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On r-hued colorings of graphs without short induced paths ☆

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ABSTRACT

For integers $k, r > 0$, a (k, r)-coloring of a graph G is a proper coloring on the vertices of G with k colors such that every vertex v of degree $d(v)$ is adjacent to vertices with at least $\min\{d(v), r\}$ different colors. The r-hued chromatic number, denoted by $\chi_r(G)$, is the smallest integer k for which a graph G has a (k, r)-coloring. We prove the following:
(i) If G is a P_4-free graph, then $\chi_r(G) \leq \chi(G) + 2(r-1)$, and this bound is best possible.
(ii) If G is a P_5-free bipartite graph, then $\chi_r(G) \leq r\chi(G)$, and this bound is best possible.
(iii) If G is a P_5-free graph, then $\chi_2(G) \leq 2\chi(G)$, and this bound is best possible.

1. Introduction

Throughout this paper, for an integer $k > 0$, define $[k] = \{1, 2, \ldots, k\}$. We study finite and simple graphs, and follow [7] for undefined notations and terms. Thus $\delta(G)$, $\Delta(G)$ and $\chi(G)$ denote the minimum degree, the maximum degree and the chromatic number of a graph G respectively. If $c : V(G) \mapsto [k]$ is a mapping, then define $c(S) = \{c(u) : u \in S\}$. Define the **neighborhood** of a vertex v in G to be $N_G(v) = \{w : w \in V, vw \in E\}$, and let $d_G(v) = |N_G(v)|$ and $N_G[v] = N_G(v) \cup \{v\}$. If $S \subseteq V$ or $S \subseteq E$, then $G[S]$ is the subgraph of G induced by S. Let $G - S = G[V(G) \setminus S]$ (if $S \subseteq V(G)$) or $G - S = G[E(G) \setminus S]$ (if $S \subseteq E(G)$). If $S \subseteq V(G)$, then let $N_S(v) = S \cap N_G(v)$. If $E(G[S]) = \emptyset$, then S is a **stable set** (or an **independent set**) of G. Following [7], we define a **clique** of a graph G to be a set of mutually adjacent vertices of G. A clique K of a graph G is maximal if K is not properly contained in another clique of G. The **clique number** of G, denoted by $\omega(G)$, is the maximum size of a clique of G.

Definition 1.1. Let k and r be positive integers. A (k, r)-coloring of a graph G is a mapping $c : V(G) \mapsto [k]$ satisfying both the following:
(C1) $c(u) \neq c(v)$, for every edge $uv \in E(G)$;
(C2) $|c(N_G(v))| \geq \min\{d_G(v), r\}$, for every $v \in V(G)$.

For a fixed integer $r > 0$, the r-hued chromatic number of G, denoted by $\chi_r(G)$, is the smallest k such that G has a (k, r)-coloring. The concept was first introduced in [18,22], where $\chi_2(G)$ is called the dynamic chromatic number of G. By the definition of $\chi_r(G)$, it follows immediately that $\chi(G) = \chi_1(G)$, and so the r-hued coloring is a generalization of the classical graph coloring. For any integers $i > j > 0$, any (k, i)-coloring of G is also a (k, j)-coloring of G, and so if

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$1 \le j < i \le \Delta \le h$, then $\chi(G) \le \chi_j(G) \le \chi_i(G) \le \chi_\Delta(G) = \chi_h(G)$, where $\Delta = \Delta(G)$. The study of r-hued colorings has drawn lots of attention, as seen in [2–4,8,10,11,13–22,24–26], among others.

In [18,22], it has been indicated that $\chi_2(G) - \chi(G)$ can be arbitrarily large. It is of interest to understand, for an integer $r \ge 2$, the relationship between $\chi_r(G)$ and $\chi(G)$ in different families of graphs. Let H be a graph. A graph G is H-**free** if G does not have an induced subgraph isomorphic to H. In particular, a $K_{1,3}$-free graph is also called a claw-free graph. Throughout this paper, for an integer $n \ge 3$, let C_n denote a cycle on n vertices and P_n denote a path on n vertices. There have been investigations on the relationship between $\chi_2(G)$ and $\chi(G)$ in different families of graphs. Among them are the following.

Theorem 1.2 ([17]). *Let G be a claw-free graph. Each of the following holds.*
(i) $\chi_2(G) \le \chi(G) + 2$.
(ii) If G is connected, then $\chi_2(G) = \chi(G) + 2$ *if and only if G is a cycle of length 5 or an even cycle of length not a multiple of 3.*

Theorem 1.3 ([19]). *Let G be a claw-free graph. Then* $\chi_3(G) \le \max\{\chi(G) + 3, 7\}$. *This bound is best possible.*

Theorem 1.4 ([5]). *Let* $k \ge 35$ *be an integer, and let G be a k-regular* C_4*-free graph. Then* $\chi_2(G) \le \chi(G) + 2\lceil 4\ln(k) + 1\rceil$.

