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Spanning tree packing number and eigenvalues of graphs with given girth

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ABSTRACT

Let $\tau(G)$ and $\kappa'(G)$ denote the spanning tree packing number and the edge-connectivity of a graph G, respectively. Cioabă and Wong (2012) in [5] conjectured an explicit relationship between $\tau(G)$ and the second largest adjacency eigenvalue $\lambda_2(G)$ of a regular graph. Gu et al. (2016) in [12] presented a more general conjecture on a simple graph G. This conjecture was proved by Liu et al. (2014) in [21] by showing that for any simple graph G with minimum degree $\delta \geq 2k \geq 4$, if $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \geq k$. Similar results involving the algebraic connectivity $\mu_{n-1}(G)$ and the second largest signless Laplacian eigenvalue $q_2(G)$ of a graph G were also obtained. In this paper, we determine a Moore function $f(\delta, g)$ for a graph G with minimum degree δ and girth g, and prove that if G is a simple graph of order n with minimum degree $\delta \geq 2k \geq 4$ and girth g, then

(i) If $\lambda_2(G) < \delta - \frac{2k-1}{f(\delta,g)}$, then $\tau(G) \geq k$.
(ii) If $\mu_{n-1}(G) > \frac{2k-1}{f(\delta,g)}$, then $\tau(G) \geq k$.
(iii) If $q_2(G) < 2\delta - \frac{2k-1}{f(\delta,g)}$, then $\tau(G) \geq k$.

The edge-connectivity analogue results are also obtained.

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Former results in Gu et al. (2016) [12], Li et al. (2013) [18], Liu et al. (2014) [20] and Liu et al. (2014) [21] are extended.

1. Introduction

Spanning tree packing number and edge-connectivity of a network have been used as measures of reliability, strength and survivability in case of attack or edge failure in networks modeled as a graph, as seen [7,11,14,22,25], among others. In particular, determining the spanning tree packing number in a graph is closely related to the design of efficient and robust communication protocols, as seen in a seminal article by Itai and Rodeh [15]. Linear algebra techniques, including eigenvalues and eigenvectors methods and matrix decompositions, have been applied as useful tools in network investigations, as seen [6,8,25]. Recently, the study of the relationship between spanning tree packing number, edge-connectivity and eigenvalues of a graph has been drawing the attention of quite a few researchers.

In this paper, we consider finite and simple graphs. In particular, $\Delta(G), \delta(G)$ and $\kappa'(G)$ denote the maximum degree, the minimum degree and the **edge-connectivity** of a graph G, respectively. The girth of a graph G, is defined as

$$g(G) = \begin{cases} \min\{|E(C)| : \ C \text{ is a cycle of } G\} & \text{if } G \text{ is not acyclic,} \\ \infty & \text{if } G \text{ is acyclic.} \end{cases}$$

Let $\overline{d}(G)$ be the average degree of G, and $\tau(G)$ be the maximum number of edge-disjoint spanning trees contained in G, which is also called **spanning tree packing number**. A literature review on $\tau(G)$ can be found in [24]. As in [1], for a vertex subset $S \subseteq V(G)$, $G[S]$ is the subgraph of G induced by S. We follow [1] for undefined terms and notation.

Let G be a simple graph with vertex set $\{v_1, \ldots, v_n\}$. The adjacency matrix of G is an $n \times n$ matrix $A(G) = (a_{ij})$, where a_{ij} is the number of edges joining v_i and v_j in G. As G is simple, $A(G)$ is symmetric $(0,1)$-matrix. Eigenvalues of G are the eigenvalues of $A(G)$. We use $\lambda_i(G)$ to denote the ith largest eigenvalues of G, and so $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. Let $D(G)$ be the diagonal degree matrix of G. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are the Laplacian matrix and the signless Laplacian matrix of G, respectively. We use $\mu_i(G)$ and $q_i(G)$ to denote the ith largest eigenvalue of $L(G)$ and $Q(G)$, respectively. The value $\mu_{n-1}(G)$ is known as the **algebraic connectivity** of G.

Fiedler [10] initiated the investigation between graph connectivity and graph eigenvalues. Motivated by Kirchhoff's matrix tree theorem [16] and by a problem of Seymour (see Reference [19] of [5]), Cioabă and Wong [5] initially conjectured an explicit relationship between $\tau(G)$ and $\lambda_2(G)$ of a regular graph. This conjecture was later extended to general graphs.

Conjecture 1.1 *(Gu et al. [12], Li and Shi [18] and Liu et al. [20]). Let k be an integer with $k \geq 2$ and G be a simple graph with minimum degree $\delta \geq 2k$. If $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \geq k$.*

Several studies made progresses towards Conjecture 1.1, as seen in [12,18,20,21]. The conjecture was finally settled in [21].

Theorem 1.2 *(Liu et al. [21]). Let $k \geq 2$ be an integer, and G be a graph with $\delta \geq 2k \geq 4$. Each of the following holds.*
(i) If $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \geq k$.
(ii) If $\mu_{n-1}(G) > \frac{2k-1}{\delta+1}$, then $\tau(G) \geq k$.
(iii) If $q_2(G) < 2\delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \geq k$.

For a subset $X \subseteq V(G)$, $d(X)$ denotes the number of edges with one end in X and the other in $V(G) - V(X)$. A technical and important tool for proving Conjecture 1.1 and the main result in this paper is the fundamental theorem on spanning tree packing number of a graph G, which was obtained by Nash-Williams [23] and Tutte [27], respectively.

