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Modulo 5-orientations and degree sequences

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ABSTRACT

In connection to the 5-flow conjecture, a modulo 5-orientation of a graph G is an orientation of G such that the indegree is congruent to outdegree modulo 5 at each vertex. Jaeger conjectured that every 9-edge-connected multigraph admits a modulo 5-orientation, whose truth would imply Tutte's 5-flow conjecture. In this paper, we study the problem of modulo 5-orientation for given multigraphic degree sequences. We prove that a multigraphic degree sequence $d = (d_1, \ldots, d_n)$ has a realization G with a modulo 5-orientation if and only if $d_i \neq 1, 3$ for each i. In addition, we show that every multigraphic sequence $d = (d_1, \ldots, d_n)$ with $\min_{1 \le i \le n} d_i \ge 9$ has a 9-edge-connected realization which admits a modulo 5-orientation for every possible boundary function. This supports the above mentioned conjecture of Jaeger.

1. Introduction

Graphs considered in this paper are finite and loopless. As in [2], a graph is *simple* if it does not contain parallel edges or loops. For a graph which may contain parallel edges, we call it a *multigraph*. For a positive integer k, let $[k] = \{1, 2, \ldots, k\}$ and $\mathbb{Z}_k$ be the set of all integers modulo k, as well as the (additive) cyclic group of order k. Following [2], $\kappa'(G)$ denotes the edge-connectivity of a graph G. Denote a cycle with n vertices by C_n. For vertex subsets $U, W \subset V(G)$, let $[U, W]_G = \{uw \in E(G) | u \in U \text{ and } w \in W\}$. When $U = \{u\}$ or $W = \{w\}$, we use $[u, W]_G$ or $[U, w]_G$ for $[U, W]_G$, respectively. The subscript G may be omitted when G is understood from the context. For a graph G and integer $k > 0$, kG denotes the graph obtained from G by replacing each edge with k parallel edges joining the same pair of vertices.

Let $D = D(G)$ denote an orientation of G. For each $v \in V(G)$, let $E_D^+(v)$ ($E_D^-(v)$, resp.) be the set of all arcs directed out from (into, resp.) v. As in [2], $d_D^+(v) = |E_D^+(v)|$ and $d_D^-(v) = |E_D^-(v)|$ denote the out-degree and the in-degree of v under the orientation D, respectively. If a graph G has an orientation D such that $d_D^+(v) \equiv d_D^-(v) \pmod{k}$ for every vertex $v \in V(G)$, then we say that G admits a *modulo k-orientation*. Let $\mathcal{M}_k$ denote the family of all graphs with a modulo k-orientation. Note that, for even k, a graph admits a modulo k-orientation if and only if every vertex has even degree.

Let Γ be an Abelian group, let D be an orientation of G and $f : E(G) \to \Gamma$. The pair (D, f) is a *Γ-flow* in G if the net in-flow equals the net out-flow at every vertex. That is, for any vertex $v \in V(G)$,

$$\sum_{e \in E_D^+(v)} f(e) = \sum_{e \in E_D^-(v)} f(e).$$

A flow (D, f) is *nowhere-zero* if $f(e) \neq 0$ for every $e \in E(G)$. If $\Gamma = \mathbb{Z}$ and $-k < f(e) < k$ then (D, f) is called a *k-flow*. Tutte's flow conjectures are perhaps some of the most fascinating conjectures in graph theory. Tutte's 3-flow conjecture states that

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every 4-edge-connected graph admits a nowhere-zero 3-flow, which is equivalent to saying that every 4-edge-connected graph admits a modulo 3-orientation (see [2]). The celebrated 5-flow conjecture [15] states that *every bridgeless graph admits a nowhere-zero* 5*-flow*. It is well known that the 5-flow conjecture is equivalent to the statement *every* 3*-edge-connected graph* G *admits a nowhere-zero* $\mathbb{Z}_5$*-flow*. It was observed by Jaeger [6] that if the graph $3G$ has a modulo 5-orientation, then G admits a nowhere-zero $\mathbb{Z}_5$-flow. Specifically, let D be a modulo 5-orientation of $3G$ and $f = 1$ be a constant mapping from $E(3G)$ to 1. Then the sum of this flow (D, f) of $3G$ would give a nowhere-zero $\mathbb{Z}_5$-flow of G, and this led Jaeger [6] to propose the following stronger conjecture, whose truth implies Tutte's 5-flow conjecture.

Conjecture 1.1 (*[6]*). *Every* 9*-edge-connected multigraph admits a modulo* 5*-orientation.*

Jaeger [6] also proposed a more general Circular Flow Conjecture that every $4p$-edge-connected multigraph admits a modulo $(2p + 1)$-orientation, however it was disproved for all $p \geq 3$ in [5].

The concept of strongly $\mathbb{Z}_5$-connectedness is introduced in [10] serving as contractible configurations for modulo 5-orientations (see also [9]). For a graph G, let $Z(G, \mathbb{Z}_5) = \{b : V(G) \to \mathbb{Z}_5 \mid \sum_{v \in V(G)} b(v) \equiv 0 \pmod 5\}$. A graph G is **strongly** $\mathbb{Z}_5$**-connected** if, for every $b \in Z(G, \mathbb{Z}_5)$, there is an orientation D such that $d_D^+(v) - d_D^-(v) \equiv b(v) \pmod 5$ for every vertex $v \in V(G)$. Let $\langle \mathcal{S}\mathbb{Z}_5 \rangle$ denote the family of all strongly $\mathbb{Z}_5$-connected graphs. Conjecture 1.1 is further strengthened to the following conjecture in [9].

Conjecture 1.2 (*[9]*). *Every* 9*-edge-connected multigraph is strongly* $\mathbb{Z}_5$*-connected.*

Conjectures 1.1 and 1.2 are confirmed for 12-edge-connected multigraphs by Lovász, Thomassen, Wu and Zhang [12]. We also note that, by a result in [11], the truth of Conjecture 1.2 would imply another conjecture of Jaeger et al. [7] which states that every 3-edge-connected graph is $\mathbb{Z}_5$-connected. A graph is called $\mathbb{Z}_5$*-connected* if for any $b \in Z(G, \mathbb{Z}_5)$, there is an orientation D and a mapping $f : E(G) \mapsto \{1, 2, 3, 4\}$ such that for every vertex $v \in V(G)$,

$$\sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e) \equiv b(v) \pmod 5.$$

Denote $\langle \mathbb{Z}_5 \rangle$ to be the family of all $\mathbb{Z}_5$-connected graphs.

An integral degree sequence $d = (d_1, d_2, \ldots, d_n)$ is called graphic (multigraphic, resp.) if there is a simple graph (multigraph, resp.) G so that the degree sequence of G equals d; such a graph G is called a realization of d. Graphic and multigraphic sequences with certain flow and group connectivity properties have been extensively studied [3,11,13,14,16,17]. Specifically, all graphic sequences with nowhere-zero 3-flow or 4-flow realization are characterized by Luo et al. [13,14], respectively. The problem of characterizing all degree sequences with $\mathbb{Z}_3$-connected properties is proposed and studied by Yang et al. [17], and solved by Dai and Ying [3]. In general, the $\mathbb{Z}_k$-connected realization problem is characterized for $k = 4$ by Wu et al. [16], and it is eventually resolved in [11] for every k.

In this paper, we study the degree sequences with realizations that are strongly $\mathbb{Z}_5$-connected or have modulo 5-orientation properties. Our main results are the following characterizations.

Theorem 1.3. *For any multigraphic sequence* $d = (d_1, d_2, \ldots, d_n)$*,* d *has a modulo* 5*-orientation realization if and only if* $d_i \notin \{1, 3\}$ *for every* $1 \leq i \leq n$*.*

Theorem 1.4. *For any multigraphic sequence* $d = (d_1, d_2, \ldots, d_n)$*,* d *has a strongly* $\mathbb{Z}_5$*-connected realization if and only if* $\sum_{i=1}^n d_i \geq 8n - 8$ *and* $\min_{i \in [n]} d_i \geq 4$*.*

In addition, we obtain the following theorem, which provides partial evidences for Conjectures 1.1 and 1.2.