Theorem 1.5 ([1]). *If G is a* P_4*-free graph, then* $\chi_2(G) \le \chi(G) + 2$.

These results motivate our current study. In this paper, we study the dependency of $\chi_r(G)$ and $\chi(G)$ among P_4-free graphs and P_5-free graphs, for any integer $r \ge 2$. The main results are the following.

Theorem 1.6. *Let G be a connected* P_4*-free graph. Each of the following holds.*
(i) $\chi_r(G) \le \chi(G) + 2(r-1)$*, and*
(ii) $\chi_r(G) = \chi(G) + 2(r-1)$ *if and only if* $G = K_{s,t}$ *for some integers* $s \ge r$ *and* $t \ge r$.

Theorem 1.7. *Let* $r \ge 2$ *be an integer and G be a connected* P_5*-free bipartite graph. Then* $\chi_r(G) \le r\chi(G)$.

Theorem 1.8. *Let G be a connected* P_5*-free graph. Then* $\chi_2(G) \le 2\chi(G)$.

Theorems 1.7 and 1.8 are best possible in the sense that there exist infinitely many P_5-free bipartite graphs reaching the bounds. In fact, For any integers $m \ge n \ge r \ge 2$, the complete bipartite graph $K_{m,n}$ is P_5-free and by Theorem 2.3 of [17] (see Lemma 2.3 in Section 2), $\chi_r(K_{m,n}) = r\chi(K_{m,n})$. Theorem 1.6 will be proved in Section 2, and Theorems 1.7 and 1.8 will be justified in the last section.

2. r-hued colorings of P_4-free graphs

We start with a few more notations and terms to be used in this section. If G is a simple graph, then $\overline{G}$ denote the complement of G. Let A and B be disjoint nonempty vertex sets. We use $K(A, B)$ to denote a complete bipartite graph with vertex bipartition A and B. Thus $K(A, B)$ is isomorphic to $K_{|A|,|B|}$. Now we assume that A, B are two disjoint vertex subsets of a graph G. Following [7], we define $E[A, B] = \{uv \in E(G) | u \in A \text{ and } v \in B\}$. If $|E[A, B]| = |A||B|$, we say that A **is complete to** B; if $E[A, B] = \emptyset$, we say that A **is anti-complete to** B.

A graph G is **perfect** if for any induced subgraph H of G, $\chi(H) = \omega(H)$. The famous Strong Perfect Graph Theorem characterizes all perfect graphs.

Theorem 2.1 ([9]). *A graph is perfect if and only if it contains no* C_k *nor* $\overline{C_k}$ *as an induced subgraph, for any odd integer* $k \ge 5$.

To proceed our proof, we display some properties of P_4-free graphs. As when $k \ge 5$ is an odd integer, every C_k and every $\overline{C_k}$ contains an induced P_4. It follows from the Strong Perfect Graph Theorem that

$$\text{every } P_4\text{-free graph must be perfect.} \tag{1}$$

While determining whether a graph is 3-colorable or not is an NP-complete problem, it is known that k-coloring problem of P_4-free graphs can be solved in polynomial time since a P_4-free graph has a special structural property, as stated below.

Theorem 2.2 ([23]). *If G is a* P_4*-free graph, then* $V(G)$ *can be divided into two disjoint subsets A and B, such that either A is complete to B or A is anti-complete to B.*

By Theorem 2.2, it follows that if G is a connected P_4-free graph, then $V(G)$ can be divided into two disjoint subsets A and B such that A is complete to B. Hence we have

$$\chi(G) = \chi(G[A]) + \chi(G[B]). \tag{2}$$

By (1), $\chi(G) = \omega(G)$. However, if $G = K(A, B)$, then for any $r \ge 2$, we have $\chi_r(G[A]) = \chi_r(G[B]) = 1$. This special case was formerly studied in [17].

Lemma 2.3 ([17]). *For any integer $r \geq 1$, we have $\chi_r(K_{s,t}) = \min\{2r, s+t, r+s, r+t\}$.*

Thus when $r \geq 2$, the above-mentioned relationship in (2) is not applicable as $\chi_r(G)$ and $\chi_r(G[A]) + \chi_r(G[B])$ may be different.

Corollary 2.4. *Let G be a connected P_4-free graph. Then $\chi_2(G) \leq \chi(G) + 2$, where the equality holds if and only if $G = K_{s,t}$ for some integers $s \geq r$ and $t \geq r$.*