Theorem 1.3 *(Nash-Williams [23] and Tutte [27]). Let G be a connected graph and let $k > 0$ be an integer. Then $\tau(G) \geq k$ if and only if for any partition $(V_1, \ldots, V_t)$ of $V(G)$, $\sum_{i=1}^{t} d(V_i) \geq 2k(t-1)$.*

As consequences of Theorem 1.3, the relationship between $\tau(G)$ and $\kappa'(G)$ has been investigated, as seen in [11] and [17], among others. A characterization was proved in [3].

Theorem 1.4 *(Catlin et al. [3]). Let $k \geq 1$ be an integer. Then $\kappa'(G) \geq 2k$ if and only if for any subset $X \subseteq E(G)$ with $|X| \leq k$, $\tau(G - X) \geq k$.*

Based on the relationship between $\tau(G)$ and $\kappa'(G)$ in Theorem 1.4, researchers initially attempted to prove Conjecture 1.1 by building the relationship between $\kappa'(G)$ and eigenvalues. Cioabă in [4] initiated the investigation on the relationship between $\kappa'(G)$ and adjacency eigenvalues of regular graphs. From then on, a number of results have been obtained.

Theorem 1.5. *Let d and k be integers with $d \geq k \geq 2$, and let G be a simple graph on n vertices with $\delta \geq k$.*
(i) (Cioabă [4]) If G is d-regular and $\lambda_2(G) \leq d - \frac{(k-1)n}{(d+1)(n-d-1)}$, then $\kappa'(G) \geq k$.
(ii) (Cioabă [4]) If G is d-regular and $\lambda_2(G) < d - \frac{2(k-1)}{d+1}$, then $\kappa'(G) \geq k$.
(iii) (Gu et al. [12]) If $\lambda_2(G) < \delta - \frac{2(k-1)}{\delta+1}$, then $\kappa'(G) \geq k$.
(iv) (Li and Shi [18], Liu et al. [20]) If $\lambda_2(G) \leq \delta - \frac{(k-1)n}{(\delta+1)(n-\delta-1)}$, then $\kappa'(G) \geq k$.
(v) (Liu, Lu and Tian [19]) If $\mu_{n-1}(G) \geq \frac{(k-1)n}{(\delta+1)(n-\delta-1)}$, then $\kappa'(G) \geq k$.
(vi) (Liu, Lu and Tian, [19]) If $q_2(G) \leq 2\delta - \frac{(k-1)n}{(\delta+1)(n-\delta-1)}$, then $\kappa'(G) \geq k$.

These results motivate the current research. Known results in the literature focused on regular graphs and simple graphs, and hence it is natural to ask whether we will have a different range of the eigenvalues to predict the values of $\tau(G)$ or $\kappa'(G)$, when we are restricted to certain graph families such as bipartite graphs or triangle free graphs. Therefore, we introduce the parameter girth to investigate directly other general graph families. The approach of using girth in the study of eigenvalues and edge-connectivity was earlier taken by Liu, Lu and Tian in [19].

The main goal of this study is to investigate, when the girth of a graph G is known, the relationship between the eigenvalues of G and $\tau(G)$. As a byproduct, similar investigation for the relationship between eigenvalues of G and $\kappa'(G)$ is also conducted. Motivated by the methods deployed in [21], for any graph G with the adjacency matrix A and the diagonal degree matrix D, we define $\lambda_i(G,a)$ to be the ith largest eigenvalue of $aD+A$, where $a \geq -1$ is a real number. For given integers δ and g with $\delta > 0$ and $g \geq 3$, let $t = \lfloor \frac{g-1}{2} \rfloor$, and define the Moore function as follows.

$$f(\delta, g) = \begin{cases} 2t+1 & \text{if } \delta = 2 \text{ and } g = 2t+1, \\ 1+\delta+\sum_{i=2}^{t}(\delta-1)^i & \text{if } \delta \geq 3 \text{ and } g = 2t+1, \\ 2t+2 & \text{if } \delta = 2 \text{ and } g = 2t+2, \\ 2+2(\delta-1)^t+\sum_{i=1}^{t-1}(\delta-1)^i & \text{if } \delta \geq 3 \text{ and } g = 2t+2. \end{cases} \tag{1}$$

Theorems 1.6 and 1.7 are our main results. When $a = 0, -1$ and 1, respectively, Theorem 1.6 yields the relationship between $\kappa'(G)$ and the second largest adjacency eigenvalue $\lambda_2(G)$, the algebraic connectivity $\mu_{n-1}(G)$ and the second largest signless Laplacian eigenvalue $q_2(G)$, respectively.

Theorem 1.6. *Let g and k be integers with $g \geq 3$ and $k \geq 2$, $a \geq -1$ be a real number, and G be a simple graph of order n with minimum degree $\delta \geq k \geq 2$ and girth g. Each of the following holds.*

(i) If $\lambda_2(G,a) \leq (a+1)\delta - \dfrac{(k-1)n}{f(\delta,g)(n-f(\delta,g))}$, then $\kappa'(G) \geq k$.

(ii) If $\lambda_2(G,a) < (a+1)\delta - \dfrac{2(k-1)}{f(\delta,g)}$, then $\kappa'(G) \geq k$.

As $f(\delta,3) = \delta+1$ and when $a = 0, -1$ and 1, respectively, Theorem 1.6 extends Theorem 1.5. We present other applications of Theorem 1.6 in Section 3.