Theorem 1.5. *For any multigraphic sequence* $d = (d_1, d_2, \ldots, d_n)$ *with* $\min_{i \in [n]} d_i \geq 9$*,* d *has a* 9*-edge-connected strongly* $\mathbb{Z}_5$*-connected realization.*

Theorem 1.5 also leads to the following corollary.

Corollary 1.6. *For any multigraphic sequence* $d = (d_1, d_2, \ldots, d_n)$ *with* $\min_{i \in [n]} d_i \geq 8$*,* d *has a* 8*-edge-connected modulo* 5*-orientation realization.*

The rest of the paper is organized as follows. In section 2, we present some necessary preliminaries. Our main results are proved in section 3.

2. Preliminaries

For an edge set $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X, and then deleting the resulting loops. If H is a subgraph of G, then we use G/H for $G/E(H)$. As K_1 is strongly $\mathbb{Z}_5$-connected, for any graph G, every vertex lies in a maximal strongly $\mathbb{Z}_5$-connected subgraph. Let $H_1, H_2, \ldots, H_c$ denote the collection of all maximal subgraphs in the graph G. Then $G' = G/(\cup_{i=1}^c E(H_i))$ is called the $\langle \mathcal{S}\mathbb{Z}_5 \rangle$*-reduction* of G. If G is strongly $\mathbb{Z}_5$-connected, then its $\langle \mathcal{S}\mathbb{Z}_5 \rangle$-reduction is K_1, a singleton.

The following lemma is a summary of some basic properties stated in [8,9] and [10].

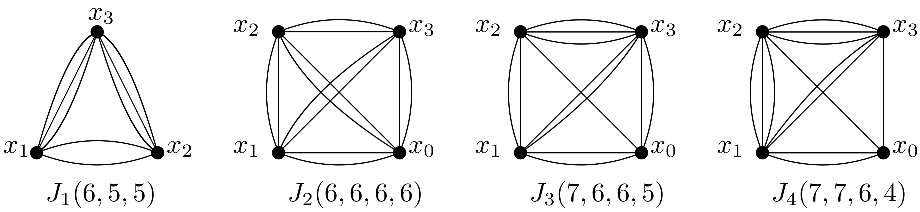

Fig. 1. The graphs in Lemma 2.7.

Lemma 2.1 ([8–10]). *Each of the following holds.*

(i) If $H \in \langle \mathbb{Z}_5 \rangle$ and $G/H \in \langle \mathbb{Z}_5 \rangle$, then $G \in \langle \mathbb{Z}_5 \rangle$.

(ii) A cycle of length n is in $\langle \mathbb{Z}_5 \rangle$ if and only if $n \le 4$.

(iii) Let mK_2 denote the loopless graph with two vertices and m parallel edges. Then mK_2 is strongly $\mathbb{Z}_5$-connected if and only if $m \ge 4$.

(iv) $G \in \mathcal{M}_5$ if and only if its $\langle \mathcal{S}\mathbb{Z}_5 \rangle$-reduction $G' \in \mathcal{M}_5$.

(v) $G \in \langle \mathcal{S}\mathbb{Z}_5 \rangle$ if and only if its $\langle \mathcal{S}\mathbb{Z}_5 \rangle$-reduction $G' = K_1$.

The following theorem is a special case of the results stated in [11].

Theorem 2.2 ([11]). *Let G be a graph. Then each of the following holds.*

(i) $G \in \langle \mathbb{Z}_5 \rangle$ if and only if $3G \in \langle \mathcal{S}\mathbb{Z}_5 \rangle$.

(ii) If $G \in \langle \mathcal{S}\mathbb{Z}_5 \rangle$, then G contains four edge-disjoint spanning trees, and in particular, $|E(G)| \ge 4|V(G)| - 4$.

For a realization G of a multigraphic degree sequence $d = (d_1, d_2, \ldots, d_n)$, if G is a realization of d with $V(G) = \{v_1, \ldots, v_n\}$ such that $d_G(v_i) = d_i$, then v_i is called the d_i-vertex for each $i \in [n]$. As a rearrangement of a degree sequence does not change its realizations, we will just focus on nonincreasing multigraphic sequence in the rest of the paper for convenience.

Theorem 2.3 (Hakimi [4]). *Let $d = (d_1, d_2, \ldots, d_n)$ be a nonincreasing integral sequence with $n \ge 2$ and $d_n \ge 0$. Then d is a multigraphic sequence if and only if $\sum_{i=1}^{n} d_i$ is even and $d_1 \le d_2 + \cdots + d_n$.*

Theorem 2.4 (Boesch and Harary [1]). *Let $d = (d_1, \ldots, d_n$ be a nonincreasing integral sequence with $n \ge 2$ and $d_n \ge 0$. Let j be an integer with $2 \le j \le n$ such that $d_j \ge 1$. Then the sequence $(d_1, d_2, \ldots, d_n)$ is multigraphic if and only if the sequence $(d_1 - 1, d_2, \ldots, d_{j-1}, d_j - 1, d_{j+1}, \ldots, d_n)$ is multigraphic.*

Let G be a graph with $uv \in E(G)$ and let w be a vertex different from u and v, where w may or may not be in $V(G)$. Define $G^{(w,uv)}$ to be the graph containing w obtained from $G - uv$ by adding new edges wu and wv. We also say that $G^{(w,uv)}$ is obtained from G by *inserting the edge uv to w* in this paper. The following observation is straightforward, which indicates the inserting operation would preserve the edge connectivity.

Lemma 2.5. *Let G be a connected graph.*

(i) Let $w \in V(G) \setminus \{u, v\}$ and $G' = G^{(w,uv)}$. Then $\kappa'(G') \ge \kappa'(G)$.

(ii) Let $w \notin V(G)$ be a new vertex and $e_1, \ldots, e_t \in E(G)$. Then the graph G' obtained from G by inserting the edges $e_1, \ldots, e_t$ to w satisfies $\kappa'(G') \ge \min\{\kappa'(G), 2t\}$.

Proof. (i) Let $[X, X^c]_{G'}$ be an edge cut of G'. Observe that either $|[X, X^c]_{G'}| = |[X, X^c]_G|$ or $|[X, X^c]_{G'}| = |[X, X^c]_G| + 2$ depending on the position of u, v, w in X or X^c. So $|[X, X^c]_{G'}| \ge |[X, X^c]_G| \ge \kappa'(G)$, and thus $\kappa'(G') \ge \kappa'(G)$.

(ii) The proof of (ii) is similar to (i). ■

Let $x_1x_2, x_2x_3 \in E(G)$. We use $G_{[x_2,x_1x_3]}$ to denote the graph obtained from $G - \{x_1x_2, x_2x_3\}$ by adding a new edge x_1x_3. The operation to get $G_{[x_2,x_1x_3]}$ from G is referred as to *lift* the edges x_1x_2, x_2x_3 in G. The next lemma follows from the definition of strongly $\mathbb{Z}_5$-connectedness.

Lemma 2.6. *Let x_1, x_2, x_3 and $G_{[x_2,x_1x_3]}$ be the same notation as defined above. If $G_{[x_2,x_1x_3]} \in \langle \mathcal{S}\mathbb{Z}_5 \rangle$, then $G \in \langle \mathcal{S}\mathbb{Z}_5 \rangle$.*

The next lemma shows that the small graphs depicted in Fig. 1 could play a crucial role in the inductive arguments of our proofs.

