Linear Algebra and its Applications 579 (2019) 72–88

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Vertex-connectivity and eigenvalues of graphs ☆

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ARTICLE INFO

Article history:
Received 16 October 2018
Accepted 22 May 2019
Available online 28 May 2019
Submitted by S.M. Cioaba

MSC:
05C50
05C40

Keywords:
Vertex-connectivity
Algebraic connectivity
Adjacency eigenvalue
Laplacian eigenvalue
Signless Laplacian eigenvalue

ABSTRACT

Let $\kappa(G)$, $\mu_{n-1}(G)$, $\lambda_2(G)$ and $q_2(G)$ denote the vertex-connectivity, the algebraic connectivity, the second largest adjacency eigenvalue, and the second largest signless Laplacian eigenvalue of G, respectively. In this paper, we prove that for an integer $k > 0$ and any simple graph G of order n with maximum degree Δ and minimum degree $\delta \geq k$, the vertex-connectivity $\kappa(G) \geq k$ if $\mu_{n-1}(G) > \mathcal{H}_2(\Delta, \delta, k)$ or $\lambda_2(G) < \delta - \mathcal{H}_2(\Delta, \delta, k)$ or $q_2(G) < 2\delta - \mathcal{H}_2(\Delta, \delta, k)$, where $\mathcal{H}_2(\Delta, \delta, k) = \frac{(k-1)n\Delta}{(n-k+1)(k-1)+4(\delta-k+2)(n-\delta-1)}$, which improves the result in [Appl. Math. Comput. 344–345 (2019) 141–149] and the result in [Electron. J. Linear Algebra 34 (2018) 428–443]. Analogue results involving $\mu_{n-1}(G)$, $\lambda_2(G)$ and $q_2(G)$ to characterize vertex-connectivity of regular graphs, triangle-free graphs and graphs with fixed girth are also presented.

☆ The research of Zhen-Mu Hong is supported by NSFC (No. 11601002), Outstanding Young Talents International Visiting Program of Anhui Provincial Department of Education (No. gxgwfx2018031) and Key Projects in Natural Science Research of Anhui Provincial Department of Education (No. KJ2016A003). The research of Zheng-Jiang Xia is supported by Key Projects in Natural Science Research of Anhui Provincial Department of Education (No. KJ2018A0438). The research of Hong-Jian Lai is supported by NSFC (Nos. 11771039 and 11771443).

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https://doi.org/10.1016/j.laa.2019.05.030
0024-3795/

1. Introduction

We only consider finite and simple graphs in this paper. Undefined notation and terminologies will follow Bondy and Murty [3]. Let $G = (V, E)$ be a graph of order n. We use $\kappa(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex-connectivity, the minimum degree and the maximum degree of a graph G, respectively. For a vertex subset $S \subseteq V(G)$, $G[S]$ is the subgraph of G induced by S.

Let $G = (V, E)$ be a simple graph with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G)$. The adjacency matrix of G is defined to be a $(0, 1)$-matrix $A(G) = (a_{ij})_{n\times n}$, where $a_{ij} = 1$ if v_i and v_j are adjacent, $a_{ij} = 0$ otherwise. As G is simple and undirected, $A(G)$ is a symmetric $(0, 1)$-matrix. The adjacency eigenvalues of G are the eigenvalues of $A(G)$. Denoted by $D(G) = \operatorname{diag}\{d_G(v_1), d_G(v_2), \ldots, d_G(v_n)\}$, the degree diagonal matrix of G, where $d_G(v_i)$ denotes the degree of v_i. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are called the Laplacian matrix and the signless Laplacian matrix of G, respectively. We use $\lambda_i(G)$, $\mu_i(G)$ and $q_i(G)$ to denote the ith largest eigenvalue of $A(G)$, $L(G)$ and $Q(G)$, respectively. The second smallest Laplacian eigenvalue $\mu_{n-1}(G)$ is called algebraic connectivity by Fiedler [8]. In [1], Abiad et al. raised the following research problem.

Problem 1.1 *(Abiad et al. [1]).* For a d-regular simple graph or multigraph G and for $2 \le k \le d$, what is the best upper bound for $\lambda_2(G)$ which guarantees $\kappa'(G) \ge k$ or $\kappa(G) \ge k$?

The edge-connectivity problem was earlier investigated by Cioabă [6], and has been intensively studied by many researchers, as found in [1,5–7,9,10,12–15,17,18], among others. For the vertex-connectivity of simple graphs, the following results have been proved in [1,16]. There are corresponding results when multiple edges are allowed, as seen in O [19] and Abiad et al. [1].

Theorem 1.2 *(Abiad et al. [1]). Let d and k be integers with $d \ge k \ge 2$ and G be a d-regular simple graph of order n. Let*

$$f(d,k) = \begin{cases} d+1, & \text{if } k = 2; \\ d+2-k, & \text{if } k \ge 3. \end{cases}$$

If $\lambda_2(G) < d - \frac{(k-1)dn}{2f(d,k)(n-f(d,k))}$, then $\kappa(G) \ge k$.

Theorem 1.3 *(Liu et al. [16]). Let G be a simple graph of order n with maximum degree Δ and minimum degree $\delta \ge k \ge 2$. Let $\alpha = \left\lceil \frac{1}{2}(\delta + 1 + \sqrt{(\delta+1)^2 - 2(k-1)\Delta}) \right\rceil$, and*

$$\phi(\delta, \Delta, k) = \begin{cases} (\delta - k + 2)(n - \delta + k - 2), & \text{if } \Delta \ge 2(\delta - k + 2); \\ \alpha(n - \alpha), & \text{if } \delta \le \Delta < 2(\delta - k + 2). \end{cases}$$

If $\mu_{n-1}(G) > \frac{(k-1)n\Delta}{2\phi(\delta,\Delta,k)}$, *or* $\lambda_2(G) < \delta - \frac{(k-1)n\Delta}{2\phi(\delta,\Delta,k)}$, *or* $q_2(G) < 2\delta - \frac{(k-1)n\Delta}{2\phi(\delta,\Delta,k)}$, *then* $\kappa(G) \geq k$.

Theorem 1.4 *(Liu et al. [16]). Let* G *be a* d*-regular simple graph of order* n *with* $d \geq k \geq 2$*. Let* $\beta = \left\lceil \frac{1}{2}(d+1+\sqrt{(d+1)^2-2(k-1)d}) \right\rceil$, *and*

$$\varphi(d,k) = \begin{cases} (d+1)(n-d-1), & if\ k=2; \\ (d-k+2)(n-d+k-2), & if\ k \geq 3 \text{ and } d \leq 2k-4; \\ \beta(n-\beta), & if\ k \geq 3 \text{ and } d > 2k-4. \end{cases}$$

If $\mu_{n-1}(G) > \frac{(k-1)nd}{2\varphi(d,k)}$, *or* $\lambda_2(G) < d - \frac{(k-1)nd}{2\varphi(d,k)}$, *or* $q_2(G) < 2d - \frac{(k-1)nd}{2\varphi(d,k)}$, *then* $\kappa(G) \geq k$.

