subseteq D_3(G_0)$, contrary to (11). Thus we assume that $\{e, e_1, e_2\} \cap (E(\Gamma) - E(W_5)) \neq \emptyset$ and $\{e, e_1, e_2\} \cap (E(W_5) - E(\Gamma)) \neq \emptyset$.

Assume that $e \in E(W_5)$. If $e_1 \in E(W_5)$, then $e_2 \in E(\Gamma) - E(W_5)$. By Theorem 2.3(ii), $(W_5 - e)(e_1)$ is collapsible. By Theorem 2.3(vi), $\Gamma(e_2)$ is collapsible. Thus G_0^* is collapsible, a contradiction. If $e_1, e_2 \in E(\Gamma)$, by Theorem 2.3(vi), $\Gamma(e_1, e_2)$ is collapsible. Thus G_0^* is collapsible, a contradiction again. So $e \in E(\Gamma) - E(W_5)$. As $\{e, e_1, e_2\} \cap (E(W_5) - E(\Gamma)) \neq \emptyset$, we assume that $e_1 \in E(W_5) - E(\Gamma)$.

Assume that $e_2 \in E(\Gamma)$. As $(W_5 - w_4w_5)(e_1)$ is collapsible, let T_5 be a spanning $(v(e_1), w_4)$-trail in $W_5(e_1)$. Let $f_1 \in E_\Gamma(w_4) - \{w_4w_5, e\}$. By the choice of G, $(\Gamma - e)(e_2, f_1)$ has a dominating $(v(e_2), v(f_1))$-trail T_6 with $V((\Gamma - e)(e_2, f_1)) - V(T_6) \subseteq D_3(\Gamma)$. Let $E_1 = \begin{cases} E(T_6) - \{w_4v(f_1)\}, & \text{if } w_4v(f_1) \in E(T_6) \\ E(T_6) \cup \{w_4v(f_1)\}, & \text{if } w_4v(f_1) \notin E(T_6) \end{cases}$. Then the subgraph T_7 induced by $E(T_5) \cup E_1$ is a dominating $(v(e_1), v(e_2))$-trail in G_0^* with $V(G_0^*) - V(T_7) \subseteq D_3(G_0)$, contrary to (11). So $e_2 \in E(W_5) - E(\Gamma)$.

Let $f_2 \in E_\Gamma(w_4) - \{w_4w_5, e\}$. By the choice of G, $(\Gamma - e)(f_2, w_4w_5)$ has a dominating $(v(f_2), v(w_4w_5))$-trail T_8 with $V((\Gamma - e)(f_2, w_4w_5)) - V(T_8) \subseteq D_3(\Gamma)$. Let $M = \{w_4v(f_2), w_4v(w_4w_5)\}$ and $E_2 = (E(T_8) - M) \cup (M - E(T_8))$. By Theorem 2.3(vi), $W_5(e_1, e_2)$ is collapsible. Thus $W_5(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail T_9. So the subgraph T_{10} induced by $(E(T_9) - E_2) \cup (E_2 - E(T_9))$ is a dominating $(v(e_1), v(e_2))$-trail in G_0^* with $V(G_0^*) - V(T_{10}) \subseteq D_3(G_0)$, contrary to (11). ■

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Declaration of competing interest

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