Lemma 2.7. *Each of the graphs J_1, J_2, J_3, J_4 in Fig. 1 is strongly $\mathbb{Z}_5$-connected.*

Proof. (i) Let $b \in Z(J_1, \mathbb{Z}_5)$. If $b(x_1) \ne 0$, we lift two edges x_3x_1, x_1x_2 in J_1 to obtain the graph $J_{1[x_1,x_2x_3]}$, say H. Since $|[x_1, \{x_2, x_3\}]_H| = 3$ and $b(x_1) \ne 0$, we can modify the boundary $b(x_1)$ with the three edges in $[x_1, \{x_2, x_3\}]_H$. Specifically, orient 2, 0, 3, 1 edges toward x_1 when $b(x_1) = 4, 3, 2, 1$, respectively. By Lemma 2.1(iii) and $|[x_2, x_3]_H| = 4$, we can also modify the boundaries $b(x_2), b(x_3)$ with four parallel edges x_2x_3. By symmetry, we assume that $b(x_1) = b(x_2) = 0$, then $b(x_3) = 0$ since $b \in Z(J_1, \mathbb{Z}_5)$. Orient all the edges in $[x_1, \{x_2, x_3\}]_{J_1}$ toward x_1 and orient all the edges in $[x_2, \{x_1, x_3\}]_{J_1}$ from x_2 to obtain an orientation of J_1, which agrees with the boundary $b(x_1) = b(x_2) = b(x_3) = 0$. Therefore J_1 is strongly $\mathbb{Z}_5$-connected by definition.

(ii) Let $b \in Z(J_2, \mathbb{Z}_5)$. If $b(x_0) = 0$, we lift three pairs of edges $\{x_2x_0, x_0x_3\}$, $\{x_2x_0, x_0x_1\}$ and $\{x_3x_0, x_0x_1\}$ from J_2 to obtain the graph $3K_3$. By Lemma 2.1(v) and since $J_1 \in \langle \mathcal{S}\mathbb{Z}_5\rangle$ is a spanning subgraph of $3K_3$, we have $3K_3 \in \langle \mathcal{S}\mathbb{Z}_5\rangle$, which implies that the boundary b at each vertex can be modified in J_2. If $b(x_0) = 2$ or 3, we lift the edges pair $\{x_2x_0, x_0x_3\}$ twice to obtain the graph G_1 and then orient the parallel edges from x_0 to x_1 or from x_1 to x_0 in G_1, respectively. By Lemma 2.1(iii), we could modify the boundary $b(x_1)$ by two pairs of parallel edges x_1x_2, x_1x_3 and then modify the boundaries $b(x_2)$ and $b(x_3)$ by the four parallel edges between x_2 and x_3. Thus the obtained orientation agrees with the boundary b. So we have $b(x_i) \in \{1, 4\}$ for each i, and by symmetry, we may assume that $b(x_0) = b(x_2) = 1$ and $b(x_1) = b(x_3) = 4$. To agree with the boundary b in this case, we orient two pairs of parallel edges x_1x_0, x_3x_0 toward x_0, two pairs of parallel edges x_1x_2, x_3x_2 toward x_2, two parallel edges x_0x_2 with opposite directions and two parallel edges x_1x_3 with opposite directions. Therefore, all possible boundaries b are examined, and so J_2 is strongly $\mathbb{Z}_5$-connected by definition.

(iii) Let $b \in Z(J_3, \mathbb{Z}_5)$. If $b(x_0) \neq 0$, lift two edges x_2x_0, x_0x_3 to obtain $J_{3[x_0,x_2x_3]}$, say L. Since $b(x_0) \neq 0$ and $|[x_0, \{x_1, x_3\}]_L| = 3$, we can modify the boundary $b(x_0)$ with the three edges in $[x_0, \{x_1, x_3\}]_L$. As $|[x_1, \{x_2, x_3\}]_L| = 4$ and by Lemma 2.1(iii), we can modify the boundary $b(x_1)$. Furthermore, as $|[x_2, x_3]_L| = 4$ and by Lemma 2.1(iii), we can modify the boundaries $b(x_2)$ and $b(x_3)$. Thus we assume that $b(x_0) = 0$. We lift the two edges x_2x_1, x_1x_3 to obtain L. Orient the five edges incident with x_0 out from x_0 in L. If $b(x_1) = 0, 1, 3$ in L we orient two edges from x_1 toward x_2, x_3, two edges from x_2, x_3 toward x_1, one edge from x_1 to x_2 and one edge from x_3 to x_1, respectively. If $b(x_1) = 4, 2$, reverse the above obtained orientation in L corresponding to $b(x_0) = 1, 3$, respectively. Then modify the boundaries $b(x_2)$ and $b(x_3)$, by Lemma 2.1(iii) and $|[x_2, x_3]_L| = 4$. Thus J_3 is strongly $\mathbb{Z}_5$-connected.

(iv) Since J_4 contains J_1 as a subgraph, $J_4/J_1 = 4K_2$ and $J_1 \in \langle \mathcal{S}\mathbb{Z}_5\rangle$, we conclude that J_4 is strongly $\mathbb{Z}_5$-connected by Lemma 2.1(iii)(v). ■

3. Proofs of main results

We shall present the proof of Theorem 1.4 first, which will be used in the proof of Theorem 1.3.

3.1. Proof of Theorem 1.4

Define $\mathcal{F}_n = \{(d_1, \ldots, d_n) : \sum_{i=1}^n d_i = 8n - 8 \text{ and } \min_{i\in[n]}\{d_i\} \geq 4\}$.

Lemma 3.1. *Let $d = (d_1, d_2, \ldots, d_n) \in \mathcal{F}_n$ be a nonincreasing sequence. Then d is multigraphic. Moreover, each of the following holds.*

(i) If $n \geq 4$ and $(d_{n-1}, d_n) \in \{(5, 5), (6, 5)\}$, then there exist $(d'_1, \ldots, d'_{n-2}) \in \mathcal{F}_{n-2}$ and nonnegative integer c_j such that for each $1 \leq j \leq n - 2$, $d_j = d'_j + c_j$ and

$$\sum_{j=1}^{n-2} c_j = \begin{cases} 6, & \text{if } (d_{n-1}, d_n) = (5, 5); \\ 5, & \text{if } (d_{n-1}, d_n) = (6, 5). \end{cases} \tag{1}$$

(ii) If $n \geq 5$ and $(d_{n-2}, d_{n-1}, d_n) \in \{(7, 7, 5), (6, 6, 6), (7, 6, 6), (7, 7, 6)\}$, then there exist $(d'_1, \ldots, d'_{n-3}) \in \mathcal{F}_{n-3}$ and nonnegative integer c_j such that for each $1 \leq j \leq n - 3$, $d_j = d'_j + c_j$ and

$$\sum_{j=1}^{n-3} c_j = \begin{cases} 5, & \text{if } (d_{n-2}, d_{n-1}, d_n) = (7, 7, 5); \\ 6, & \text{if } (d_{n-2}, d_{n-1}, d_n) = (6, 6, 6); \\ 5, & \text{if } (d_{n-2}, d_{n-1}, d_n) = (7, 6, 6); \\ 4, & \text{if } (d_{n-2}, d_{n-1}, d_n) = (7, 7, 6). \end{cases} \tag{2}$$

Proof. Since $d_n \geq 4$, we have $\sum_{i=2}^n d_i \geq 4n - 4$. Then $d_1 \leq \sum_{i=1}^n d_i - (4n - 4) = 4n - 4 \leq \sum_{i=2}^n d_i$. By Theorem 2.3, d is multigraphic.