Theorem 1.7. *Let g and k be integers with $g \geq 3$ and $k \geq 2$, $a \geq -1$ be a real number, and G be a simple graph of order n with minimum degree $\delta \geq 2k \geq 4$ and girth g. If $\lambda_2(G,a) < (a+1)\delta - \dfrac{2k-1}{f(\delta,g)}$, then $\tau(G) \geq k$.*

Likewise, when $a = 0, -1$ and 1, respectively, Theorem 1.7 reveals the relationship between $\tau(G)$ and $\lambda_2(G)$, $\mu_{n-1}(G)$ and $q_2(G)$, respectively. As $f(\delta,3) = \delta+1$, Corollary 1.8 extends Theorem 1.2.

Corollary 1.8. *Let g and k be integers with $g \geq 3$ and $k \geq 2$, and G be a simple graph of order n with minimum degree $\delta \geq 2k \geq 4$ and girth g. Each of the following holds.*
(i) If $\lambda_2(G) < \delta - \frac{2k-1}{f(\delta,g)}$, then $\tau(G) \geq k$.
(ii) If $\mu_{n-1}(G) > \frac{2k-1}{f(\delta,g)}$, then $\tau(G) \geq k$.
(iii) If $q_2(G) < 2\delta - \frac{2k-1}{f(\delta,g)}$, then $\tau(G) \geq k$.

The arguments adopted in this paper are refinements and improvements of those presented in [19–21]. In the next section, we display the interlacing technique, a common tool in spectral theory of matrices. The proofs of the main results are in the subsequent sections.

2. Preliminaries

The main tool in our paper is the eigenvalue interlacing technique described below.

Given two non-increasing real sequences $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$ and $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_m$ with $n > m$, the second sequence is said to *interlace* the first one if $\theta_i \geq \eta_i \geq \theta_{n-m+i}$ for $i = 1, 2, \ldots, m$. The interlacing is *tight* if exists an integer $k \in [0, m]$ such that $\theta_i = \eta_i$ for $1 \leq i \leq k$ and $\theta_{n-m+i} = \eta_i$ for $k+1 \leq i \leq m$.

Lemma 2.1 *(Cauchy interlacing [2]). Let A be a real symmetric matrix and B be a principal submatrix of A. Then the eigenvalues of B interlace the eigenvalues of A.*

Consider an $n \times n$ real symmetric matrix

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,m} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m,1} & M_{m,2} & \cdots & M_{m,m} \end{pmatrix},$$

whose rows and columns are partitioned according to the partition $X_1, X_2, \ldots, X_m$ of $\{1, 2, \ldots, n\}$. The *quotient matrix* $R(M)$ of the matrix M is the $m \times m$ matrix whose entries are the average row sums of the blocks $M_{i,j}$ of M. The partition is *equitable* if each block $M_{i,j}$ of M has constant row (and column) sum.

Lemma 2.2 *(Brouwer and Haemers [2,13]). Let M be a real symmetric matrix. Then the eigenvalues of any quotient matrix of M interlace the ones of M. Furthermore, if the interlacing is tight, then the partition is equitable.*

3. Proof of Theorem 1.6

Following [1], for disjoint subsets X and Y of $V(G)$, let $E(X, Y)$ be the set of edges with one end in X and the other end in Y. Define

$$e(X, Y) = |E(X, Y)| \text{ and } d(X) = e(X, V(G) - X).$$

Tutte [26] initiated the cage problem, which seeks, for any given integers d and g with $d \geq 2$ and $g \geq 3$, the smallest possible number of vertices $n(d, g)$ such that there exists a d-regular simple graph with girth g. A tight lower bound (often referred as the Moore bound) on $n(d, g)$ can be found in [9].

Lemma 3.1 *(Exoo and Jajcay [9]). For given integers $d \geq 2$ and $g \geq 3$, let $t = \lfloor \frac{g-1}{2} \rfloor$. Then*

$$n(d, g) \geq N(d, g) = \begin{cases} 1 + d\sum_{i=0}^{t-1}(d-1)^i & g = 2t+1, \\ 2\sum_{i=0}^{t}(d-1)^i & g = 2t+2. \end{cases}$$

We start our arguments with a technical lemma. For a subset $X \subseteq V(G)$, define $|X| = |V(G[X])|$, $\overline{X} = V(G) - X$ and $N_G(X) = \{u \in \overline{X} : \exists\, v \in X \text{ such that } uv \in E(G)\}$. If $X = \{v\}$, then we use $N_G(v)$ for $N_G(\{v\})$. When G is understood from the context, we often omit the subscript G. Recall that the Moore function $f(\delta, g)$ is defined as in (1).

Lemma 3.2. *Let G be a simple graph with minimum degree $\delta = \delta(G) \geq 2$ and girth $g = g(G) \geq 3$, and X be a vertex subset of G. If $d(X) < \delta$, then $|X| \geq f(\delta, g)$.*

Proof. For notational convenience, we use X to denote both a vertex subset of G as well as $G[X]$, the subgraph induced by the vertices of X.

Claim 3.3. *X contains at least a cycle.*

By contradiction, assume that X is acyclic. Then $|E(X)| \leq |X| - 1$, and so

$$\delta \cdot |X| \leq \sum_{v \in X} d_G(v) = 2|E(X)| + d(X) \leq 2(|X| - 1) + \delta - 1,$$

leading to a contradiction $|X| \leq \frac{\delta-3}{\delta-2} < 1$. This proves Claim 3.3.

By Claim 3.3, X must contain a cycle with length at least g. We shall justify the lemma by making a sequence of claims.