Proof. By Theorem 2.2, $V(G)$ can be divided into two disjoint subsets A and B such that A is complete to B. Let $G_1 = G[A]$, $G_2 = G[B]$, and for $i \in \{1, 2\}$, let $\omega_i = \omega(G_i)$. By symmetry, we assume that $\omega_1 \geq \omega_2$. By (2)p4-perfect, $\chi(G) = \omega_1 + \omega_2$. If $\omega_2 \geq 2$, then any proper k-coloring of G is also a $(k, 2)$-coloring of G, implying that $\chi_2(G) = \chi(G)$ in this case. Suppose that $\omega_1 \geq 2$ and $\omega_2 = 1$. Let c_1 be an $(\omega_1, 1)$-coloring of G_1, and extend c_1 to c by coloring all the vertices of B with $\min\{2, |B|\}$ new colors. If $|B| = 1$, then for any vertex $v \in A$ with $d_G(v) = 1$, $|c(N_G(v))| = 1$; for any vertex $u \in A$ with $d_G(u) \geq 2$, $|c(N_G(u))| = |c_1(N_G(u))| + 1$ when $|c_1(N_G(u))| = 1$ and $|c(N_G(u))| \geq |c_1(N_G(u))|$ when $|c_1(N_G(u))| \geq 2$. Since A is complete to B and $\omega(G) = \omega_1 + 1$, it follows by Definition 1.1 that c is a $(\chi(G), 2)$-coloring of G. If $|B| \geq 2$, as A is complete to B and $\omega(G) = \omega_1 + 1$, then it follows by Definition 1.1 that c is a $(\chi(G) + 1, 2)$-coloring of G. Finally, if $\omega_1 = \omega_2 = 1$, then G is a complete bipartite graph, and the corollary follows from Lemma 2.3 immediately. ■

Proof of Theorem 1.6. We argue by contradiction and assume that

$$\text{there exists a counterexample to Theorem 1.6 with } r \text{ being minimized.} \tag{3}$$

For this value of r, we choose a connected P_4-free graph G such that G is a counterexample to Theorem 1.6.

Let $k = \chi(G)$. By Theorem 2.2, $V(G)$ can be partitioned into two subsets A and B such that A is complete to B. Let $G_1 = G[A]$, $G_2 = G[B]$, $\omega_1 = \omega(G_1)$ and $\omega_2 = \omega(G_2)$. Since G is P_4-free, both $G[A]$ and $G[B]$ are P_4-free graphs. By symmetry, we assume that $|A| \geq |B|$. Let $c : V(G) \to [k]$ be a proper coloring of G. By (1), we have $\chi(G_1) = \omega_1$ and $\chi(G_2) = \omega_2$. By (2),

$$\chi(G) = \chi(G_1) + \chi(G_2) = \omega_1 + \omega_2, |c(A)| = \omega_1 \text{ and } |c(B)| = \omega_2. \tag{4}$$

Claim 1. *Each of the following holds.*
(i) $r \geq 3$.
(ii) $\max\{\omega_1, \omega_2\} \geq 2$.
(iii) $\min\{\omega_1, \omega_2\} < r$.
(iv) $|A| \geq r$.

Proof. By Corollary 2.4, and by the fact that G is a counterexample to Theorem 1.6, we conclude that (i) must hold.

If $\max\{\omega_1, \omega_2\} = 1$, then G is a complete bipartite graph. By Lemma 2.3, we have $\chi_r(K_{s,t}) \leq 2r = \chi(G) + 2(r-1)$, where equality holds if and only if $\min\{s, t\} \geq r$. Hence (ii) follows.

Assume that $\min\{\omega_1, \omega_2\} \geq r$. By (1) and (2), we have $\chi(G) = \omega_1 + \omega_2$. Let c be a $(k, 1)$-coloring of G. Then as A is complete to B in G and as $\min\{\omega_1, \omega_2\} \geq r$, every vertex v is adjacent to a clique of size at least r in G. It follows by Definition 1.1 that c is a (k, r)-coloring of G, contrary to the assumption that G is a counterexample.

If $|A| \leq r-1$, then $|B| \leq |A| \leq r-1$, then we use $|V(G)| = |A| + |B| \leq 2(r-1)$ colors so that distinct vertices will be colored differently. Hence this is an (n, r)-coloring of G with $n \leq 2(r-1)$. This justifies (iv), and completes the proof of the claim. ■

Claim 2. *If $|B| \geq r$, then $\chi_r(G) \leq \chi(G) + \max\{(r-\omega_1), 0\} + \max\{(r-\omega_2), 0\}$.*

Proof. Let $h_1 = r - \omega_1$ and $h_2 = r - \omega_2$. If both $h_1 \leq 0$ and $h_2 \leq 0$, then as A is complete to B, it follows from Definition 1.1 that any proper k-coloring c is also a (k, r)-coloring of G. Hence in this case, we have $\chi_r(G) = \chi(G)$.

Case 1. Both $h_1 > 0$ and $h_2 > 0$.