(i) Denote $k = 16 - d_{n-1} - d_n$. If $n \geq 4$, then by $\sum_{i=1}^n d_i = 8n - 8$, we have

$$\sum_{i=1}^n d_i = 8n - 8 \geq 4(n - 2) + 16 = 4(n - 2) + (d_n + d_{n-1}) + k.$$

Thus there exists a minimal integer $i_0 \in [n-2]$ such that $\sum_{j=1}^{i_0} d_j \geq 4i_0+k$. Let $c_j = d_j-4$ for $1 \leq j \leq i_0-1$, $c_{i_0} = k-\sum_{j=1}^{i_0-1} d_j$ and $c_j = 0$ if $i_0 + 1 \leq j \leq n - 2$. Let $d'_j = d_j - c_j$ for each $1 \leq j \leq n - 2$. Then the degree sequence $(d'_1, \ldots, d'_{n-2}) \in \mathcal{F}_{n-2}$

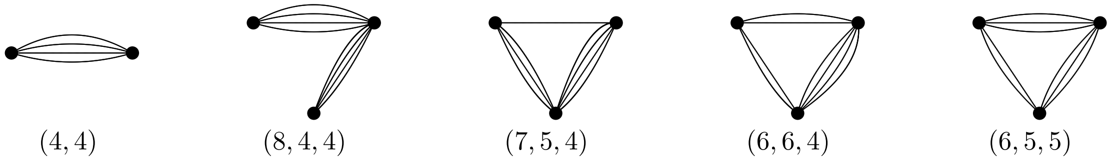

Fig. 2. The graphs in Lemma 3.2.

since

$$\sum_{j=1}^{n-2} d'_j = \sum_{j=1}^{n-2} d_j - \sum_{j=1}^{n-2} c_j = \sum_{j=1}^{n-2} d_j - k = \sum_{j=1}^{n} d_j - 16 = 8(n-2),$$

and $d'_j \geq 4$ for each $1 \leq j \leq n-2$. Moreover, Eq. (1) is satisfied as well.

(ii) The proof is similar to (i). Denote $t = 24 - d_{n-2} - d_{n-1} - d_n$. If $n \geq 5$, then by $\sum_{i=1}^{n} d_i = 8n-8$, we obtain

$$\sum_{i=1}^{n} d_i = 8n - 8 \geq 4(n-3) + 24 = 4(n-3) + (d_n + d_{n-1} + d_{n-2}) + t.$$

Thus there exists a minimal integer $i_0 \in [n-3]$ such that $\sum_{j=1}^{i_0} d_j \geq 4i_0 + t$. Let $c_j = d_j - 4$ for $1 \leq j \leq i_0 - 1$, $c_{i_0} = t - \sum_{j=1}^{i_0-1} d_j$ and $c_j = 0$ if $i_0 + 1 \leq j \leq n-3$. Let $d'_j = d_j - c_j$ for $1 \leq j \leq n-3$. Then $(d'_1, \ldots, d'_{n-3}) \in \mathcal{F}_{n-3}$ as

$$\sum_{j=1}^{n-3} d'_j = \sum_{j=1}^{n-3} d_j - \sum_{j=1}^{n-3} c_j = \sum_{j=1}^{n-3} d_j - t = \sum_{j=1}^{n} d_j - 24 = 8(n-3),$$

and $d'_j \geq 4$ for each $1 \leq j \leq n-3$. Furthermore, Eq. (2) holds as well. ■

To prove Theorem 1.4, we verify the following key Lemma first.

Lemma 3.2. *For any nonincreasing multigraphic sequence $d = (d_1, d_2, \ldots, d_n)$ with $\sum_{i=1}^{n} d_i = 8n-8$ and $d_n \geq 4$, d has a strongly $\mathbb{Z}_5$-connected realization.*

Proof. We apply induction on n. If $2 \leq n \leq 3$, then all the degree sequences satisfying the assumption $\sum_{i=1}^{n} d_i = 8n-8$ and $d_n \geq 4$ are depicted below in Fig. 2.

It follows from Lemma 2.1(iii)(v) and Lemma 2.7 that each graph above is strongly $\mathbb{Z}_5$-connected, and so Lemma 3.2 holds if $2 \leq n \leq 3$. Thus we assume that $n \geq 4$ and Lemma 3.2 holds for integers smaller than n. Notice that $4 \leq d_n \leq 7$, since $\sum_{i=1}^{n} d_i = 8n-8$.

Case 1: $d_n = 4$.

Since $\sum_{i=1}^{n-1} d_i = 8n - 12 \geq 4(n-1) + 4$, similar to the proof of Lemma 3.1, there exist a sequence $d' = (d'_1, \ldots, d'_{n-1})$ and nonnegative integer c_i for each $i \in [n-1]$ such that $\sum_{i=1}^{n-1} c_i = 4$, $d_i = d'_i + c_i$ and $d'_i \geq 4$. Then $\sum_{i=1}^{n-1} d'_i = 8(n-1) - d_n - \sum_{i=1}^{n-1} c_i = 8(n-2)$. By Lemma 3.1, d' is multigraphic and d' has a strongly $\mathbb{Z}_5$-connected realization G' by induction on n. Let G be the graph obtained from G' by adding one new vertex v_n and c_i edges joining the vertex v_n with d'_i-vertex for each $1 \leq i \leq n-1$. As $G/G' = 4K_2 \in \langle \mathcal{S}\mathbb{Z}_5 \rangle$ and $G' \in \langle \mathcal{S}\mathbb{Z}_5 \rangle$, G is a strongly $\mathbb{Z}_5$-connected realization of d by Lemma 2.1(iii)(v).

Case 2: $d_n = 5$ or $d_n = 6$.

In this case, we shall divide our discussion according to (d_{n-1}, d_n) or (d_{n-2}, d_{n-1}, d_n).

If $(d_{n-1}, d_n) \in \{(5,5), (6,5)\}$, by Lemma 3.1(i), there exists $d' = (d'_1, d'_2, \ldots, d'_{n-2}) \in \mathcal{F}_{n-2}$ such that $d_i = d'_i + c_i$ where $\sum_{i=1}^{n-2} c_i = 6$ if $(d_{n-1}, d_n) = (5,5)$ and $\sum_{i=1}^{n-2} c_i = 5$ if $(d_{n-1}, d_n) = (6,5)$. By Lemma 3.1, d' is multigraphic. By induction on n, d' has a strongly Z_5-connected realization G'. Construct the graph G from G' by adding two new vertices v_{n-1}, v_n with $\lceil \frac{16 - \sum_{i=1}^{n-2} c_i}{5} \rceil$ parallel edges $v_n v_{n-1}$ and for each $i \in [n-2]$, joining c_i edges from the d'_i-vertex to $\{v_{n-1}, v_n\}$ to obtain a new graph G as a d-realization. Since $G/G' = J_1$ (see Fig. 1), $G' \in \langle \mathcal{S}\mathbb{Z}_5 \rangle$ and $J_1 \in \langle \mathcal{S}\mathbb{Z}_5 \rangle$ by Lemma 2.7, we conclude that G is a strongly $\mathbb{Z}_5$-connected realization of d by Lemma 2.1(v).

If $n \geq 5$ and $(d_{n-2}, d_{n-1}, d_n) \in \{(7,7,5), (6,6,6), (7,6,6), (7,7,6)\}$, by Lemma 3.1(ii), there exists $d' = (d'_1, d'_2, \ldots, d'_{n-3}) \in \mathcal{F}_{n-3}$ satisfying $d_i = d'_i + c_i$ and Eq. (2). Since $\sum_{i=1}^{n-3} d'_i = 8(n-4)$ and $\min_{i \in [n-3]} d'_i \geq 4$ and by Lemma 3.1, d' is multigraphic. Then d' has a strongly $\mathbb{Z}_5$-connected realization G', by induction on n.