Claim 3.4. *Each of the following holds.*
(i) If $g \geq 3$, then there exists a vertex $u_0 \in X$ such that $N(u_0) \cap \overline{X} = \emptyset$.
(ii) If $g \geq 3$, then X contains a path $P = u_0 u_1 u_2 \cdots u_{g-3}$ such that for any $i \in \{0, 1, 2, ..., g-3\}$, $N(u_i) \cap \overline{X} = \emptyset$, i.e., the neighborhood of each vertex is contained in X.

If (i) does not hold, then for every vertex $v \in X$, we always have $N(v) \cap \overline{X} \neq \emptyset$. Fix a vertex $v_0 \in X$. Then

$$
\begin{aligned}
d(X) &= |N(v_0) \cap \overline{X}| + |e(X - \{v_0\}, \overline{X})| \geq |N(v_0) \cap \overline{X}| + |X - \{v_0\}| \\
&\geq |N(v_0) \cap \overline{X}| + |N(v_0) \cap X| = d(v_0) \geq \delta,
\end{aligned}
$$

contrary to the fact $d(X) < \delta$. Hence (i) follows.

We shall prove (ii) by induction on g. By (i), (ii) holds if $g = 3$. Assume that $g \geq 4$ and (ii) holds for smaller values of g. Thus X contains a path $P' = u_0 u_1 \cdots u_{g-4}$ such that for any $i \in \{0, 1, 2, ..., g-4\}$, $N(u_i) \cap \overline{X} = \emptyset$. Let $N' = \{u' \in N(u_0) : N(u') \cap \overline{X} \neq \emptyset\}$ and $N'' = \{u'' \in N(u_{g-4}) : N(u'') \cap \overline{X} \neq \emptyset\}$. Since $g(G) = g$, for any $w \in N(u_0)$, $N(w) \cap V(P') = \{u_0\}$, and for any $w \in N(u_{g-4})$, $N(w) \cap V(P') = \{u_{g-4}\}$. As $u_{g-4} \in X$ and $|N(u_{g-4}) - V(P')| \geq \delta - 1 \geq d(X) \geq |N''|$, we have either $|N(u_{g-4}) - V(P')| > |N''|$ or $|N(u_{g-4}) - V(P')| = |N''|$. If $|N(u_{g-4}) - V(P')| > |N''|$, then there must exist a vertex $u_{g-3} \in N(u_{g-4}) - (V(P') \cup N'')$, and hence a path $P = u_0 u_1 u_2 \cdots u_{g-3}$ satisfying (ii) is found, and so (ii) holds by induction in this case. Next we assume that $|N(u_{g-4}) - V(P')| = d(X) = |N''|$. By the definition of N', for any $u' \in N'$, there must exist a vertex $w' \in \overline{X}$ such that $u'w' \in E(G)$. Hence we conclude that $N' = \emptyset$. Note that $|N(u_0) - V(P')| \geq \delta - 1 \geq 1$, and so there must exist a vertex $u_{-1} \in N(u_0) - V(P')$ such that $N(u_{-1}) \cap \overline{X} = \emptyset$. This implies that, letting $v_i = u_{i-1}$ for $0 \leq i \leq g-3$, we obtain a path $P = v_0 v_1 \cdots v_{g-3}$ such that for any $i \in \{0, 1, 2, ..., g-3\}$, $N(v_i) \cap \overline{X} = \emptyset$. Hence (ii) is also proved by induction in this case. This justifies the claim.

Let $t = \lfloor \frac{g-1}{2} \rfloor$. Assume that $g = 2t + 1$ is odd, by Lemma 3.1 and by Claim 3.4(ii), if $\delta \geq 3$, then $1 \leq d(X) \leq \delta - 1$, and so

$$
\begin{aligned}
|X| &\geq 1 + \delta \sum_{i=0}^{t-1} (\delta - 1)^i - d(X) - d(X)(\delta - 1) - \cdots - d(X)(\delta - 1)^{t-2} \\
&\geq 1 + \delta \sum_{i=0}^{t-1} (\delta - 1)^i - \sum_{i=1}^{t-1} (\delta - 1)^i = 1 + \delta + \sum_{i=2}^{t} (\delta - 1)^i = f(\delta, g). \qquad (2)
\end{aligned}
$$

If $\delta = 2$, then by Claim 3.4(ii), $|X| \geq g = 2t + 1$.

By the same reason, assume that $g = 2t + 2$ is even. If $\delta \geq 3$, then $1 \leq d(X) \leq \delta - 1$, and hence we have

$$
\begin{aligned}
|X| &\geq 2 \sum_{i=0}^{t} (\delta - 1)^i - d(X) - d(X)(\delta - 1) - \cdots - d(X)(\delta - 1)^{t-2} \\
&\geq 2 \sum_{i=0}^{t} (\delta - 1)^i - \sum_{i=1}^{t-1} (\delta - 1)^i = 2 + 2(\delta - 1)^t + \sum_{i=1}^{t-1} (\delta - 1)^i = f(\delta, g). \qquad (3)
\end{aligned}
$$

If $\delta = 2$, then by Claim 3.4(ii), $|X| \geq g = 2t + 2$.