Let $a_1, a_2, \ldots, a_{\omega_1} \in A$ such that $c(\{a_1, a_2, \ldots, a_{\omega_1}\}) = c(A)$, and choose h_1 vertices

$$a_{\omega_1+1}, a_{\omega_1+2}, \ldots, a_{\omega_1+h_1}$$

from $A - \{a_1, a_2, \ldots, a_{\omega_1}\}$; and let $b_1, b_2, \ldots, b_{\omega_2} \in B$ such that $c(\{b_1, b_2, \ldots, b_{\omega_2}\}) = c(B)$, and choose h_2 vertices $b_{\omega_2+1}, \ldots, b_{\omega_2+h_2}$ from $B - \{b_1, b_2, \ldots, b_{\omega_2}\}$. Define $c' : V(G) \mapsto [k + h_1 + h_2]$ by

$$c'(x) = \begin{cases} c(x) & \text{if } x \in (A - \{a_{\omega_1+1}, \ldots, a_{\omega_1+h_1}\}) \cup (B - \{b_{\omega_2+1}, \ldots, b_{\omega_2+h_2}\}) \\ k+i & \text{if } x = a_{\omega_1+i}, \text{ where } 1 \leq i \leq h_1 \\ k+h_1+j & \text{if } x = b_{\omega_2+j}, \text{ where } 1 \leq j \leq h_2 \end{cases}.$$

Since c is a proper k-coloring, c' is also a proper $(k + h_1 + h_2)$-coloring. If $x \in A$, then $|c'(N(x))| \geq |c'(B)| = \omega_2 + h_2 = r$. If $y \in B$, then $|c'(N(y))| \geq |c'(A)| = \omega_1 + h_1 = r$. Hence in this case, c' is a proper $(k + h_1 + h_2, r)$-coloring of G. Thus $\chi_r(G) \leq k + h_1 + h_2 = \chi(G) + (r - \omega_1) + (r - \omega_2)$. This proves that Claim 2 holds in this case.

Case 2. Either $h_1 \le 0$ or $h_2 \le 0$.

If $h_1 > 0$ and $h_2 \le 0$, similarly as in Case 1, we may choose and recolor h_1 vertices in A by h_1 new colors $\{k+1, k+2, \dots, k+h_1\}$. And we do not need to recolor vertices in B. Define $c'' : V(G) \mapsto [k+h_1]$ by

$$c''(x) = \begin{cases} c(x) & \text{if } x \in (A - \{a_{\omega_1+1}, \dots, a_{\omega_1+h_1}\}) \cup B \\ k+j & \text{if } x = a_{\omega_1+j}, \text{ where } 1 \le j \le h_1 \end{cases}.$$

Since c is a proper k-coloring, c'' is also a proper $(k+h_1)$-coloring. As A is complete to B, for each $y \in B$, $A \subset N(y)$ and so $|c''(N(y))| \ge |c''(A)| = \omega_1 + h_1 = r$. If $x \in A$, then as $\omega_2 \ge r$, $|c''(N(x))| \ge |c(B)| \ge r$. Thus by definition, c'' is a proper $(k+h_1, r)$-coloring of G, and so $\chi_r(G) \le k + h_1 = \chi(G) + (r - \omega_1)$. If $h_1 \le 0$ and $h_2 > 0$, the proof is similar. This proves that Claim 2 holds in this case as well, and completes the proof of Claim 2. ■

Claim 3. *If $|B| < r$, then $\chi_r(G) \le \chi(G) + 2r - 3$.*

Proof. Let $b = |B|$. Then $|A| \ge r$ and $1 \le b \le r-1$. Define $r' = r - b$. Then $r' < r$. By (3), we have $\chi_{r'}(G_1) \le \omega_1 + 2(r'-1)$. Let $k' = \max\{\omega_1 + 2(r'-1), r\}$ and c_2 be a (k', r')-coloring of G_1. Denote $B = \{z_1, z_2, \dots, z_b\}$ and define $c'_2 : V(G) \to [k'+b]$ as follows:

$$c'_2(x) = \begin{cases} c_2(x) & \text{if } x \in A \\ k'+i & \text{if } x = z_i, \text{ where } 1 \le i \le b \end{cases}.$$

Since $k' \ge r$ and A is complete to B, for any vertex $v \in B$, $|c'_2(N_G(v))| \ge k' \ge r$; for any vertex $u \in A$, $|c'_2(N_G(u))| = |c_2(N_G(u))| + b \ge r$. It follows by Definition 1.1 that c'_2 is a $(k'+b, r)$-coloring of G. As

$$\begin{aligned} k' + b &\le \max\{\chi(G_1) + 2(r-b) - 2, r\} + b \\ &\le \max\{\chi(G) + 2r - 2 - b, r + b\} \le \chi(G) + 2r - 3, \end{aligned}$$

this justifies Claim 3. ■

By Claim 1, $\omega_1 + \omega_2 \ge 3$. It follows by Claims 2 and 3 that if G is a P_4-free graph and if $\omega_1 + \omega_2 \ge 3$, then $\chi_r(G) < \chi(G) + 2(r-1)$. This, together with Lemma 2.3, implies Theorem 1.6. ■

3. r-hued colorings of P_5-free graphs

In this section, we investigate the relationship between $\chi_r(G)$ and $\chi(G)$ for a P_5-free graph G. We start with an example.

Example 3.1. Let $k \ge 2$ and $r \ge 1$ be integers. There exists a family $\mathcal{F}$ of connected P_5-free graphs, such that every graph $G \in \mathcal{F}$ satisfies $\chi_r(G) = r\chi(G)$.