If $(d_{n-2}, d_{n-1}, d_n) = (7,7,5)$, let $A = \{v \in V(G') : v \text{ is a } d'_i \text{-vertex with } c_i > 0 \text{ and } i \in [n-3]\}$. We construct a graph G from G' by adding three new vertices v_{n-2}, v_{n-1}, v_n and 12 edges such that $|[v_n, v_{n-1}]_G| = 3$, $|[v_{n-2}, v_{n-1}]_G| = 4$, $|[v_n, A]_G| = 2$, $|[v_{n-2}, A]_G| = 3$ to obtain a new graph G so that G is a d-realization. By Lemmas 2.1 and 2.7(iii)(v), as $G' \in \langle \mathcal{S}\mathbb{Z}_5 \rangle$

and $G/G'/[v_{n-1}, v_{n-2}]_G = J_1 \in \langle \mathcal{S}\mathbb{Z}_5\rangle$, we have $G \in \langle \mathcal{S}\mathbb{Z}_5\rangle$, which provides a strongly $\mathbb{Z}_5$-connected realization of d. Similarly, if $(d_{n-2}, d_{n-1}, d_n) \in \{(6, 6, 6), (7, 6, 6), (7, 7, 6)\}$, we accordingly construct a graph G such that $G/G' \in \{J_2, J_3, J_4\}$ respectively, and $x_0 \in V(J)$ with $J \in \{J_2, J_3, J_4\}$ (see Fig. 1) is the vertex onto which G' is contracted in G/G'. Thus d has a realization G. By Lemma 2.1(v) and Lemma 2.7, G is a strongly $\mathbb{Z}_5$-connected realization of d.

The remaining case is $n = 4$ and $\sum_{i=1}^{4} d_i = 24$, and then $(d_1, d_2, d_3, d_4) = (6, 6, 6, 6)$. By Lemma 2.7, the graph J_2 (see Fig. 1) is the desired graph.

Case 3: $d_n = 7$.

If $d_n = 7$, by $\sum_{i=1}^{n} d_i = 8n - 8$, then $d_n = d_{n-1} = \cdots = d_{n-6} = 7$, which implies that $n \geq 7$. Thus

$$\sum_{i=1}^{n-4} d_i = 8n - 8 - 28 \geq 4(n - 4) + 4.$$

By a similar argument as in Lemma 3.1, there exist a degree sequence $d' = (d'_1, \cdots, d'_{n-4})$ and nonnegative integer c_i such that $d_i = d'_i + c_i$ and $d'_i \geq 4$ for $1 \leq i \leq n - 4$, where $\sum_{i=1}^{n-4} c_i = 4$. Thus

$$\sum_{i=1}^{n-4} d'_i = \sum_{i=1}^{n} d_i - \sum_{i=n-3}^{n} d_i - \sum_{i=1}^{n-4} c_i = 8(n - 1) - 28 - 4 = 8(n - 5).$$

By Lemma 3.1, d' is multigraphic. By induction on n, d' has a strongly $\mathbb{Z}_5$-connected realization G'. We construct the graph G from G' and $3C_4$ by adding c_i edges between d'_i-vertex and vertices of $3C_4$ such that $d_G(x) = 7$ for any $x \in V(3C_4)$ so that G is a d-realization. By Lemma 2.1(ii) and Theorem 2.2(i), $3C_4 \in \langle \mathcal{S}\mathbb{Z}_5\rangle$. By Lemma 2.1(iii) (v) and $(G/G')/3C_4 = 4K_2 \in \langle \mathcal{S}\mathbb{Z}_5\rangle$, G is a strongly $\mathbb{Z}_5$-connected d-realization. This completes the proof. ■

Now we are ready to prove Theorem 1.4.

Theorem 1.4. *For any nonincreasing multigraphic sequence $d = (d_1, d_2, \ldots, d_n)$, d has a strongly $\mathbb{Z}_5$-connected realization if and only if $\sum_{i=1}^{n} d_i \geq 8n - 8$ and $d_n \geq 4$.*

Proof. To prove the necessity, by Theorem 2.2(ii) and Lemma 2.1(iii), if $G \in \langle \mathcal{S}\mathbb{Z}_5\rangle$ with degree sequence $(d_1, d_2, \ldots, d_n)$, then $\sum_{i=1}^{n} d_i \geq 8n - 8$ and $d_n \geq 4$.

For sufficiency, suppose the contrary that the nonincreasing multigraphic sequence $(d_1, d_2, \ldots, d_n)$ is a counterexample with $\sum_{i=1}^{n} d_i$ minimized. By Lemma 3.2, $\sum_{i=1}^{n} d_i > 8n - 8$ and $d_n \geq 4$. If $d_2 = 4$, then by Theorem 2.3, we have $\sum_{i=1}^{n} d_i \leq 2\sum_{i=2}^{n} d_i = 8n - 8$, a contradiction. Thus we assume that $d_2 \geq 5$ and let $(d'_1, d'_2, d'_3 \cdots, d'_n) = (d_1 - 1, d_2 - 1, d_3, \ldots, d_n)$. By Theorem 2.4, $(d'_1, \ldots, d'_n)$ is multigraphic, and so by the minimality of d, $(d'_1, \ldots, d'_n)$ has a strongly $\mathbb{Z}_5$-connected realization G'. Then we obtain the graph G as a d-realization from G' by adding one edge between the d'_1-vertex and the d'_2-vertex. Since $G' \in \langle \mathcal{S}\mathbb{Z}_5\rangle$, it follows from Lemma 2.1(v) that $G \in \langle \mathcal{S}\mathbb{Z}_5\rangle$, a contradiction. ■

3.2. Proof of Theorem 1.3

Theorem 1.3. *For any nonincreasing multigraphic sequence $d = (d_1, d_2, \ldots, d_n)$, d has a modulo 5-orientation realization if and only if $d_i \notin \{1, 3\}$ for every $1 \leq i \leq n$.*

Proof. To prove the necessity, let $(d_1, \ldots, d_n)$ be any multigraphic sequence, by the definition of modulo 5-orientation, we achieve $d_i \notin \{1, 3\}$ for every $1 \leq i \leq n$.

For sufficiency, suppose the contrary that the nonincreasing multigraphic sequence $d = (d_1, \ldots, d_n)$ is a counterexample with $m = \sum_{i=1}^{n} d_i$ minimized. By Theorem 2.3, $d_1 \leq \sum_{i=2}^{n} d_i$.

Claim A. $d_1 \leq \sum_{i=2}^{n} d_i - 4$.

By contradiction, we assume that $d_1 \in \{\sum_{i=2}^{n} d_i - 2, \sum_{i=2}^{n} d_i\}$.

If $d_1 = \sum_{i=2}^{n} d_i$, then d has a unique realization G by setting v_1 as the center vertex adjacent to the vertices $v_2, \ldots, v_n$ with $d_2, \ldots, d_n$ multiple edges, respectively. Now we are to prove that G has a modulo 5-orientation D. For each $2 \leq i \leq n - 1$, if d_i is even, then we orient one half of the edges from v_i toward v_1 and orient rest edges from v_1 to v_i. If d_i is odd, we assign $\frac{d_i+5}{2}$ edges with the orientation from v_i into vertex v_1 and $\frac{d_i-5}{2}$ edges with opposite direction. Thus G is a modulo 5-orientation realization of d, a contradiction.

Assume that $d_1 = \sum_{i=2}^{n} d_i - 2$. From the above oriented graph G with degree sequence $(\sum_{i=2}^{n} d_i, d_2, \ldots, d_n)$, we pick up one directed edge oriented into the vertex v_1, denoted by e_1, and another edge oriented out from v_1, denoted by e_2, where $e_1 \cap e_2 = \{v_1\}$. Let G' be the graph obtained from G by lifting two edges e_1, e_2 to become a new edge. It is easy to see that G' preserves the modulo 5-orientation and that G' has degree sequence $d = (\sum_{i=2}^{n} d_i - 2, d_2, \ldots, d_n)$. This contradicts to the assumption that d is a counterexample.

Claim B. $d_n \notin \{2, 4\}$ *and* $n \geq 4$.