This completes the proof of the lemma. $\square$

3.1. Proof of Theorem 1.6(i)

Let k be an integer with $k \geq 2$. By contradiction, we assume that $\kappa'(G) = r \leq k-1$. Then there exists a partition (X, Y) with $Y = V(G) - X$ such that $e(X, Y) = r \leq k-1 \leq \delta - 1 < \delta$. Let $|X| = n_1$ and $|Y| = n_2$. By Lemma 3.2 and as $n_1 + n_2 = n$, then $f(\delta, g) \leq \min\{n_1, n_2\} \leq \frac{n}{2} \leq n - f(\delta, g)$. Hence we have

$$n_1 n_2 = n_1(n - n_1) \geq f(\delta, g)(n - f(\delta, g)). \tag{4}$$

Let $\bar{d_1} = \frac{1}{n_1}\sum_{v\in X} d(v)$, $\bar{d_2} = \frac{1}{n_2}\sum_{v\in Y} d(v)$. Then $\bar{d_1}, \bar{d_2} \geq \delta$. Accordingly, the quotient matrix $R(aD + A)$ of $aD + A$ on the partition (X, Y) becomes:

$$R(aD + A) = \begin{pmatrix} (a+1)\bar{d_1} - \frac{r}{n_1} & \frac{r}{n_1} \\ \frac{r}{n_2} & (a+1)\bar{d_2} - \frac{r}{n_2} \end{pmatrix}.$$

As the characteristic polynomial of $R(aD + A)$ is

$$\lambda^2 - [(a+1)\bar{d_1} - \frac{r}{n_1} + (a+1)\bar{d_2} - \frac{r}{n_2}]\lambda + [(a+1)\bar{d_1} - \frac{r}{n_1}][(a+1)\bar{d_2} - \frac{r}{n_2}] - \frac{r^2}{n_1 n_2},$$

we have, by a direct computation,

$$\begin{aligned}
&\lambda_2(R(aD + A)) \\
&= \frac{1}{2}\{[(a+1)\bar{d_1} - \frac{r}{n_1} + (a+1)\bar{d_2} - \frac{r}{n_2}] \qquad (5)\\
&\quad -\sqrt{[(a+1)\bar{d_1} - \frac{r}{n_1} + (a+1)\bar{d_2} - \frac{r}{n_2}]^2 - 4[(a+1)\bar{d_1} - \frac{r}{n_1}][(a+1)\bar{d_2} - \frac{r}{n_2}] + \frac{4r^2}{n_1 n_2}}\} \\
&\geq \min\{(a+1)\bar{d_1}, (a+1)\bar{d_2}\} - \frac{rn}{n_1 n_2} \\
&\geq (a+1)\delta - \frac{(k-1)n}{f(\delta, g)(n - f(\delta, g))}.
\end{aligned}$$

By Lemma 2.2, $\lambda_2(G, a) \geq \lambda_2(R(aD + A)) \geq (a+1)\delta - \frac{(k-1)n}{f(\delta,g)(n-f(\delta,g))}$. By assumption, $\lambda_2(G, a) \leq (a+1)\delta - \frac{(k-1)n}{f(\delta,g)(n-f(\delta,g))}$, and so we have $\lambda_2(G, a) = \lambda_2(R(aD + A)) = (a+1)\delta - \frac{(k-1)n}{f(\delta,g)(n-f(\delta,g))}$. It follows that all the inequalities in (5) must be equalities. Hence $r = k-1$ and $\bar{d_1} = \bar{d_2} = \delta$, implying that G must be a δ-regular graph, and so $\lambda_1(G, a) = (a+1)\delta$. By algebraic manipulations,

$$
\begin{aligned}
&\lambda_1(R(aD+A))\\
=&\frac{1}{2}\{[(a+1)\delta-\frac{r}{n_1}+(a+1)\delta-\frac{r}{n_2}]\\
+&\sqrt{[(a+1)\delta-\frac{r}{n_1}+(a+1)\delta-\frac{r}{n_2}]^2-4[(a+1)\delta-\frac{r}{n_1}][(a+1)\delta-\frac{r}{n_2}]+\frac{4r^2}{n_1n_2}}\}\\
=&\frac{1}{2}\{[2(a+1)\delta-\frac{r}{n_1}-\frac{r}{n_2}]+\sqrt{[(a+1)\delta-\frac{r}{n_1}-((a+1)\delta-\frac{r}{n_2})]^2+\frac{4r^2}{n_1n_2}}\}\\
=&\frac{1}{2}\{[2(a+1)\delta-\frac{r}{n_1}-\frac{r}{n_2}]+\sqrt{(\frac{r}{n_1}-\frac{r}{n_2})^2+\frac{4r^2}{n_1n_2}}\}\\
=&\frac{1}{2}\{[2(a+1)\delta-\frac{r}{n_1}-\frac{r}{n_2}]+(\frac{r}{n_1}+\frac{r}{n_2})\}\\
=&(a+1)\delta.
\end{aligned}
$$

Therefore, the interlacing is tight. By Lemma 2.2, the partition is equitable. This means that every vertex in X has the same number of neighbors in Y. However, by Claim 3.4(i) of Lemma 3.2, there exists at least one vertex in X without a neighbor in Y. This implies that $r = e(X, Y) = k - 1 = 0$, contrary to the assumption that $k \geq 2$. □

3.2. Corollaries of Theorem 1.6(i)

By contradiction, assume that $\kappa'(G) \leq k - 1$. Since $f(\delta, g) \leq \min\{n_1, n_2\} \leq \frac{n}{2} \leq n - f(\delta, g)$, it follows from the proof of Theorem 1.6(i) that

$$
\lambda_2(G,a) \geq (a+1)\delta - \frac{(k-1)n}{f(\delta,g)(n-f(\delta,g))} \geq (a+1)\delta - \frac{2(k-1)}{f(\delta,g)}, \tag{6}
$$

contrary to the assumption of Theorem 1.6(ii). Hence Theorem 1.6(ii) follows.