For convenience, in this example, we often use $[k]$ for $\mathbb{Z}_k$, the additive group of integers modulo k. For positive integers $n_1, n_2, \dots, n_k$, ($n_i \ge r, i = 1, 2, \dots, k$), let $K = K_{n_1,n_2,\dots,n_k}$ denote a complete k-partite graph such that the k partite vertex sets are $V_1, V_2, \dots, V_k$ with $|V_i| = n_i$, $1 \le i \le k$. Let $U = \{u_1, u_2, \dots, u_k\}$ be a set of vertices with $U \cap V(K) = \emptyset$; and let $n = \sum_{i=1}^{k} n_i + k$. Obtain a graph $G = G(n, k, r)$ from K and U by joining u_i to every vertex in V_i but not to any other vertices, for each i with $1 \le i \le k$. Thus $n = |V(K)| + |U| = |V(G)|$. Let $\mathcal{F}$ be the collection of all graphs $G(n, k, r)$ for some values n, k, r with $n \ge k \ge r \ge 1$. Proposition 3.2 indicates that every graph $G \in \mathcal{F}$ satisfies $\chi_r(G) = r\chi(G)$.

Proposition 3.2. *For any graph $G \in \mathcal{F}$, each of the following holds.*
(i) $\chi(G) = \omega(G) = k$.
(ii) $\chi_r(G) = rk$.
(iii) G is P_5-free.

Proof. Let $G \in \mathcal{F}$. Then for some integers n and k, we have $G = G(n, k, r)$. We shall use the same notations above. For each i with $1 \le i \le k$, fix a vertex $w_i \in V_i$; and let $W = \{w_1, w_2, \dots, w_k\}$. Since K is a complete k-partite graph, $G[W]$ is isomorphic to K_k.

(i) By definition of G, $G[W]$ is a k-clique of G and so $\chi(G) \ge \omega(G) = k$. Let $c : V(G) \mapsto [k]$ be so defined that $c(V_i) = i$ and $c(u_i) = i + 1 \pmod k$. Since K is a k-partite graph, each V_i is a stable set; since $N_G(u_i) = V_i$, it follows that c is a proper k-coloring of G. This proves (i).

(ii) Suppose that $\ell = \chi_r(G)$ and let $c : V(G) \mapsto [\ell]$ be a (k, r)-coloring of G. Since $G[W]$ is isomorphic to K_k, we may assume that for each i with $1 \le i \le k$, $c(w_i) = i$.

Fix an i with $1 \le i \le k$. Since $n_i \ge r$ and $N_G(u_i) = V_i$, there must be a vertex subset $Z_i \subseteq V_i$ such that $|c(Z_i)| = |Z_i| = r$. Randomly pick a vertex $z_i \in Z_i$, and let $Z = \{z_1, z_2, \dots, z_k\}$. As K is a complete k-partite graph, $G[Z]$ is isomorphic to K_k and so $|c(Z)| = k$. It follows that $\ell \ge |c(\cup_{i=1}^{k} Z_i)| = rk$.

To justify (ii), it suffices to present a (rk, r)-coloring of G. Construct a mapping $c : V(G) \mapsto [rk]$ as follows. For $1 \le i \le k$, define $c(V_i) = \{(i-1)r + 1, (i-1)r + 2, \dots, (i-1)r + r\}$ and $c(u_i) = (i-1)r + r + 1$. As K is a complete k-partite graph with $k \geq r$, the restriction of c to $V(K)$ is a (rk, r)-coloring. Since $N_G(u_i) = V_i$, and since $|c(V_i)| = r$, it follows that c is indeed a (rk, r)-coloring. This proves that $\ell = \chi_r(G) \leq rk$, and so completes the proof of (ii).

(iii) Let $P = x_1x_2x_3...x_t$ be a longest induced path in G. Since K is a complete k-partite graph, and since P is induced, we must have $|V(P) \cap V(K)| \leq 3$ and $|V(P) \cap V(K)| = 3$ if and only if $V(P) \cap V(K) = \{x_{i-1}, x_i, x_{i+1}\}$ for some i with $1 < i < 5$ such that x_{i-1} and x_{i+1} are in the same partite set of K. If x_{i-1} and x_{i+1} are both in a V_j, then we must have $t = 3$ and $P = x_{i-1}x_ix_{i+1}$ since $N(u_j) = V_j$. If $|V(P) \cap V(K)| = 2$, then as P is a longest induced path, $V(P) \cap V(K) = \{x_{i-1}, x_i\}$. We may assume, without lot of generality, that $x_{i-1} \in V_1$ and $x_i \in V_2$. It follows that $P = u_1x_{i-1}x_iu_2$. Hence in any case, $|V(P)| = 4$ and so G must be P_5-free. ■

Proposition 3.2 leads to the following problem.