By contradiction, assume that $d_n = 2t$ *for some* $t \in \{1, 2\}$. *Let* $d' = (d'_1, d'_2, \ldots, d'_{n-1}) = (d_1, d_2, \ldots, d_{n-1})$. *Since* $d_1 \leq \sum_{i=2}^{n} d_i - 4$ *by Claim A, we have* $d'_1 \leq \sum_{i=2}^{n-1} d'_i$. *By Theorem 2.3,* d' *is multigraphic. Since* $\sum_{i=1}^{n-1} d'_i < m$ *and by the minimality of m,* d' *has a modulo 5-orientation realization* G'. *We pick up t directed edges* $e_1, \ldots, e_t$ *in the modulo 5-orientation of* G'. *Let G be the graph obtained from* G' *by inserting the edges* $e_1, \ldots, e_t$ *to a new vertex* v_n. *This would extend the modulo 5-orientation of* G' *to the graph G. However, it is clear that G is a d-realization, a contradiction.*

The case of $n = 2$ *is obvious. Let* $n = 3$. *Since* $d_3 \geq 5$, *we have* $d_1 + d_2 + d_3 \geq 15$, *and so* $d_1 + d_2 + d_3 \geq 16$ *by parity. By Theorem 1.4 and since* $16 = 8(n-1)$, *d has a strongly* Z_5*-connected realization, and therefore a modulo 5-orientation realization, a contradiction.*

Claim C. $d_1 \leq \sum_{i=2}^{n} d_i - 6$ *and* $d_n \neq 6$.

Suppose to the contrary that $d_1 = \sum_{i=2}^{n} d_i - 4$ *(by Claim A). Similar to the proof of Claim A, let G be a* $(\sum_{i=2}^{n} d_i, d_2, \ldots, d_n)$*-realization with center vertex* v_1 *adjacent to the vertices* $v_2, \ldots, v_n$ *with* $d_2, \ldots, d_n$ *multiple edges, respectively. Since* $d_{n-1} \geq d_n \geq 5$ *by Claim B, we lift the edges pair* $\{v_1v_{n-1}, v_1v_n\}$ *twice to obtain a graph* G'. *Then* $G'[\{v_1, v_{n-1}, v_n\}]$ *contains the graph* J_1 *(see Fig. 1), and therefore has a modulo 5-orientation by Lemma 2.7. Since* $|[v_1, v_i]_{G'}| \geq 5$ *for each* $2 \leq i \leq n-2$, *we can extend the modulo 5-orientation of* $G'[\{v_1, v_{n-1}, v_n\}]$ *to the entire graph* G' *by Lemma 2.1(iii). This shows that* G' *is a modulo 5-orientation d-realization, a contradiction.*

Using a similar argument as employed in the proof of Claim B, we obtain $d_n \neq 6$. *Since* $(d_1, d_2, \ldots, d_{n-1})$ *is multigraphic provided that* $d_n = 6$ *and* $d_1 \leq \sum_{i=2}^{n} d_i - 6$. *That is, we can insert three edges in* G' *to a new vertex* v_n *to form the desired graph G.*

Now, as $d_n \geq 5$ and by Theorem 1.4, we have

$$\sum_{i=1}^{n} d_i \leq 8n - 10. \tag{3}$$

Claim D. $d_n \neq 5$.

If $n = 4$ *and* $d_4 = 5$, *then by* $\sum_{i=1}^{4} d_i \leq 22$, $d = (d_1, d_2, d_3, d_4) \in \{(5, 5, 5, 5), (7, 5, 5, 5), (6, 6, 5, 5)\}$. *If* $(d_1, d_2, d_3, d_4) \in \{(5, 5, 5, 5), (6, 6, 5, 5)\}$, *we obtain the desired graph G from* J_1 *in Fig. 1 by replacing the vertex* x_3 *with 2 or 3 parallel edges, separately. If* $(d_1, d_2, d_3, d_4) = (7, 5, 5, 5)$, *then we have the graph G from* J_1 *by inserting the parallel edges* x_1x_2 *to a new vertex* x_4 *and adding one new edge* x_3x_4. *In any case, it is easy to check that G is a modulo 5-orientation d-realization, a contradiction.*

If $n \geq 5$ *and* $d_n = d_{n-1} = 5$, *then let* $d' = (d'_1, d'_2, \ldots, d'_{n-2}) = (d_1, d_2, \ldots, d_{n-2})$. *Since*

$$d_1 + 5(n-1) \leq d_1 + \sum_{i=2}^{n} d_i \leq 8n - 10,$$

we obtain $d_1 \leq 3n - 5$. *Since* $n \geq 5$, $d'_1 \leq 3n - 5 \leq 5(n-3) \leq \sum_{i=2}^{n-2} d'_i$. *By Theorem 2.3,* d' *is multigraphic. By induction,* d' *has a modulo 5-orientation realization* G'. *Pick up a directed edge uv in the graph* G'. *Construct the graph G from* G' *by adding distinct vertices* v_{n-1}, v_n, *deleting oriented edge uv and adding oriented edges* uv_{n-1}, v_nv *and 4 parallel edges* v_nv_{n-1}. *Thus G is the desired graph by Lemma 2.1(iii), a contradiction.*

Otherwise, since $d_n = 5$ *and* $\sum_{i=1}^{n} d_i$ *is even, there exists an odd* $d_i \geq 7$ *for some* $1 \leq i \leq n-1$. *Let* $d' = (d'_1, \ldots, d'_i, \ldots, d'_{n-1}) = (d_1, \ldots, d_i - 5, \ldots, d_{n-1})$. *Since* $n \geq 5$, *we have* $d'_1 = d_1 \leq 3n - 5 \leq 5(n-3) + 2 \leq \sum_{i=2}^{n-1} d'_i$. *By Theorem 2.3 and induction, let* G' *be a modulo 5-orientation realization of* d'. *Construct the graph G from* G' *by adding a new vertex* v_n *such that* v_n *is adjacent to the* d'_i*-vertex with 5 parallel edges. By Lemma 2.1(iii), G is a modulo 5-orientation d-realization, a contradiction.*

Claim E. $d_n \neq 7$.

If $d_n = 7$, *then* $d_n = d_{n-1} = \cdots = d_{n-6} = 7$ *by Eq. (3), which implies that* $n \geq 7$. *Let* $d' = (d'_1, \ldots, d'_{n-4}) = (d_1, \ldots, d_{n-4})$. *Since* $d_1 + 7(n-1) \leq \sum_{i=1}^{n} d_i \leq 8n - 10$, *we obtain* $d'_1 \leq n - 3 \leq 7(n-5) \leq \sum_{i=2}^{n-4} d'_i$. *By Theorem 2.3 and induction,* d' *has a modulo 5-orientation realization* G'. *Let* u_1v_1, u_2v_2 *be two directed distinct edges in* G'. *We construct the graph G from* G' *and* $3C_4$ *with vertices* v_j, $n-3 \leq j \leq n$, *by deleting* u_1v_1, u_2v_2 *and adding oriented edges* $u_1v_{n-3}, v_{n-2}v_1, u_2v_{n-1}, v_nv_2$. *By Lemma 2.1(ii) and (i),* $3C_4$ *is strongly* $\mathbb{Z}_5$*-connected. Thus the modulo 5-orientation of* G' *is easily extended to the graph G as a d-realization, a contradiction.*

Therefore, it follows from Claims A–E that $d_n \geq 8$, and so $\sum_{i=1}^{n} d_i \geq 8n$, a contradiction to Eq. (3). The proof is completed. ■

3.3. Proof of Theorem 1.5

A graph is called cubic if it is 3-regular. For a cubic graph G, a $Y-\Delta$ *operation* on a vertex v is to replace the vertex v with a triangle, where each edge incident with v in G becomes an edge incident to a vertex of the triangle. It is clear that applying

$Y - \Delta$ operation on a cubic graph still results a cubic graph, and it follows from Lemma 2.1(i)(ii) that any graph obtained from K_4 by $Y - \Delta$ operation is $\mathbb{Z}_5$-connected. We will use this observation (and in fact a stronger property) in the proof of Theorem 1.5. Before presenting the proof, we shall handle some special cases first. If a sequence d consists of the terms $d_1, \ldots, d_t$ having multiplicities $m_1, \ldots, m_t$, we may write $d = (d_1^{m_1}, \ldots, d_t^{m_t})$ for convenience.