Theorem 1.5 *(Liu et al. [16]). Let* G *be a simple bipartite graph of order* n *with maximum degree* Δ *and minimum degree* $\delta \geq k \geq 2$*. Let* $\gamma = \left\lceil \delta + \sqrt{\delta^2-(k-1)\Delta} \right\rceil$, *and*

$$\psi(\delta,\Delta,k) = \begin{cases} (2\delta-k+1)(n-2\delta+k-1), & if\ \Delta \geq 2\delta-k+1; \\ \gamma(n-\gamma), & if\ \delta \leq \Delta < 2\delta-k+1. \end{cases}$$

If $\mu_{n-1}(G) > \frac{(k-1)n\Delta}{2\psi(\delta,\Delta,k)}$, *or* $\lambda_2(G) < \delta - \frac{(k-1)n\Delta}{2\psi(\delta,\Delta,k)}$, *or* $q_2(G) < 2\delta - \frac{(k-1)n\Delta}{2\psi(\delta,\Delta,k)}$, *then* $\kappa(G) \geq k$.

Motivated by these former results, we aim to find better bounds on $\lambda_2(G)$, $\mu_{n-1}(G)$ or $q_2(G)$ which assure that $\kappa(G) \geq k$. The main tool of these former results is quotient matrix and Cauchy Interlacing Theorem. The interlacing may not be tight, so losing information is inevitable. Based on the corollary of the following Courant-Weyl inequalities, as seen on page 29 of [4], we focus on establishing the lower bounds on $\mu_{n-1}(G)$.

Theorem 1.6 *(Courant-Weyl Inequalities). Let* A *and* B *be Hermitian matrices of order* n*, and let* $1 \leq i,j \leq n$.

(i) *If* $i+j \leq n+1$, *then* $\lambda_i(A)+\lambda_j(B) \geq \lambda_{i+j-1}(A+B)$.
(ii) *If* $i+j \geq n+1$, *then* $\lambda_i(A)+\lambda_j(B) \leq \lambda_{i+j-n}(A+B)$.

Corollary 1.7. *Let* $t \geq 0$ *be a real number and* G *be a graph of order* n *with minimum degree* δ*. If* $q_2(G) < 2\delta - t$, *then* $\lambda_2(G) < \delta - t$; *if* $\lambda_2(G) < \delta - t$, *then* $\mu_{n-1}(G) > t$.

Proof. Let A, L, Q and D be the adjacency matrix, Laplacian matrix, signless Laplacian matrix and degree diagonal matrix of G. Since $A+D = Q$, by Theorem 1.6 (ii), $\lambda_2(A)+\lambda_n(D) \leq \lambda_2(Q)$. Hence, $\lambda_2(G) \leq q_2(G) - \delta$. Therefore, if $q_2(G) < 2\delta - t$, then $\lambda_2(G) \leq q_2(G) - \delta < \delta - t$.

Since $A + L = D$, by Theorem 1.6 (i), $\lambda_2(A) + \lambda_{n-1}(L) \geq \lambda_n(D) = \delta$. Hence, $\mu_{n-1}(G) \geq \delta - \lambda_2(G)$. Thus, if $\lambda_2(G) < \delta - t$, then $\mu_{n-1}(G) \geq \delta - \lambda_2(G) > t$. □

By Corollary 1.7, if $q_2(G) < 2\delta - t$ or $\lambda_2(G) < \delta - t$, then $\mu_{n-1}(G) > t$. Therefore, we believe that improving the bound on $\mu_{n-1}(G)$ would be a more effective way to improve the bounds on $\lambda_2(G)$ and $q_2(G)$. On the other hand, based on the fact that if $\mu_{n-1}(G) > 0$ then $\kappa(G) \geq 1$, it is natural to consider the function $f(k)$ such that $\mu_{n-1}(G) > f(k)$ to assure that $\kappa(G) \geq k$. In this paper, we mainly use the property of the eigenvector corresponding to algebraic connectivity $\mu_{n-1}(G)$ of G to get better bounds on $\mu_{n-1}(G)$. By Corollary 1.7, the results involving $\lambda_2(G)$ and $q_2(G)$ are trivial to obtain. For simplicity, we first present some functions that will appear in the following discussions.

Definition 1.8. For integers n, Δ, δ, d, k with $\delta \geq k$ and $d \geq k$, define

(i) $\mathcal{H}_1(\Delta, \delta, k) = \frac{(k-1)(n-k+1)\Delta}{4(\delta-k+2)(n-\delta-1)}$;
(ii) $\mathcal{H}_2(\Delta, \delta, k) = \frac{(k-1)n\Delta}{(n-k+1)(k-1)+4(\delta-k+2)(n-\delta-1)}$;
(iii) $\mathcal{H}_3(\Delta, \delta, k) = \frac{(k-1)(n-k+1)\Delta}{4(2\delta-k+1)(n-2\delta)}$;
(iv) $\mathcal{H}_4(\Delta, \delta, k) = \frac{(k-1)n\Delta}{(n-k+1)(k-1)+4(2\delta-k+1)(n-2\delta)}$.

The main results of this paper are presented as Theorems 1.9, 1.10, and Theorems 1.12–1.15. Using a lemma of Alon and Milman [2], we obtain Theorem 1.9, which improves Theorem 1.3.

Theorem 1.9. *Let k be an integer and G be a simple graph of order n with maximum degree Δ and minimum degree $\delta \geq k \geq 2$ and $n \geq 2k - 2$. If $\mu_{n-1}(G) > \mathcal{H}_1(\Delta, \delta, k)$, then $\kappa(G) \geq k$.*

Applying an inequality of Fiedler [8], Theorem 1.10 is obtained, which improves Theorem 1.3 and Theorem 1.9.

Theorem 1.10. *Let k be an integer and G be a simple graph of order n with maximum degree Δ and minimum degree $\delta \geq k \geq 2$. If $\mu_{n-1}(G) > \mathcal{H}_2(\Delta, \delta, k)$, then $\kappa(G) \geq k$.*

For any d-regular graph G, $\mu_{n-1}(G) = d - \lambda_2(G) = 2d - q_2(G)$. Setting $\Delta = \delta = d$ in Theorem 1.10, we get the following corollary for d-regular graphs, which improves Theorem 1.2 and Theorem 1.4.