For real numbers a and b with $\frac{a}{b} \geq -1$, let $\lambda_i(G, a, b)$ be the ith largest eigenvalue of the matrix $aD + bA$. Thus $\lambda_i(G, a, 1) = \lambda_i(G, a)$.

Corollary 3.5. *Let a and b be real numbers with $b \neq 0$ and $\frac{a}{b} \geq -1$, k be an integer with $k \geq 2$, and G be a simple graph with order n, girth g and minimum degree $\delta \geq k$. Each of the following holds.*
(i) $b > 0$ and $\lambda_2(G, a, b) \leq (a + b)\delta - \frac{b(k-1)n}{f(\delta,g)(n-f(\delta,g))}$, then $\kappa'(G) \geq k$.
(ii) $b < 0$ and $\lambda_{n-1}(G, a, b) \geq (a + b)\delta - \frac{b(k-1)n}{f(\delta,g)(n-f(\delta,g))}$, then $\kappa'(G) \geq k$.

Proof. As $aD + bA = b(\frac{a}{b}D + A)$, it follows by definition that

$$
\begin{cases}
\text{if } b>0, & \text{then } \lambda_i(G,a,b) = b\lambda_i(G,\frac{a}{b}); \text{ and}\\
\text{if } b<0, & \text{then } \lambda_{n-i+1}(G,a,b) = b\lambda_i(G,\frac{a}{b}).
\end{cases} \tag{7}
$$

Hence Corollary 3.5 follows from Theorem 1.6(i). □

Choosing $a \in \{0, -1, 1\}$ and $b = 1$ in Corollary 3.5, we have the following special case.

Corollary 3.6. *Let k be an integer with $k \geq 2$, and G be a simple graph with order n, girth g and minimum degree $\delta \geq k$. Each of the following holds.*
(i) If $\lambda_2(G) \leq \delta - \frac{(k-1)n}{f(\delta,g)(n-f(\delta,g))}$, then $\kappa'(G) \geq k$.
(ii) If $\mu_{n-1}(G) \geq \frac{(k-1)n}{f(\delta,g)(n-f(\delta,g))}$, then $\kappa'(G) \geq k$.
(iii) If $q_2(G) \leq 2\delta - \frac{(k-1)n}{f(\delta,g)(n-f(\delta,g))}$, then $\kappa'(G) \geq k$.

By the definition of $N(d, g)$ in Lemma 3.1, we have $N(\delta, 2t+1) = 1 + \delta \sum_{i=0}^{t-1}(\delta-1)^i$ for odd $g = 2t+1$, and $N(\delta, 2t+2) = 2\sum_{i=0}^{t}(\delta-1)^i$ for even $g = 2t+2$. In fact, the girth has also been used in [19] to study the edge connectivity and the (signless) Laplacian eigenvalues of graphs. Liu et al. [19] obtained the following result by using a different method.

Theorem 3.7 *(Liu, Lu and Tian, [19]). Let $\delta \geq k \geq 2$ be two integers, and G be a connected graph of order n, girth g and minimum degree δ. Each of the following holds.*
(i) If $\mu_{n-1}(G) \geq \frac{(k-1)n}{g(n-g)}$, then $\kappa'(G) \geq k$. Moreover, if $\delta \geq 3$ and $\mu_{n-1}(G) \geq \frac{(k-1)n}{\frac{4}{9}N(\delta,g)(n-\frac{4}{9}N(\delta,g))}$, then $\kappa'(G) \geq k$.
(ii) If $q_2(G) \leq 2\delta - \frac{(k-1)n}{g(n-g)}$, then $\kappa'(G) \geq k$. Moreover, if $\delta \geq 3$ and $q_2(G) \leq 2\delta - \frac{(k-1)n}{\frac{4}{9}N(\delta,g)(n-\frac{4}{9}N(\delta,g))}$, then $\kappa'(G) \geq k$.

Remark 3.8. The results in Corollary 3.6 and those in Theorem 3.7 address the relationship between eigenvalues and edge-connectivity of a graph G. We are to compare these results. Let $t \geq 1$ be an integer. The following are observed.
(i) Suppose that $g = 2t+1$ is odd. If $\delta = 2$, then $f(\delta, g) = 2t+1 = g$. If $\delta \geq 3$, then $f(\delta, g) = 1 + \delta + \sum_{i=2}^{t}(\delta-1)^i > 2t+2 > g$, and $f(\delta, g) = 1 + \delta + \sum_{i=2}^{t}(\delta-1)^i > \frac{4}{9}(1 + \delta\sum_{i=0}^{t-1}(\delta-1)^i) = \frac{4}{9}N(\delta, g)$.
(ii) Suppose that $g = 2t+2$ is even. If $\delta = 2$, then $f(\delta, g) = 2t+2 = g$. If $\delta \geq 3$, then $f(\delta, g) = 2 + 2(\delta-1)^t + \sum_{i=1}^{t-1}(\delta-1)^i > 2t+4 > g$, and $f(\delta, g) = 2 + 2(\delta-1)^t + \sum_{i=1}^{t-1}(\delta-1)^i > \frac{4}{9}(2\sum_{i=0}^{t}(\delta-1)^i) = \frac{4}{9}N(\delta, g)$.