Problem 3.3. For integers $k > 0$, $r \geq 2$ and $t \geq 4$, determine a best possible function $f(k, r, t)$ such that for every connected P_t-free graph G with $\chi(G) = k$, we have $\chi_r(G) \leq f(k, r, t)$. More specifically, is there a best possible value $c = c(r, t)$ such that for every connected P_t-free graph G, we have $\chi_r(G) \leq c(r, t)\chi(G)$? In particular, can $c(r, 5) = r$?

Theorem 1.6 indicates that $f(k, r, 4) = k + 2(r - 1)$, answering the problem when $t = 4$ for any r and k. In this section we will prove Theorems 1.7 and 1.8. Theorem 1.8 suggests $c(2, 5) = 2$, providing evidences for $c(r, 5) = r$.

A subgraph H of G is **dominating** if every vertex of G is either in $V(H)$ or is adjacent to a vertex in H. A subset $V' \subseteq V(G)$ is dominating if $G[V']$ is dominating. Bacso and Tuza [6] proved the following result about P_5-free graphs.

Theorem 3.4 ([6]). *If G is a connected P_5-free graph, then G has a dominating clique or a dominating P_3.*

Using Theorem 3.4, Hoang et al. in [12] indicated that for P_5-free graphs, the k-coloring problem can be solved in polynomial time. We will also apply this structural property of P_5-free graphs to investigate the relationship between $\chi_r(G)$ and $\chi(G)$ for a P_5-free graph G.

Lemma 3.5. *Let r and s be integers with $r \geq 2$ and $s \geq 3$, G be a connected graph with a dominating clique K with $|V(K)| = s$. Let $k = \chi_{r-1}(G - V(K)) + s$. Then G has a $(k, 1)$-coloring $c : V(G) \mapsto [k]$ such that for any vertex $v \in V(G) - V(K)$,*

$$|c(N_G(v))| \geq \min\{d_G(v), r\}. \tag{5}$$

Proof. Let $k_1 = \chi_{r-1}(G - V(K))$ and $k = k_1 + s$. We first let $c_1 : V(G - V(K)) \mapsto [k_1]$ be a $(k_1, r - 1)$-coloring of $G - V(K)$. Extend c_1 to $c : V(G) \mapsto [k]$ by coloring $V(K)$ with $s = |V(K)|$ new colors in $\{k_1 + 1, k_1 + 2, \ldots, k_1 + s\}$.

For each vertex $v \in V(G) - V(K)$, since c_1 is a $(k_1, r - 1)$-coloring of $G - V(K)$,

$$|c_1(N_{G-V(K)}(v))| \geq \min\{r - 1, d_{G-V(K)}(v)\}.$$

As $|c(N_G(v) \cap V(K))| = |N_G(v) \cap V(K)|$ and K is a dominating clique of G, it follows that (5) must hold. This proves the lemma. ■

Corollary 3.6. *Let r and s be integers with $r \geq 2$ and $s \geq 3$, G be a connected graph with a dominating clique K with $|V(K)| = s$. If $r \leq s$ and if $\chi_{r-1}(G - V(K)) \leq (r - 1)\chi(G)$, then $\chi_r(G) \leq r\chi(G)$.*

Proof. By Lemma 3.5, G has a $(k, 1)$-coloring $c : V(G) \mapsto [k]$ such that for any vertex $v \in V(G) - V(K)$, (5) holds. Since K is a complete graph on $s \geq r$ vertices, we have $\chi(G) \geq |V(K)| = s$, and every vertex $v \in V(K)$ also satisfies (5). Hence c is a (k, r)-coloring of G, and so $\chi_r(G) \leq \chi_{r-1}(G - V(K)) + s \leq (r - 1)\chi(G) + \chi(G) = r\chi(G)$. ■

3.1. r-hued colorings of P_5-free bipartite graphs

For a subset $S \subseteq V(G)$, define $N_G(S) = \cup_{v \in S} N_G(v)$. Recall that $K(A, B)$ denote the complete bipartite graph with vertex bipartition (A, B). We start with a few definitions and lemmas.

Definition 3.7. Let $P_3 = w_1w_2w_3$ be a dominating path of a connected graph G. For $i = 1, 2, 3$, define $V_i = \{v \in V(G) : vw_i \in E(G)\}$.

With the notation in Definition 3.7, we have the following observation, which follows from Definition 3.7 and from the fact that a bipartite graph contains no cycles of odd length.

Observation 3.8. *Suppose G is bipartite and P_5-free with $w_1w_2w_3$ being a dominating path. Each of the following holds.*

(i) Either $V_1 \subseteq V_3$ or $V_3 \subseteq V_1$.

(ii) $E(G[V_1 \cup V_3]) = \emptyset$, and $E(G[V_2]) = \emptyset$.

(iii) For any $v \in V_1 \cup V_3$, $N_G(v) \subseteq V_2$.

(iv) For any $v \in V_2$, $N_G(v) \subseteq V_1 \cup V_3$.