Corollary 1.11. *Let k be an integer and G be a d-regular simple graph of order n with $d \geq k \geq 2$. If $\mu_{n-1}(G) > \mathcal{H}_2(d, d, k)$, or equivalently $\lambda_2(G) < d - \mathcal{H}_2(d, d, k)$, or equivalently $q_2(G) < 2d - \mathcal{H}_2(d, d, k)$, then $\kappa(G) \geq k$.*

Applying a result of Brouwer and Haemers [4], we get the following result with respect to μ_1 and μ_{n-1}.

Theorem 1.12. *Let G be a connected graph of order n with minimum degree $\delta \geq k \geq 2$. If*

$$\frac{\mu_1}{\mu_{n-1}} < r + \sqrt{r^2 - 1} \text{ or equivalently } \frac{\mu_{n-1}}{\mu_1} > r - \sqrt{r^2 - 1},$$

then $\kappa(G) \geq k$, where $r = \frac{2(\delta-k+2)(n-\delta-1)}{n(k-1)} + 1$.

For triangle-free graphs, we get Theorems 1.13–1.15, where Theorem 1.13 improves Theorem 1.5, and Theorem 1.14 improves Theorem 1.13.

Theorem 1.13. *Let k be an integer and G be a simple triangle-free graph of order n with maximum degree Δ and minimum degree $\delta \geq k \geq 2$ and $n \geq 2k - 2$. If $\mu_{n-1}(G) > \mathcal{H}_3(\Delta, \delta, k)$, then $\kappa(G) \geq k$.*

Theorem 1.14. *Let k be an integer and G be a simple triangle-free graph of order n with maximum degree Δ and minimum degree $\delta \geq k \geq 2$. If $\mu_{n-1}(G) > \mathcal{H}_4(\Delta, \delta, k)$, then $\kappa(G) \geq k$.*

Theorem 1.15. *Let G be a connected triangle-free graph of order n with minimum degree $\delta \geq k \geq 2$. If*

$$\frac{\mu_1}{\mu_{n-1}} < s + \sqrt{s^2 - 1} \text{ or equivalently } \frac{\mu_{n-1}}{\mu_1} > s - \sqrt{s^2 - 1},$$

then $\kappa(G) \geq k$, where $s = \frac{2(2\delta-k+1)(n-2\delta)}{n(k-1)} + 1$.

In Section 2, we display some preliminaries and mechanisms, including the bounds of Laplacian eigenvalues and the scale of the connected component of $G - S$ when deleting vertex subset S in G. These will be applied in the proofs of the main results, to be presented in Section 3 and Section 4. In the last section, we investigate the relationship between vertex-connectivity and algebraic connectivity of graphs with fixed girth.

2. Preliminaries

Lemma 2.1 *(Alon and Milman [2]). Let $G = (V, E)$ be a graph of order n. Let X and Y be two disjoint subsets of V such that each vertex of X has distance at least ρ to each vertex of Y. Let E_X (resp. E_Y) be the set of edges of G with both ends in X (resp. in Y). Then*

$$\mu_{n-1}(G) \leq \frac{1}{\rho^2}\left(\frac{1}{|X|} + \frac{1}{|Y|}\right)(|E| - |E_X| - |E_Y|).$$

Let $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, and let G be a graph on vertices $1, 2, \ldots, n$. Then x can be considered as a function defined on the vertex set of G, that is, for any vertex i, we map it to $x_i = x(i)$. We often say that x_i is a value of vertex i given by x. Fiedler [8] derived a useful expression for $\mu_{n-1}(G)$ as follows.

Lemma 2.2 *(Fiedler [8]). Let G be a graph of order n. Then*

$$\mu_{n-1}(G) = \min \frac{n \sum\limits_{ij \in E} (x_i - x_j)^2}{\sum\limits_{i,j \in V, i<j} (x_i - x_j)^2},$$

where the minimum is taken over all non-constant vectors $x \in \mathbb{R}^n$.

Lemma 2.3. *Let G be a simple graph of order n with minimum degree δ. Let S be an arbitrary minimum vertex-cut with κ vertices and X be the vertex set of a minimum component of $G - S$, and $Y = V - (S \cup X)$. Then*

$$|X| \cdot |Y| \geq (\delta - \kappa + 1)(n - \delta - 1).$$

Proof. Since each vertex in X is adjacent to at most $|X| - 1$ vertices of X and κ vertices of S,

$$\delta|X| \leq \sum_{x \in X} d_G(x) \leq |X|(|X| + \kappa - 1),$$

which implies $|X| \geq \delta - \kappa + 1$. Note that $|X| \leq |Y|$ and $|X| + |Y| = n - \kappa$. Therefore,

$$\delta - \kappa + 1 \leq |X| \leq |Y| \leq n - \delta - 1.$$

The result follows. $\square$

Lemma 2.4. *Let G be a simple triangle-free graph of order n with minimum degree $\delta \geq 1$. Let S be an arbitrary minimum vertex-cut with κ vertices and X be the vertex set of a minimum component of $G - S$, and $Y = V - (S \cup X)$. If $\kappa < \delta$, then*

$$|X| \cdot |Y| \geq (2\delta - \kappa)(n - 2\delta).$$

Proof. If $x \in X$, then $\delta \leq d_G(x) \leq |X| + |S|$. The assumption $\kappa < \delta$ implies that x has at least one neighbor $y \in X$. Since G is triangle-free, we deduce that $N_G(x) \cap N_G(y) = \emptyset$, where $N_G(x)$ is the neighbor set of x. As $N_G(x) \cup N_G(y) \subseteq X \cup S$, it follows that

$$|X| + |S| = |X \cup S| \geq |N_G(x) \cup N_G(y)| = |N_G(x)| + |N_G(y)| \geq 2\delta$$

and thus $|X| \geq 2\delta - |S| = 2\delta - \kappa$. Combining this with $|X| \leq |Y|$ and $|X| + |Y| = n - \kappa$, we arrive at

$$2\delta - \kappa \le |X| \le |Y| \le n - 2\delta.$$

The result follows. $\square$

Lemma 2.5 *(Haemers [11]). Let X and Y be disjoint sets of vertices of graph G, such that there is no edge between X and Y. Then*

$$\frac{|X||Y|}{(n-|X|)(n-|Y|)} \le \left(\frac{\mu_1 - \mu_{n-1}}{\mu_1 + \mu_{n-1}}\right)^2.$$

For applications, a useful Lemma can be derived from Lemma 2.5 as follows.