Lemma 3.3. *Each of the integral multigraphic sequences* $(17, 9^3), (14, 9^4), (16, 9^4), (16, 9^6)$ *has a* 9*-edge-connected strongly* $\mathbb{Z}_5$*-connected realization.*

Proof. For $d = (17, 9^3)$, we construct a graph G as d-realization from J_1 in Fig. 1 by adding a new vertex x_4 with 2 parallel edges x_1x_4 and 7 multiple edges x_2x_4, respectively, then adding 3, 2 multiple edges x_3x_2, x_1x_2, respectively. It is routine to check that G is 9-edge-connected, i.e. for any $S \subset V(G)$ with $|S| = 1$ or 2, we have $|[S, V(G) \setminus S]_G| \geq 9$. By Lemmas 2.7 and 2.1(iii)(v), G is a strongly $\mathbb{Z}_5$-connected d-realization.

For $d = (16, 9^6)$, we construct the graph G_0 from two disjoint copies of $3K_4$ with labeled vertices v', v'' respectively, by identifying vertices v', v'' to a new vertex and lifting the two edges e_1, e_2, where e_1, e_2 are adjacent to v', v'' in each $3K_4$. It is easy to check that G_0 is 9-edge-connected. Since G_0 contains J_2 (see Fig. 1) as a subgraph and by Lemmas 2.7 and 2.1(v), G_0 is a strongly $\mathbb{Z}_5$-connected d-realization.

For $d = (16, 9^4)$, we obtain the desired graph G_1 gained from J_1 in Fig. 1 by adding two new vertices x_4, x_5 with edges x_1x_4, x_2x_4 and 3, 3, 3, 7 parallel edges $x_3x_5, x_1x_5, x_2x_5, x_4x_5$, respectively. For any $S \subset V(G_1)$, it is easy to check that $|[S, V(G_1) \setminus S]| \geq 9$. Thus G_1 is a 9-edge-connected strongly $\mathbb{Z}_5$-connected d-realization by Lemma 2.7 and Lemma 2.1(iii)(v).

For $d = (14, 9^4)$, we have the desired graph G_2 obtained from above G_1 by lifting the two edges x_3x_5 and x_4x_5. Let $S \subset V(G_2)$. It is routine to verify that $|[S, V(G_2) \setminus S]_{G_2}| \geq 9$ for any $S \subset V(G_2)$. Therefore G_2 is a 9-edge-connected strongly $\mathbb{Z}_5$-connected d-realization by Lemmas 2.7 and 2.1(iii)(v). ■

Theorem 1.5. *For any nonincreasing multigraphic sequence* $d = (d_1, d_2, \ldots, d_n)$ *with* $\min_{i\in[n]} d_i \geq 9$*,* d *has a* 9*-edge-connected strongly* $\mathbb{Z}_5$*-connected realization.*

Proof. Let $d = (d_1, d_2, \ldots, d_n)$ be a nonincreasing multigraphic sequence with $d_n \geq 9$. By Theorem 2.3, we have $d_1 \leq \sum_{i=2}^{n} d_i$. If $n = 2$, then $d_1 = d_2$ and it is obvious to verify this statement by Lemma 2.1(iii). We argue by induction on $m = \sum_{i=1}^{n} d_i$ and assume that $n \geq 3$ and that Theorem 1.5 holds for smaller value of m. We are to construct a 9-edge-connected strongly $\mathbb{Z}_5$-connected d-realization.

Case 1: $d_1 = 9$.

Since $d_n \geq 9$, we have $(d_1, d_2, \ldots, d_n) = (9, 9, \ldots, 9)$. Since $\sum_{i=1}^{n} d_i$ is even and $n \geq 3$, this implies that n is even and $n \geq 4$. We obtain a graph G' by applying $Y - \Delta$ operation on the complete graph K_4 several times until the cubic graph processes n vertices. By Lemma 2.1(i)(ii), $G' \in \langle \mathbb{Z}_5 \rangle$. Let $G = 3G'$. Then $G \in \langle \mathcal{S}\mathbb{Z}_5 \rangle$ by Theorem 2.2(i). Since G' is 3-edge-connected, G is a 9-edge-connected strongly $\mathbb{Z}_5$-connected d-realization.

Case 2: $d_2 \geq 10$.

In this case, $d_1 \geq d_2 \geq 10$, and we let $d' = (d_1 - 1, d_2 - 1, d_3, \ldots, d_n)$. By Theorem 2.4, d' is multigraphic. By induction on m, d' has a 9-edge-connected strongly $\mathbb{Z}_5$-connected realization G'. Construct the graph G from G' by adding one edge joining $(d_1 - 1)$-vertex and $(d_2 - 1)$-vertex in graph G'. By Lemma 2.1(v), G is also a 9-edge-connected strongly $\mathbb{Z}_5$-connected realization of d.

Now, we consider the remaining case.

Case 3: $d_1 \geq 10$ and $d_2 = \cdots = d_n = 9$.

If $d_1 \geq 18$, we let $d' = (d_1 - 9, d_2, \ldots, d_{n-1})$. Then d' is multigraphic as $d_1 - 9 \leq \sum_{i=2}^{n-1} d_i$ and by Theorem 2.3. By induction on m, there exists a 9-edge-connected strongly $\mathbb{Z}_5$-connected graph G' as d'-realization. Construct the graph G by adding one new vertex v_n and 9 parallel edges joining v_n and $(d_1 - 9)$-vertex in G'. By Lemma 2.1(iii)(v), G is the desired graph. Combining Case 1, we assume that $10 \leq d_1 \leq 17$ below.

Case 3.1: d_1 is odd.

Since $\sum_{i=1}^{n} d_i$ is even, n is even and $n \geq 4$. If $n = 4$ and $11 \leq d_1 \leq 15$, we let $d_1 - 9 = 2q$, where $1 \leq q \leq 3$. Let v be an arbitrary vertex in $3K_4$ and let $e_1, \ldots, e_q$ be non-parallel edges not adjacent to v in $3K_4$. We obtain the graph G as d-realization from $3K_4$ by inserting the edges $e_1, \ldots, e_q$ to the vertex v. By Lemma 2.5(i), G is 9-edge connected. Since G contains J_2 as a spanning subgraph, by Lemmas 2.7 and 2.1(v), $G \in \langle \mathcal{S}\mathbb{Z}_5 \rangle$. Otherwise, $(d_1, d_2, d_3, d_4) = (17, 9, 9, 9)$, which has already been handled in Lemma 3.3.

If $n \geq 6$, we obtain a graph G' by applying $Y - \Delta$ operation on K_4 repeatedly until the cubic graph processes n vertices. Denote the last obtained vertex by v_1 in G', which is in the last generated triangle. Let $d_1 - 9 = 2q$, where $1 \leq q \leq 4$. We select q edges $e_1, \ldots, e_q$ that are coming from the edges of the basic graph K_4, which are not adjacent to v_1 in the graph G'. Obtain the graph G from $3G'$ by inserting the edges $e_1, \ldots, e_q$ to v_1. By Lemma 2.5(i), G is 9-edge-connected. To verify that G is strongly $\mathbb{Z}_5$-connected, we first observe that the graph induced by the vertices of the last generated triangle is strongly $\mathbb{Z}_5$-connected as it contains J_1 as a spanning subgraph. Then we can contract the last generated triangle and consecutively contract all the generated triangles, the remaining graph is strongly $\mathbb{Z}_5$-connected as it contains a J_2 as a spanning subgraph. By Lemma 2.1(v), G is a strongly $\mathbb{Z}_5$-connected d-realization.