Lemma 3.9. *Let G be a connected P_5-free graph with a dominating path $P_3 = w_1w_2w_3$. If G is bipartite, then either $V_2 = \{w_1, w_3\}$, or $|V_2| \geq 3$ and for any $v \in V_2 - \{w_1, w_3\}$, one of the following holds.*
(i) $N_G(v) = \{w_2\}$.
(ii) For any $u \in V_1 - V_3$, if $uv \in E(G)$, then $V_3 \subseteq N_G(v)$.
(iii) For any $u \in V_3 - V_1$, if $uv \in E(G)$, then $V_1 \subseteq N_G(v)$.

Proof. As $w_1, w_3 \in V_2$, we have $|V_2| \geq 2$. Assume that $|V_2| \geq 3$ and (i) does not hold, we are to show that one of (ii) and (iii) must hold. By symmetry, it suffices to justify (ii).

Suppose that there exists a vertex $u \in V_1 - V_3$ with $uv \in E(G)$. For any $u' \in V_3 - \{w_2\}$, $P = uvw_2w_3u'$ is a path on 5 vertices in G. Since G is bipartite, then $uw_2, w_2u', vw_3, uu' \notin E(G)$. Also $uw_3 \notin E(G)$ and since G is P_5-free, we must have $u'v \in E(G)$. This implies that $V_3 \subseteq N_G(v)$. ■

Lemma 3.10. *Let G be a connected P_5-free bipartite graph on $n = |V(G)|$ vertices with a dominating path $P_3 = w_1w_2w_3$ such that $|V_2| \geq 3$. Adopting the notation in Definition 3.7 and defining $V_{21} = \{v \in V_2 : N_G(v) \cap (V_1 - V_3) \neq \emptyset\}$, each of the following holds.*
(i) If $V_1 = V_3$, then for any $u, u' \in V_3$, if $d_G(u') \leq d_G(u)$, then $N_G(u') \subseteq N_G(u)$; and for any $v, v' \in V_2$, if $d_G(v') \leq d_G(v)$, then $N_G(v') \subseteq N_G(v)$.
(ii) If $V_3 \subset V_1$ and $V_1 - V_3 \neq \emptyset$, then each of the following holds.
(ii-1) $G[V_{21} \cup V_3] = K(V_{21}, V_3)$ is a complete bipartite graph.
(ii-2) For any $u, u' \in V_1$, if $d_G(u') \leq d_G(u)$, then $N_G(u') \subseteq N_G(u)$; for any $v, v' \in V_2$, if $d_G(v') \leq d_G(v)$, then $N_G(v') \subseteq N_G(v)$.

Proof. (i). By Observation 3.8 (iii), for any vertex $u \in V_3 - \{w_2\}$, $d(w_2) \geq d(u)$ and $N_G(u) \subseteq N_G(w_2)$. And by Observation 3.8 (iv), if $V_1 = V_3$, then $d(w_1) = d(w_3) \geq d(v)$ and $N_G(v) \subseteq N_G(w_1) = N_G(w_3) = V_3$ for any vertex $v \in V_2 \setminus \{w_1, w_3\}$.

Suppose that $u, u' \in V_3 \setminus \{w_2\}$ with $d_G(u) \geq d_G(u')$. By contradiction, we assume that $N_G(u') - N_G(u) \neq \emptyset$. Since $d_G(u) \geq d_G(u')$ and $N_G(u') - N_G(u) \neq \emptyset$, we also have $N_G(u) - N_G(u') \neq \emptyset$. Pick a vertex $v' \in N_G(u') - N_G(u)$ and a vertex $v \in N_G(u) - N_G(u')$, where $v, v' \in V_2 \setminus \{w_1, w_3\}$. Then $P = uvw_2v'u'$ is a path on 5 vertices in G. Since G is bipartite, $uw_2, uu', w_2u' \notin E(G)$. Since G is P_5-free, one of $uv', u'v$ must be in $E(G)$, contrary to the assumptions that $v' \in N_G(u') - N_G(u)$ and $v \in N_G(u) - N_G(u')$. Hence we must have $N_G(u') \subseteq N_G(u)$.

Similarly, assume that there exist vertices $v, v' \in V_2 \setminus \{w_1, w_3\}$ with $d_G(v) \geq d_G(v')$ and $N_G(v') - N_G(v) \neq \emptyset$, then we also have $N_G(v) - N_G(v') \neq \emptyset$. Pick a vertex $u' \in N_G(v') - N_G(v)$ and a vertex $u \in N_G(v) - N_G(v')$, where $u, u' \in V_3$. Thus $Q = uvw_2v'u'$ is a path on 5 vertices in G. Since G is bipartite, $uw_2, vv', w_2u' \notin E(G)$. Since G is P_5-free, one of $uv', u'v$ must be in $E(G)$, contrary to the assumptions that $u \in N_G(v) - N_G(v')$ and $u' \in N_G(v') - N_G(v)$. This completes the proof of (i).