Lemma 2.6 *(Brouwer and Haemers [4]). Let G be a connected graph on n vertices, and let X and Y be disjoint sets of vertices, such that there is no edge between X and Y. Then*

$$\frac{|X||Y|}{n(n-|X|-|Y|)} \le \frac{(\mu_1 - \mu_{n-1})^2}{4\mu_1\mu_{n-1}}.$$

3. Vertex-connectivity and Laplacian eigenvalues in graphs

In this section, we present the proofs of Theorem 1.9, Theorem 1.10 and Theorem 1.12.

Proof of Theorem 1.9. To the contrary, suppose that $1 \le \kappa = \kappa(G) \le k-1$. Let S be an arbitrary minimum vertex-cut and X be the vertex set of a minimum component of $G - S$, and $Y = V - (S \cup X)$. By Lemma 2.3 and $1 \le \kappa \le k-1$, we obtain

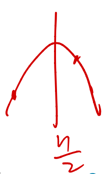
$$|X| \cdot |Y| \ge (\delta - \kappa + 1)(n - \delta - 1) \ge (\delta - k + 2)(n - \delta - 1). \tag{3.1}$$

Since each edge in $E - (E(G[X]) \cup E(G[Y]))$ is incident with at least one of the $n - |X| - |Y|$ vertices of the set S,

$$|E| - |E(G[X])| - |E(G[Y])| \le (n - |X| - |Y|)\Delta = \kappa\Delta. \tag{3.2}$$

As $1 \le \kappa \le k - 1 \le \frac{n}{2}$, we have

$$(n-\kappa)\kappa \le (k-1)(n-k+1). \tag{3.3}$$



Since each vertex of X has distance at least 2 to each vertex of Y, by Lemma 2.1

$$\mu_{n-1}(G) \le \frac{|X| + |Y|}{4|X||Y|}(|E| - |E(G[X])| - |E(G[Y])|). \tag{3.4}$$

Substituting (3.1) and (3.2) in (3.4), by (3.3) we obtain

$$
\begin{aligned}
\mu_{n-1}(G) &\leq \frac{(|X|+|Y|)\kappa\Delta}{4(\delta-k+2)(n-\delta-1)} \\
&= \frac{(n-\kappa)\kappa\Delta}{4(\delta-k+2)(n-\delta-1)} \\
&\leq \frac{(k-1)(n-k+1)\Delta}{4(\delta-k+2)(n-\delta-1)},
\end{aligned}
$$

which is a contradiction to the hypothesis. Thus $\kappa(G) \geq k$. $\square$

Remark 3.1. The lower bound on $\mu_{n-1}(G)$ of Theorem 1.9 is better than the one of Theorem 1.3 in the following cases. If $\Delta \geq 2(\delta-k+2)$ and $n \geq \delta+k$, then $n > 2k-2$ and $2(n-\delta-1) \geq n-\delta+k-2$, and so

$$\frac{(k-1)(n-k+1)\Delta}{4(\delta-k+2)(n-\delta-1)} < \frac{(k-1)n\Delta}{2(\delta-k+2)(n-\delta+k-2)}.$$

If $\delta \leq \Delta < 2(\delta-k+2)$ and $n \geq 2\delta+2$, then $2(\delta-k+2) \geq \delta+1$. Thus, for $\delta-k+2 < \alpha < \delta+1 \leq \frac{n}{2}$,

$$\frac{(k-1)(n-k+1)\Delta}{4(\delta-k+2)(n-\delta-1)} < \frac{(k-1)n\Delta}{2(\delta+1)(n-\delta-1)} < \frac{(k-1)n\Delta}{2\alpha(n-\alpha)}.$$

Proof of Theorem 1.10. To the contrary, suppose that $1 \leq \kappa = \kappa(G) \leq k-1$. Let S be an arbitrary minimum vertex-cut and X be the vertex set of a minimum component of $G-S$, and $Y = V-(S \cup X)$. By Lemma 2.3 and $1 \leq \kappa \leq k-1$, we obtain

$$\delta-\kappa+1 \leq |X| \leq |Y| \leq n-\delta-1, \tag{3.5}$$

$$|X| \cdot |Y| \geq (\delta-\kappa+1)(n-\delta-1). \tag{3.6}$$

Let $x = (x_1, x_2, \ldots, x_n)^T$ be a real vector. If $i \in X$, then set $x_i = 1$; if $i \in Y$, then set $x_i = -1$; if $i \in S$, then set $x_i = 0$. By Lemma 2.2,

$$\mu_{n-1}(G) \leq \frac{n \sum\limits_{ij \in E} (x_i - x_j)^2}{\sum\limits_{i,j \in V, i<j} (x_i - x_j)^2} \tag{3.7}$$

holds for the real vector x. Denote $E_S = E \setminus (E(G[X]) \cup E(G[Y]))$. Applying the values of the entries of x into the inequality (3.7), we obtain

$$\sum_{ij \in E} (x_i - x_j)^2 = \sum_{ij \in E_S} (x_i - x_j)^2 \leq \sum_{ij \in E_S} 1 \leq |S|\Delta \leq (k-1)\Delta, \tag{3.8}$$

$$\sum_{i,j \in V, i<j} (x_i - x_j)^2 = \sum_{i \in X, j \in S} (x_i - x_j)^2 + \sum_{i \in Y, j \in S} (x_i - x_j)^2 + \sum_{i \in X, j \in Y} (x_i - x_j)^2$$

$$
\begin{aligned}
&= \sum_{i\in X, j\in S} (1-0)^2 + \sum_{i\in Y, j\in S} ((-1)-0)^2 + \sum_{i\in X, j\in Y} (1-(-1))^2 \\
&= |X||S| + |Y||S| + 4|X||Y| \\
&= (n-\kappa)\kappa + 4|X||Y| \\
\text{by (3.6)} \qquad &\geq (n-\kappa)\kappa + 4(\delta-\kappa+1)(n-\delta-1) \\
&= -\kappa(\kappa + (3n-4\delta-4)) + 4(\delta+1)(n-\delta-1).
\end{aligned}
$$

Set $f(\kappa) = -\kappa(\kappa + (3n-4\delta-4))$. The minimum of $f(\kappa)$ is attained at $f(1)$ or $f(k-1)$ for $1 \leq \kappa \leq k-1$. By (3.5), $n \geq 2\delta - \kappa + 2 \geq 2\delta - k + 3$, which implies $f(1) - f(k-1) = (k-2)(3n-4\delta+k-4) \geq 0$. Thus

$$
\begin{aligned}
\sum_{i,j\in V, i<j} (x_i - x_j)^2 &\geq f(k-1) + 4(\delta+1)(n-\delta-1) \\
&= (n-k+1)(k-1) + 4(\delta-k+2)(n-\delta-1). \qquad (3.9)
\end{aligned}
$$