As $n_1(n-n_1)$ is an increasing function on the closed interval $[1, \frac{n}{2}]$, it follows that $f(\delta, g)(n - f(\delta, g)) \geq g(n-g)$ and $f(\delta, g)(n - f(\delta, g)) > \frac{4}{9}N(\delta, g)(n - \frac{4}{9}N(\delta, g))$. Consequently, $\frac{(k-1)n}{f(\delta,g)(n-f(\delta,g))} \leq \frac{(k-1)n}{g(n-g)}$ and $\frac{(k-1)n}{f(\delta,g)(n-f(\delta,g))} < \frac{(k-1)n}{\frac{4}{9}N(\delta,g)(n-\frac{4}{9}N(\delta,g))}$. In the sense commented above, especially when δ is sufficiently large, results in Corollary 3.6 improve those of Theorem 3.7.

From Corollary 3.6, for simple graphs, we have $g \geq 3$. By the definition of $f(\delta, g)$ in (1), we have $f(\delta, g) \geq f(\delta, 3) = \delta + 1$. Suppose that $\kappa'(G) \leq k-1$. By (4), then $n_1(n-n_1) \geq f(\delta, g)(n - f(\delta, g)) \geq (\delta+1)(n-\delta-1)$. By Corollary 3.6(i), then $\lambda_2(G) >$

$\delta - \frac{(k-1)n}{f(\delta,g)(n-f(\delta,g))} \geq \delta - \frac{(k-1)n}{(\delta+1)(n-\delta-1)} \geq \delta - \frac{2(k-1)}{\delta+1}$, contrary to the assumption of Theorem 1.5(iii) and (iv). Hence Theorem 1.5 are consequences of Corollary 3.6.

By Corollary 3.6, for bipartite graphs, we have $g \geq 4$. Note that $f(\delta, 4) = 2\delta$. By a similar analysis, Corollary 3.6 also implies the following result on bipartite graphs.

Corollary 3.9. *Let G be a bipartite graph with order n and minimum degree $\delta \geq k \geq 2$. Each of the following holds.*
(i) If $\lambda_2(G) < \delta - \frac{k-1}{\delta}$, then $\kappa'(G) \geq k$.
(ii) If $\lambda_2(G) \leq \delta - \frac{(k-1)n}{2\delta(n-2\delta)}$, then $\kappa'(G) \geq k$.
(iii) If $\mu_{n-1}(G) \geq \frac{(k-1)n}{2\delta(n-2\delta)}$, then $\kappa'(G) \geq k$.
(iv) If $q_2(G) \leq 2\delta - \frac{(k-1)n}{2\delta(n-2\delta)}$, then $\kappa'(G) \geq k$.

4. Proof of Theorem 1.7 and its Corollaries

Throughout this section, for given integers δ and g, we continue to define $f(\delta, g)$ as in (1). We utilize the arguments deployed in [21] to prove Theorem 1.7 by imposing the girth requirement. In particular, the following technical lemma will also be used, with an additional condition $a \geq -1$ to justify the algebraic manipulation needed in the proof of the lemma.

Lemma 4.1 *(Lemma 3.2 of [21]). Let $a \geq -1$ be a real number and G be a simple graph with minimum degree δ. For any two disjoint nonempty vertex subsets X and Y, if $\lambda_2(G,a) \leq (a+1)\delta - \max\{\frac{d(X)}{|X|}, \frac{d(Y)}{|Y|}\}$, then*

$$[e(X,Y)]^2 \geq [(a+1)\delta - \frac{d(X)}{|X|} - \lambda_2(G,a)][(a+1)\delta - \frac{d(Y)}{|Y|} - \lambda_2(G,a)]|X||Y|.$$

Proof of Theorem 1.7. Let $V_1, \dots, V_t$ be an arbitrary partition of $V(G)$. Without loss of generality, we assume that $d(V_1) \leq d(V_2) \leq \cdots \leq d(V_t)$. By Theorem 1.3, it suffices to show that $\sum_{i=1}^{t} d(V_i) \geq 2k(t-1)$. The inequality holds trivially if $t = 1$. Hence we assume that $t \geq 2$. If $d(V_1) \geq 2k$, then $\sum_{i=1}^{t} d(V_i) \geq 2kt > 2k(t-1)$. Thus we assume that $d(V_1) \leq 2k - 1$.

Let s be the largest integer such that $d(V_s) \leq 2k-1$. Then $d(V_s) \leq 2k-1 < \delta$, where $1 \leq s \leq t$. And if $s < t$, then $d(V_{s+1}) \geq 2k$. By Lemma 3.2, $|V_i| \geq f(\delta, g)$ for $1 \leq i \leq s$. It follows that for any i with $1 < i \leq s$,

$$\lambda_2(G,a) < (a+1)\delta - \frac{2k-1}{f(\delta,g)} \leq (a+1)\delta - \max\{\frac{d(V_1)}{|V_1|}, \frac{d(V_i)}{|V_i|}\}. \tag{8}$$

By (8) and Lemma 4.1, then

$$\begin{aligned}[e(V_1,V_i)]^2 \geq &[(a+1)\delta - \frac{d(V_1)}{|V_1|} - \lambda_2(G,a)][(a+1)\delta - \frac{d(V_i)}{|V_i|} - \lambda_2(G,a)]|V_1|\cdot|V_i| \\ > &[\frac{2k-1}{f(\delta,g)} - \frac{d(V_1)}{|V_1|}]|V_1|[\frac{2k-1}{f(\delta,g)} - \frac{d(V_i)}{|V_i|}]|V_i| \\ \geq &[2k-1-d(V_1)][2k-1-d(V_i)] \\ \geq &[2k-1-d(V_i)]^2.\end{aligned}$$