Case 3.2: d_1 is even.

Since $\sum_{i=1}^{n} d_i$ is even, n is odd and $n \geq 3$. When $n = 3$, we have $d = (d_1, d_2, d_3) = (d_1, 9^2)$ and it is straightforward to obtain a 9-edge connected d-realization G containing the graph J_1. If $n = 5$ and $d_1 = 14$ or $d_1 = 16$ or $n = 7$ and $d_1 = 16$, then the multigraphic sequences are $(14, 9^4)$, $(16, 9^4)$, $(16, 9^6)$, which are all verified by Lemma 3.3.

The remaining cases are as follows: $n \geq 9$, or $n = 7$ and $10 \leq d_1 \leq 14$, or $n = 5$ and $10 \leq d_1 \leq 12$. We construct a graph G' by applying $Y - \Delta$ operation on K_4 repeatedly until the cubic graph processes $n - 1$ vertices. Let $E' \subset E(G')$ consist the edges of the base graph K_4 and one edge in each generated triangle in G'. Thus $|E'| \geq 8$ if $n \geq 9$; $|E'| = 7$ if $n = 7$; $|E'| = 6$ if $n = 5$. Let $d_1 = 2q$. Note that $|E'| \geq q$. We select the edges $e_1, \ldots, e_q$ in E' and obtain the graph G from $3G'$ by inserting the edges $e_1, \ldots, e_q \in E'$ to a new vertex v_1. By Lemma 2.5(ii), G is 9-edge connected. Clearly, G is a d-realization. To see that G is strongly $\mathbb{Z}_5$-connected, we first recall that J_1 and J_2 are strongly $\mathbb{Z}_5$-connected by Lemma 2.7. By contracting J_1 and $3K_3$ in the generated triangles of G consecutively, the resulting graph consists of 5 vertices, namely v_1 and the remaining 4 vertices induced a graph containing J_2. We then contract J_2 and the resulting $2q$ parallel edges to obtain K_1. This shows that G is a strongly $\mathbb{Z}_5$-connected by Lemma 2.1(v). The proof is completed. ■

Proof of Corollary 1.6. We assume that $d = (d_1, \ldots, d_n)$ is a nonincreasing multigraphic sequence with $d_n \geq 8$. By Theorem 2.3, $d_1 \leq \sum_{i=2}^{n} d_i$. The case of $n = 2$ is trivial. Assume that $n \geq 3$. Suppose to the contrary that $(d_1, \ldots, d_n)$ is a counterexample with $m = \sum_{i=1}^{n} d_i$ minimized.

If $d_1 \geq 10$, let $d' = (d'_1, d'_2, \ldots, d'_n) = (d_1 - 2, d_2, \ldots, d_n)$. If $d_1 - 2 = d'_1 \geq d'_2 = d_2$, then $d'_1 \leq d_1 \leq \sum_{i=2}^{n} d_i = \sum_{i=2}^{n} d'_i$. Otherwise, $d'_1 = d_1 - 2 < d'_2$, then $\max_{i\in[n]}\{d'_i\} = d_2 \leq d_1 \leq d_1 - 2 + \sum_{i=3}^{n} d_i = d'_1 + \sum_{i=3}^{n} d'_i$, since $n \geq 3$. Hence d' is multigraphic in any case by Theorem 2.3. Let G' be a 8-edge-connected modulo 5-orientation d'-realization by the minimality. We obtain the desired graph G from G' by inserting one edge to the $(d_1 - 2)$-vertex in G'. By Lemma 2.5(i), G is also a 8-edge-connected modulo 5-orientation d-realization, a contradiction.

If $d_1 = 8$, then $d_1 = \cdots = d_n = 8$. Hence $G = 4C_n$ is a 8-edge-connected modulo 5-orientation d-realization, a contradiction. Assume that $d_1 = 9$ in the following. As $\sum_{i=1}^{n} d_i$ is even, we have $d_2 = 9$. If $d_n = 8$, we let $d' = (d'_1, d'_2, \ldots, d'_{n-1}) = (d_1, d_2, \ldots, d_{n-1})$. Then $d'_1 \leq d'_2 \leq \sum_{i=2}^{n} d'_i$, and so d' is multigraphic by Theorem 2.3. Let G' be a 8-edge-connected modulo 5-orientation d'-realization by the minimality. Let $e_i \in E(G')$, $1 \leq i \leq 4$. We obtain the desired graph G from G' by inserting the edges $e_1, \ldots, e_4$ to one new vertex v_n. By Lemma 2.5(ii), G is a 8-edge-connected modulo 5-orientation realization of d, a contradiction. Therefore, we have $d_n \geq 9$, and it follows from Theorem 1.5 that there exists a 9-edge-connected strongly $\mathbb{Z}_5$-connected graph G as a d-realization, which admits a modulo 5-orientation as well. This contradiction completes the proof of Corollary 1.6. ■

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References

[1] F. Boesch, F. Harary, Line removal algorithms for graphs and their degree lists, IEEE Trans. Circuits Syst. 23 (1976) 778–782.
[2] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, New York, 2008.
[3] X. Dai, J. Yin, A complete characterization of graphic sequences with a Z_3-connected realization, European J. Combin. 51 (2016) 215–221.
[4] S.L. Hakimi, On the realizability of a set of integers as degrees of the vertices of a graph, SIAM J. Appl. Math. 10 (1962) 496–506.
[5] M. Han, J. Li, Y. Wu, C.-Q. Zhang, Counterexamples to Jaeger's circular flow conjecture, J. Combin. Theory Ser. B 131 (2018) 1–11.
[6] F. Jaeger, Nowhere-zero flow problems, in: L. Beineke, R. Wilson (Eds.), in: Selected Topics in Graph Theory, vol. 3, Academic Press, London, New York, 1988, pp. 91–95.
[7] F. Jaeger, N. Linial, C. Payan, N. Tarsi, Group connectivity of graphs - a nonhomogeneous analogue of nowhere zero flow properties, J. Combin. Theory Ser. B 56 (1992) 165–182.
[8] H.-J. Lai, Group connectivity of 3-edge-connected chordal graphs, Graphs Combin. 16 (2000) 165–176.
[9] H.-J. Lai, Mod $(2p + 1)$-orientations and $K_{1,2p+1}$-decompositions, SIAM J. Discrete Math. 21 (2007) 844–850.
[10] H.-J. Lai, Y. Liang, J. Liu, Z. Miao, J. Meng, Y. Shao, Z. Zhang, On strongly $\mathbb{Z}_{2s+1}$-connected graphs, Discrete Appl. Math. 174 (2014) 73–80.
[11] J. Li, H.-J. Lai, R. Luo, Group connectivity strongly Z_m-connectivity and edge disjoint spanning trees, SIAM J. Discrete Math. 31 (2017) 1909–1922.
[12] L.M. Lovász, C. Thomassen, Y. Wu, C.-Q. Zhang, Nowhere-zero 3-flows and modulo k-orientations, J. Combin. Theory Ser. B 103 (2013) 587–598.
[13] R. Luo, R. Xu, W. Zang, C.-Q. Zhang, Realizing degree sequences with graphs having nowhere-zero 3-flows, SIAM J. Discrete Math. 22 (2008) 500–519.
[14] R. Luo, W. Zang, C.-Q. Zhang, Nowhere-zero 4-flows simultaneous edge-colorings and critical partial latin squares, Combinatorica 24 (2004) 641–657.
[15] W.T. Tutte, A contribution to the theory of chromatical polynomials, Canad. J. Math. 6 (1954) 80–91.
[16] Y. Wu, R. Luo, D. Ye, C.-Q. Zhang, A note on an extremal problem for group connectivity, European J. Combin. 40 (2014) 137–141.
[17] F. Yang, X. Li, H.-J. Lai, Realizing degree sequences as $\mathbb{Z}_3$-connected graphs, Discrete Math. 333 (2014) 110–119.