Substituting (3.8) and (3.9) in (3.7), we have

$$
\mu_{n-1}(G) \leq \frac{n \sum_{ij\in E} (x_i - x_j)^2}{\sum_{i,j\in V, i<j} (x_i - x_j)^2} \leq \frac{(k-1)n\Delta}{(n-k+1)(k-1) + 4(\delta-k+2)(n-\delta-1)},
$$

which is a contradiction to the hypothesis. Hence, $\kappa(G) \geq k$. □

Remark 3.2. (i) The lower bound on $\mu_{n-1}(G)$ of Theorem 1.10 is better than the one of Theorem 1.9 when $n \neq 2\delta - k + 3$. In fact, as $n - k + 1 = (n-\delta-1) + (\delta-k+2)$, it is easy to find that

$$
\frac{(k-1)n\Delta}{(n-k+1)(k-1) + 4(\delta-k+2)(n-\delta-1)} < \frac{(k-1)(n-k+1)\Delta}{4(\delta-k+2)(n-\delta-1)}
$$

is equivalent to $(n - 2\delta + k - 3)^2 > 0$.

(ii) The lower bound on $\mu_{n-1}(G)$ of Theorem 1.10 is better than the one of Theorem 1.3 in the following cases. If $\Delta \geq 2(\delta - k + 2)$ and $n \geq \delta + \frac{k+1}{2}$, then

$$
\frac{(k-1)n\Delta}{(n-k+1)(k-1) + 4(\delta-k+2)(n-\delta-1)} < \frac{(k-1)n\Delta}{2(\delta-k+2)(n-\delta+k-2)}
$$

is equivalent to $(n-\delta-k)(\delta-k+2) + (\delta+1)(n-\delta-1) > 0$, that is

$$
n > \frac{(\delta+k)(\delta-k+2) + (\delta+1)^2}{(\delta-k+2) + (\delta+1)} = \delta + k - \frac{(\delta+1)(k-1)}{2(\delta+1) - (k-1)},
$$

which holds when $n \geq \delta + \frac{k+1}{2} = \delta + k - \frac{k-1}{2}$. If $\delta \leq \Delta < 2(\delta - k + 2)$ and $n \geq 2\delta + 2$, then by Remark 3.2 (i) and Remark 3.1, $\mathcal{H}_2(\Delta, \delta, k) < \mathcal{H}_1(\Delta, \delta, k) < \phi(\delta, \Delta, k)$.

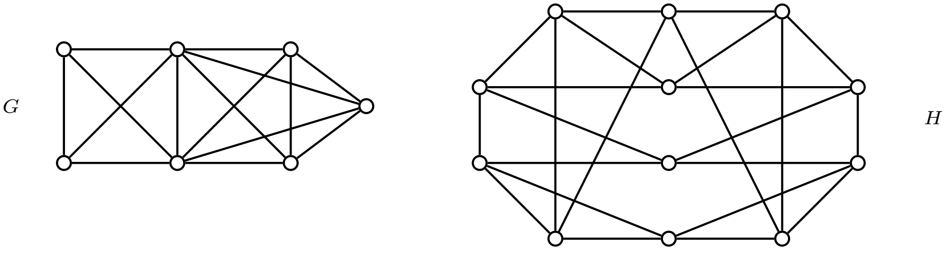


Fig. 1. The graph G (left) in Example 3.3 and graph H (right) in Example 3.4.

Proof of Theorem 1.12. To the contrary, suppose that $1 \le \kappa = \kappa(G) \le k-1$. Let S be an arbitrary minimum vertex-cut and X be the vertex set of a minimum component of $G - S$, and $Y = V - (S \cup X)$. By Lemma 2.3 and $1 \le \kappa \le k-1$, we obtain

$$|X| \cdot |Y| \ge (\delta - \kappa + 1)(n - \delta - 1) \ge (\delta - k + 2)(n - \delta - 1).$$

Combining this with $n - |X| - |Y| = \kappa \le k-1$, by Lemma 2.6,

$$\frac{(\mu_1 - \mu_{n-1})^2}{4\mu_1\mu_{n-1}} \ge \frac{|X||Y|}{n(n - |X| - |Y|)} \ge \frac{(\delta - k + 2)(n - \delta - 1)}{n(k-1)}. \tag{3.10}$$

Set $t = \frac{\mu_1}{\mu_{n-1}}$ and $r = \frac{2(\delta-k+2)(n-\delta-1)}{n(k-1)} + 1$. Substituting t and r in (3.10), we obtain $t + t^{-1} \ge 2r$. Since $t \ge 1$ and $r \ge 1$, $t \ge r + \sqrt{r^2 - 1}$ is necessary. This contradicts to the hypothesis. Therefore, $\kappa(G) \ge k$. □

Example 3.3. Let G be the graph in Fig. 1, where $n = |V(G)| = 7, \Delta(G) = 6, \delta(G) = 3, \kappa(G) = 2$ and $\mu_{n-1}(G) = 2$. We present a table to show the lower bounds on $\mu_{n-1}(G)$ of Theorems 1.3,1.9,1.10 and the upper bound on $\frac{\mu_1(G)}{\mu_{n-1}(G)}$ of Theorem 1.12.

Graph	$\mu_{n-1}(G)$	Theorem 1.3	Theorem 1.9	Theorem 1.10	$\frac{\mu_1(G)}{\mu_{n-1}(G)}$	Theorem 1.12
G	2	1.75	1	1	3.5	7

Example 3.4. Let H be the 4-regular graph in Fig. 1, where $n = |V(H)| = 12, \Delta(H) = \delta(H) = \kappa(H) = 4$ and $\mu_{n-1}(H) = 2$. We present a table to show the lower bounds on $\mu_{n-1}(H)$ of Theorems 1.4,1.9,1.10 and the upper bound on $\frac{\mu_1(H)}{\mu_{n-1}(H)}$ of Theorem 1.12.

Graph	$\mu_{n-1}(H)$	Theorem 1.4	Theorem 1.9	Theorem 1.10	$\frac{\mu_1(H)}{\mu_{n-1}(H)}$	Theorem 1.12
H	2	3.6	1.9286	1.7349	3	3.2476

4. Vertex-connectivity and Laplacian eigenvalues in triangle-free graphs

In this section, we present the proofs of Theorem 1.13, Theorem 1.14 and Theorem 1.15.