Then $e(V_1,V_i) > 2k-1-d(V_i)$, and thus $e(V_1,V_i) \geq 2k-d(V_i)$. It follows that $\sum_{i=2}^{s} e(V_1,V_i) \geq \sum_{i=2}^{s}(2k-d(V_i))$, and so as $d(V_j) \geq 2k$ for all $j \geq s+1$, we have

$$\begin{aligned}\sum_{i=1}^{t} d(V_i) &= d(V_1) + \sum_{i=2}^{s} d(V_i) + \sum_{i=s+1}^{t} d(V_i) \\ &\geq \sum_{i=2}^{s} e(V_1,V_i) + \sum_{i=2}^{s} d(V_i) + \sum_{i=s+1}^{t} d(V_i) \\ &\geq 2k(s-1) - \sum_{i=2}^{s} d(V_i) + \sum_{i=2}^{s} d(V_i) + \sum_{i=s+1}^{t} d(V_i) \\ &\geq 2k(s-1) + 2k(t-s) = 2k(t-1). \end{aligned} \tag{9}$$

Hence by Theorem 1.3, $\tau(G) \geq k$, as desired. This completes the proof of Theorem 1.7. □

The following more general result can be derived from Theorem 1.7 by arguing similarly as in [21] and using (7), within certain ranges of the real numbers a and b.

Corollary 4.2. *Let a and b be real numbers satisfying $b \neq 0$ and $\frac{a}{b} \geq -1$, k be an integer with $k \geq 2$ and G be a graph with order n, girth g and minimum degree $\delta \geq 2k$. Each of the following holds.*
(i) If $b > 0$ and $\lambda_2(G,a,b) < (a+b)\delta - \frac{b(2k-1)}{f(\delta,g)}$, then $\tau(G) \geq k$.
(ii) If $b < 0$ and $\lambda_{n-1}(G,a,b) > (a+b)\delta - \frac{b(2k-1)}{f(\delta,g)}$, then $\tau(G) \geq k$.

Thus Corollary 1.8 now follows by letting $a \in \{0,1,-1\}$ and $b = 1$ in Corollary 4.2.

Declaration of Competing Interest

We have no competing interests.

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Appendix A

The algebraic manipulations to derive (5).

$$
\begin{aligned}
&\lambda_2(R(aD+A))\\
&= \frac{1}{2}\{[(a+1)\bar{d_1} - \frac{r}{n_1} + (a+1)\bar{d_2} - \frac{r}{n_2}]\\
&\quad - \sqrt{[(a+1)\bar{d_1} - \frac{r}{n_1} + (a+1)\bar{d_2} - \frac{r}{n_2}]^2 - 4[(a+1)\bar{d_1} - \frac{r}{n_1}][(a+1)\bar{d_2} - \frac{r}{n_2}] + \frac{4r^2}{n_1n_2}}\}\\
&= \frac{1}{2}\{[(a+1)\bar{d_1} - \frac{r}{n_1} + (a+1)\bar{d_2} - \frac{r}{n_2}] - \sqrt{[(a+1)\bar{d_1} - \frac{r}{n_1} - (a+1)\bar{d_2} + \frac{r}{n_2}]^2 + \frac{4r^2}{n_1n_2}}\}\\
&= \frac{1}{2}\{[(a+1)\bar{d_1} - \frac{r}{n_1} + (a+1)\bar{d_2} - \frac{r}{n_2}] - \sqrt{[(a+1)(\bar{d_1} - \bar{d_2}) - (\frac{r}{n_1} - \frac{r}{n_2})]^2 + \frac{4r^2}{n_1n_2}}\}\\
&= \frac{1}{2}\{[(a+1)\bar{d_1} - \frac{r}{n_1} + (a+1)\bar{d_2} - \frac{r}{n_2}]\\
&\quad - \sqrt{(a+1)^2(\bar{d_1} - \bar{d_2})^2 + (\frac{r}{n_1} - \frac{r}{n_2})^2 - 2(a+1)(\bar{d_1} - \bar{d_2})(\frac{r}{n_1} - \frac{r}{n_2}) + \frac{4r^2}{n_1n_2}}\}\\
&= \frac{1}{2}\{[(a+1)(\bar{d_1} + \bar{d_2}) - \frac{r}{n_1} - \frac{r}{n_2}]\\
&\quad - \sqrt{(a+1)^2(\bar{d_1} - \bar{d_2})^2 + (\frac{r}{n_1} + \frac{r}{n_2})^2 + 2(a+1)(\bar{d_1} - \bar{d_2})(\frac{r}{n_2} - \frac{r}{n_1})}\}\\
&\geq \frac{1}{2}\{[(a+1)(\bar{d_1} + \bar{d_2}) - \frac{r}{n_1} - \frac{r}{n_2}]\\
&\quad - \sqrt{(a+1)^2(\bar{d_1} - \bar{d_2})^2 + (\frac{r}{n_1} + \frac{r}{n_2})^2 + 2(a+1)|\bar{d_1} - \bar{d_2}|(\frac{r}{n_1} + \frac{r}{n_2})}\}\\
&= \frac{1}{2}\{[(a+1)(\bar{d_1} + \bar{d_2}) - \frac{r}{n_1} - \frac{r}{n_2}] - [(a+1)|\bar{d_1} - \bar{d_2}| + (\frac{r}{n_1} + \frac{r}{n_2})]\}\\
&= \min\{(a+1)\bar{d_1}, (a+1)\bar{d_2}\} - \frac{rn}{n_1n_2}\\
&\geq (a+1)\delta - \frac{(k-1)n}{f(\delta,g)(n - f(\delta,g))}.
\end{aligned}
\tag{5}
$$

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