Proof of Theorem 1.13. To the contrary, suppose that $1 \leq \kappa = \kappa(G) \leq k-1$. Let S be a minimum vertex-cut of G and X be the vertex set of a minimum component of $G - S$, and $Y = V - (S \cup X)$. Then $|X| \leq |Y|$ and $|X| + |Y| = n - \kappa$.

Since $\kappa \leq k - 1 < \delta$, by Lemma 2.4 we obtain

$$|X| \cdot |Y| \geq (2\delta - \kappa)(n - 2\delta) \geq (2\delta - k + 1)(n - 2\delta). \tag{4.1}$$

Since each edge in $E - (E(G[X]) \cup E(G[Y]))$ is incident with at least one of the $n - |X| - |Y|$ vertices of the set S,

$$|E| - |E(G[X])| - |E(G[Y])| \leq (n - |X| - |Y|)\Delta = \kappa\Delta. \tag{4.2}$$

As $1 \leq \kappa \leq k - 1 \leq \frac{n}{2}$, we have

$$(n - \kappa)\kappa \leq (k - 1)(n - k + 1). \tag{4.3}$$

Since each vertex of X has distance at least 2 to each vertex of Y, by Lemma 2.1

$$\mu_{n-1}(G) \leq \frac{|X| + |Y|}{4|X||Y|}(|E| - |E(G[X])| - |E(G[Y])|). \tag{4.4}$$

Substituting (4.1) and (4.2) in (4.4), by (4.3) we have

$$\begin{aligned}
\mu_{n-1}(G) &\leq \frac{(|X| + |Y|)\kappa\Delta}{4(2\delta - k + 1)(n - 2\delta)} \\
&= \frac{(n - \kappa)\kappa\Delta}{4(2\delta - k + 1)(n - 2\delta)} \\
&\leq \frac{(k - 1)(n - k + 1)\Delta}{4(2\delta - k + 1)(n - 2\delta)},
\end{aligned}$$

which is a contradiction to the hypothesis. Thus $\kappa(G) \geq k$. $\square$

Remark 4.1. The lower bound on $\mu_{n-1}(G)$ of Theorem 1.13 is better than the one of Theorem 1.5 in the following cases. If $\Delta \geq 2\delta - k + 1$ and $n \geq 2\delta + k - 1$, then $n > 2k - 2$ and $2(n - 2\delta) \geq n - 2\delta + k - 1$, and so

$$\frac{(k - 1)(n - k + 1)\Delta}{4(2\delta - k + 1)(n - 2\delta)} < \frac{(k - 1)n\Delta}{2(2\delta - k + 1)(n - 2\delta + k - 1)}.$$

If $\delta \leq \Delta < 2\delta - k + 1$ and $n \geq 4\delta$, then $2\delta - k + 1 < \gamma < 2\delta \leq \frac{n}{2}$ and so

$$\frac{(k - 1)(n - k + 1)\Delta}{4(2\delta - k + 1)(n - 2\delta)} < \frac{(k - 1)n\Delta}{2 \cdot 2\delta(n - 2\delta)} < \frac{(k - 1)n\Delta}{2\gamma(n - \gamma)}.$$

Proof of Theorem 1.14. To the contrary, suppose that $1 \leq \kappa = \kappa(G) \leq k-1$. Let S be a minimum vertex-cut of G and X be the vertex set of a minimum component of $G-S$, and $Y = V - (S \cup X)$. Then $|X| \leq |Y|$ and $|X| + |Y| = n - \kappa$.

Since $\kappa \leq k-1 < \delta$, by Lemma 2.4 we obtain

$$2\delta - \kappa \leq |X| \leq |Y| \leq n - 2\delta, \tag{4.5}$$

$$|X| \cdot |Y| \geq (2\delta - \kappa)(n - 2\delta). \tag{4.6}$$

Let $x = (x_1, x_2, \ldots, x_n)^T$ be a real vector. If $i \in X$, then set $x_i = 1$; if $i \in Y$, then set $x_i = -1$; if $i \in S$, then set $x_i = 0$. Using a similar argument as in the proof of Theorem 1.10, by Lemma 2.2 we have

$$\mu_{n-1}(G) \leq \frac{n \sum\limits_{ij \in E} (x_i - x_j)^2}{\sum\limits_{i,j \in V, i<j} (x_i - x_j)^2} \leq \frac{(k-1)n\Delta}{(n-\kappa)\kappa + 4|X||Y|}. \tag{4.7}$$

By (4.6),

$$\begin{aligned} (n-\kappa)\kappa + 4|X||Y| &\geq (n-\kappa)\kappa + 4(2\delta - \kappa)(n - 2\delta) \\ &= -\kappa(\kappa + (3n - 8\delta)) + 8\delta(n - 2\delta). \end{aligned}$$

Set $f(\kappa) = -\kappa(\kappa + 3n - 8\delta) + 8\delta(n - 2\delta)$. By (4.5), $n \geq 4\delta - \kappa \geq 4\delta - k + 1$, which implies $f(1) - f(k-1) = (k-2)(3n - 8\delta + k) \geq 0$. Thus $f(\kappa) \geq \min\{f(1), f(k-1)\} = f(k-1)$ for $1 \leq \kappa \leq k-1$, and so

$$(n-\kappa)\kappa + 4|X||Y| \geq f(k-1) = (n-k+1)(k-1) + 4(2\delta - k + 1)(n - 2\delta). \tag{4.8}$$

Substituting (4.8) in (4.7), we have

$$\mu_{n-1}(G) \leq \frac{n \sum\limits_{ij \in E} (x_i - x_j)^2}{\sum\limits_{i,j \in V, i<j} (x_i - x_j)^2} \leq \frac{(k-1)n\Delta}{(n-k+1)(k-1) + 4(2\delta - k + 1)(n - 2\delta)},$$

which is a contradiction to the hypothesis. Hence, $\kappa(G) \geq k$. □

Remark 4.2. (i) The lower bound on $\mu_{n-1}(G)$ of Theorem 1.14 is better than the one of Theorem 1.13 when $n \neq 4\delta - k + 1$. In fact, as $n - k + 1 = (n - 2\delta) + (2\delta - k + 1)$, it is also easy to find that

$$\frac{(k-1)n\Delta}{(n-k+1)(k-1) + 4(2\delta - k + 1)(n - 2\delta)} < \frac{(k-1)(n-k+1)\Delta}{4(2\delta - k + 1)(n - 2\delta)}$$

is equivalent to $(n - 4\delta + k - 1)^2 > 0$.

(ii) The lower bound on $\mu_{n-1}(G)$ of Theorem 1.14 is better than the one of Theorem 1.5 in the following cases. If $\Delta \geq 2\delta - k + 1$ and $n \geq 2\delta + \frac{k-1}{2}$, then

$$\frac{(k-1)n\Delta}{(n-k+1)(k-1)+4(2\delta-k+1)(n-2\delta)} < \frac{(k-1)n\Delta}{2(2\delta-k+1)(n-2\delta+k-1)}$$

is equivalent to $(n-2\delta-k+1)(2\delta-k+1)+2\delta(n-2\delta) > 0$, that is

$$n > \frac{8\delta^2-(k-1)^2}{4\delta-(k-1)} = 2\delta + \frac{(k-1)(2\delta-k+1)}{2(2\delta-k+1)+(k-1)},$$

which holds when $n \geq 2\delta + \frac{k-1}{2}$. If $\delta \leq \Delta < 2\delta - k + 1$ and $n \geq 4\delta$, then by Remark 4.2 (i) and Remark 4.1, $\mathcal{H}_4(\Delta, \delta, k) < \mathcal{H}_3(\Delta, \delta, k) < \psi(\delta, \Delta, k)$.

Proof of Theorem 1.15. To the contrary, suppose that $1 \leq \kappa = \kappa(G) \leq k-1$. Let S be an arbitrary minimum vertex-cut and X be the vertex set of a minimum component of $G - S$, and $Y = V - (S \cup X)$. By Lemma 2.4 and $1 \leq \kappa \leq k-1$, we obtain

$$|X| \cdot |Y| \geq (2\delta-\kappa)(n-2\delta) \geq (2\delta-k+1)(n-2\delta).$$

Combining this with $n - |X| - |Y| = \kappa \leq k-1$, by Lemma 2.6,

$$\frac{(\mu_1-\mu_{n-1})^2}{4\mu_1\mu_{n-1}} \geq \frac{|X||Y|}{n(n-|X|-|Y|)} \geq \frac{(2\delta-k+1)(n-2\delta)}{n(k-1)}. \tag{4.9}$$

Set $t = \frac{\mu_1}{\mu_{n-1}}$ and $s = \frac{2(2\delta-k+1)(n-2\delta)}{n(k-1)} + 1$. Substituting t and s in (4.9), we obtain $t + t^{-1} \geq 2s$. Since $t \geq 1$ and $s \geq 1$, $t \geq s + \sqrt{s^2-1}$ is necessary. This contradicts to the hypothesis. Therefore, $\kappa(G) \geq k$. $\square$

Example 4.3. Let G be the bipartite graph in Fig. 2, where $n = |V(G)| = 14, \Delta(G) = 5, \delta(G) = \kappa(G) = 4$ and $\mu_{n-1}(G) = 2$. We present a table to show the lower bounds on $\mu_{n-1}(G)$ of Theorems 1.5, 1.13, 1.14, and the upper bound on $\frac{\mu_1(G)}{\mu_{n-1}(G)}$ of Theorem 1.15, as seen in the table below.

Graph	$\mu_{n-1}(G)$	Theorem 1.5	Theorem 1.13	Theorem 1.14	$\frac{\mu_1(G)}{\mu_{n-1}(G)}$	Theorem 1.15
G	2	2.3333	1.3750	1.3725	4.3508	4.6417

5. Vertex-connectivity and eigenvalues of graphs with fixed girth

In this section, based on a result of Liu et al. in [16], we investigate the relationship between vertex-connectivity and algebraic connectivity of graphs with fixed girth, to be shown in Theorem 5.2.

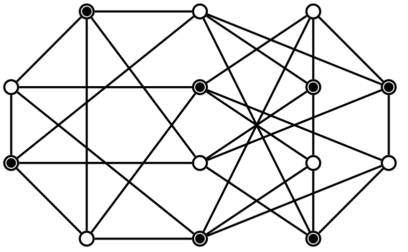


Fig. 2. The graph G in Example 4.3.

Lemma 5.1 *(Liu et al. [16]). Let G be a simple connected graph with minimum degree $\delta = \delta(G) \geq 2$ and girth $g = g(G) \geq 3$. Let C be a vertex cut of G with $|C| = c$ and A be a connected component of $G - C$. Define $t = \lceil \frac{g-1}{2} \rceil$ and*

$$\nu(\delta, g, c) = \begin{cases} 1 + (\delta - c)\sum\limits_{i=0}^{t-1}(\delta-1)^i, & \text{if } g = 2t+1 \text{ and } c \leq \delta - 1; \\ 2 + (2\delta - 2 - c)\sum\limits_{i=0}^{t-1}(\delta-1)^i, & \text{if } g = 2t+2 \text{ and } \delta \geq 3; \\ 2t+1, & \text{if } g = 2t+2 \text{ and } \delta = 2. \end{cases}$$

If $c < \delta$, then $|V(A)| \geq \nu(\delta, g, c)$.

Theorem 5.2. *Let k be an integer and G be a simple graph of order n with maximum degree Δ, minimum degree $\delta \geq k \geq 2$, girth $g = g(G) \geq 3$. Define $t = \lceil \frac{g-1}{2} \rceil$, $h = \sum_{i=0}^{t-1}(\delta-1)^i$, and*

$$\mathcal{H}(\Delta, \delta, g, k) = \frac{(k-1)n\Delta}{(n-k+1)(k-1) + 4\nu(\delta, g, k-1)(n-k+1-\nu(\delta, g, k-1))}.$$

If one of the following conditions holds, then $\kappa(G) \geq k$.

(i) $\mu_{n-1}(G) > \frac{(k-1)n\Delta}{(n-1)+4\nu(\delta,g,k-1)(n-k+1-\nu(\delta,g,k-1))}$;
(ii) $g = 2t+1$, $(4h-1)n \geq 4(h\delta+1)(2h-1) - k(2h-1)^2$ *and* $\mu_{n-1}(G) > \mathcal{H}(\Delta, \delta, g, k)$;
(iii) $g = 2t+2$, $\delta \geq 3$, $(4h-1)n \geq 8(h(\delta-1)+1)(2h-1) - k(2h-1)^2$ *and* $\mu_{n-1}(G) > \mathcal{H}(\Delta, \delta, g, k)$.

Proof. To the contrary, suppose that $1 \leq \kappa = \kappa(G) \leq k-1$. Let S be a minimum vertex-cut of G and X be the vertex set of a minimum component of $G - S$, and $Y = V - (S \cup X)$. Then $|X| \leq |Y|$ and $|X| + |Y| = n - \kappa$. Since $\kappa \leq k-1 < \delta$, by Lemma 5.1 we obtain $\nu(\delta, g, \kappa) \leq |X| \leq |Y| \leq n - \kappa - \nu(\delta, g, \kappa)$, and so

$$|X| \cdot |Y| \geq \nu(\delta, g, \kappa)(n - \kappa - \nu(\delta, g, \kappa)) \geq \nu(\delta, g, k-1)(n-k+1-\nu(\delta, g, k-1)). \quad (5.1)$$

Let $x = (x_1, x_2, \ldots, x_n)^T$ be a real vector. If $i \in X$, then set $x_i = 1$; if $i \in Y$, then set $x_i = -1$; if $i \in S$, then set $x_i = 0$. Using a similar argument as in the proof of Theorem 1.10, by Lemma 2.2 we have

$$\mu_{n-1}(G) \leq \frac{n \sum\limits_{ij \in E} (x_i - x_j)^2}{\sum\limits_{i,j \in V, i<j} (x_i - x_j)^2} \leq \frac{(k-1)n\Delta}{(n-\kappa)\kappa + 4|X||Y|}. \tag{5.2}$$

(i) By (5.1) and $(n-\kappa)\kappa \geq \min\{n-1, (n-k+1)(k-1)\} = n-1$ for $1 \leq \kappa \leq k-1$,

$$(n-\kappa)\kappa + 4|X||Y| \geq (n-1) + 4\nu(\delta, g, k-1)(n-k+1-\nu(\delta, g, k-1)). \tag{5.3}$$

Substituting (5.3) in (5.2), we obtain a contradiction to the hypothesis. Therefore, $\kappa(G) \geq k$.

(ii) If $g = 2t+1$ and $\kappa \leq k-1 < \delta$, then $\nu(\delta, g, \kappa) = h(\delta - \kappa) + 1$. Combining this with (5.1),

$$\begin{aligned}(n-\kappa)\kappa + 4|X||Y| \geq &(n-\kappa)\kappa + 4\nu(\delta, g, \kappa)(n - \kappa - \nu(\delta, g, \kappa)) \\ =&(n-\kappa)\kappa + 4(h(\delta-\kappa)+1)(n-\kappa-h(\delta-\kappa)-1) \\ =&-(2h-1)^2\kappa^2 - ((4h-1)n - 4(h\delta+1)(2h-1))\kappa \\ &+ 4(h\delta+1)(n-h\delta-1) =: f_1(\kappa).\end{aligned}$$

Since $h \geq 1$ and $(4h-1)n \geq 4(h\delta+1)(2h-1) - k(2h-1)^2$,

$$f_1(1) - f_1(k-1) = (k-2)((4h-1)n - 4(h\delta+1)(2h-1) + k(2h-1)^2) \geq 0,$$

which implies $f_1(\kappa) \geq \min\{f_1(1), f_1(k-1)\} = f_1(k-1)$. Thus

$$\begin{aligned}(n-\kappa)\kappa + 4|X||Y| &\geq f_1(k-1) \\ &= (n-k+1)(k-1) + 4\nu(\delta, g, k-1)(n-k+1-\nu(\delta, g, k-1)).\end{aligned} \tag{5.4}$$

Substituting (5.4) in (5.2), we obtain $\mu_{n-1}(G) \leq \mathcal{H}(\Delta, \delta, g, k)$, which is a contradiction to the hypothesis. Hence, $\kappa(G) \geq k$.

(iii) If $g = 2t+2$ and $\delta \geq 3$, then $\nu(\delta, g, \kappa) = h(2\delta - \kappa - 2) + 2$. Combining this with (5.1),

$$\begin{aligned}(n-\kappa)\kappa + 4|X||Y| \geq &(n-\kappa)\kappa + 4\nu(\delta, g, \kappa)(n-\kappa-\nu(\delta, g, \kappa)) \\ =&(n-\kappa)\kappa + 4(h(2\delta-\kappa-2)+2)(n-\kappa-h(2\delta-\kappa-2)-2) \\ =&-(2h-1)^2\kappa^2 - ((4h-1)n - 4(h(2\delta-2)+2)(2h-1))\kappa \\ &+ 4(h(2\delta-2)+2)(n-h(2\delta-2)-2) =: f_2(\kappa).\end{aligned}$$

Since $h \geq 1$ and $(4h-1)n \geq 8(h(\delta-1)+1)(2h-1) - k(2h-1)^2$,

$$f_2(1) - f_2(k-1) = (k-2)((4h-1)n - 4(h(2\delta-2)+2)(2h-1) + k(2h-1)^2) \geq 0,$$

which implies $f_2(\kappa) \geq \min\{f_2(1), f_2(k-1)\} = f_2(k-1)$. Thus

$$\begin{aligned}(n-\kappa)\kappa + 4|X||Y| &\geq f_2(k-1)\\ &= (n-k+1)(k-1) + 4\nu(\delta,g,k-1)(n-k+1-\nu(\delta,g,k-1)).\end{aligned} \tag{5.5}$$

Substituting (5.5) in (5.2), we obtain $\mu_{n-1}(G) \leq \mathcal{H}(\Delta,\delta,g,k)$, which is a contradiction to the hypothesis. Hence, $\kappa(G) \geq k$. □

At the end of this paper, we investigate graphs with many vertices, instead of small graphs. When n is large enough, the lower bound on $\mu_{n-1}(G)$ is close to a constant $c = c(\Delta,\delta,k)$. To show this, we present a corollary of Theorem 1.10 as an example. For other results, the proof is similar.

Corollary 5.3. *Let k be an integer and G be a simple graph of order n with maximum degree Δ and minimum degree $\delta \geq k \geq 2$. For any $\epsilon > 0$, there exists an integer N such that for any $n \geq N$, if*

$$\mu_{n-1}(G) > \frac{(k-1)\Delta}{4\delta - 3k + 7} + \epsilon,$$

then $\kappa(G) \geq k$.

Proof. Since $\delta \geq k \geq 2$ and

$$\lim_{n\to\infty} \mathcal{H}_2(\Delta,\delta,k) = \lim_{n\to\infty} \frac{(k-1)n\Delta}{(n-k+1)(k-1) + 4(\delta-k+2)(n-\delta-1)} = \frac{(k-1)\Delta}{4\delta-3k+7},$$

for any $\epsilon > 0$, there exists an integer N such that for any $n \geq N$,

$$\mathcal{H}_2(\Delta,\delta,k) \leq \frac{(k-1)\Delta}{4\delta-3k+7} + \epsilon.$$

Thus, $\mu_{n-1}(G) > \mathcal{H}_2(\Delta,\delta,k)$. By Theorem 1.10, the result follows. □

Declaration of Competing Interest

There is no competing interests.

Acknowledgements

The authors would like to thank anonymous reviewers for their valuable comments and suggestions to improve the presentation of the paper.

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