(ii). Suppose that $V_3 \subset V_1$ and $V_1 - V_3 \neq \emptyset$. By Lemma 3.9(ii), for any $v \in V_{21}$, $V_3 \subseteq N_G(v)$. Hence $G[V_{21} \cup V_3] = K(V_{21}, V_3)$, and so (ii-1) follows.

By Observation 3.8 (iv), if $V_3 \subset V_1$, then $d(w_1) \geq d(v)$ and $N_G(v) \subseteq N_G(w_1) = V_1$ for any $v \in V_2 \setminus \{w_1\}$. If $v \in V_{21}$, then by (ii-1), we have $N_G(w_3) = V_3 \subset N_G(v)$. If $v \in V_2 \setminus V_{21}$, then $N_G(v) \subset V_3 = N_G(w_3)$. Suppose $v, v' \in V_2 \setminus \{w_1, w_3\}$ with $d_G(v) \geq d_G(v')$, the proof for $N_G(v') \subseteq N_G(v)$ is similar to that for (i), so it will be omitted. As for any two vertices $u, u' \in V_1$ with $d_G(u) \geq d_G(u')$, the proof for $N_G(u') \subseteq N_G(u)$ is also similar to that for (i). Thus (ii-2) is justified. ■

Lemma 3.11. *Let G be a bipartite P_5-free graph with the vertex bipartition (U, V). If G has a dominating path P_3, then G has a $(2r, r)$-coloring $c : V(G) \mapsto [2r]$ in such a way that $c(U) \subseteq [r]$ and $c(V) \subseteq [2r] - [r]$. In particular, $\chi_r(G) \leq 2r$.*

Proof. It suffices to prove Lemma 3.11 for connected graphs. Hence we assume that G is a connected bipartite P_5-free graph with a dominating P_3. Let $V(P_3) = \{w_1, w_2, w_3\}$ and define $V_i = \{v \in V(G) | vw_i \in E(G)\}$, for $i = 1, 2, 3$ as in Definition 3.7. Set $U = V_1 \cup V_3$ and $V = V_2$. By Observation 3.8 (ii) – (iv), G is a bipartite graph with (U, V) being its vertex bipartition. By Lemma 3.9, either $V_2 = \{w_1, w_3\}$, or $|V_2| \geq 3$ and for any $v \in V_2 - \{w_1, w_3\}$, one of Lemma 3.9(i), (ii) and (iii) must hold.

Assume first that $V_2 = \{w_1, w_3\}$. Then $V(G) = \{w_1, w_3\} \cup V_1 \cup V_3$. Without loss of generality, we may assume $V_3 \subseteq V_1$. Then G is a bipartite graph with partite sets $\{w_1, w_3\}$ and V_1. Let $c : V(G) \mapsto [r+2]$ be a $(r+2, 1)$-coloring of G so that $c(V_1) \subseteq [r]$ with $|c(V_i)| = \min\{|V_i|, r\}$ for $i \in \{1, 3\}$ and $c(V_2) = c(\{w_1, w_3\}) = \{r+1, r+2\}$. Thus Lemma 3.11 holds.

Next we assume that $|V_2| \geq 3$. In the rest of the proof, we shall adopt the notation in Definition 3.7 and in Lemma 3.10. By Observation 3.8(i) and by symmetry, we may assume either $V_3 = V_1$ or $V_3 \subset V_1$.

Denote $V_1 = \{u_1, u_2, \ldots, u_h\}$ and $V_2 = \{v_1, v_2, \ldots, v_\ell\}$ such that

$$d_G(u_1) \geq d_G(u_2) \geq \cdots \geq d_G(u_h), \text{ and } d_G(v_1) \geq d_G(v_2) \geq \cdots \geq d_G(v_\ell).$$

Then by Lemma 3.10(i) and (ii), we have

$$V_1 \supseteq N_G(v_1) \supseteq N_G(v_2) \supseteq \ldots \supseteq N_G(v_\ell), \text{ and } V_2 \supseteq N_G(u_1) \supseteq N_G(u_2) \supseteq \ldots \supseteq N_G(u_h). \tag{6}$$

By (6), it is possible to relabel $V_1 = \{x_1, x_2, \ldots, x_h\}$ so that for each i with $1 \leq i \leq \ell$, there exists a subscript $n_i \leq h$ such that $N_G(v_i) = \{x_1, x_2, \ldots, x_{n_i}\}$. Similarly, we can relabel $V_2 = \{y_1, y_2, \ldots, y_\ell\}$ so that for each j with $1 \leq j \leq h$, there exists a subscript $k_j \leq \ell$ such that $N_G(u_j) = \{y_1, y_2, \ldots, y_{k_j}\}$. Define $c : V(G) \mapsto [2r]$ to be a mapping satisfying the following.

(A) For $i = 1, 2, \ldots, h$, choose $j = j(i)$ with $1 \le j \le r$ and $i \equiv j \pmod r$, and define $c(x_i) = j$. Thus $c(V_1) \subseteq [r]$ and $|c(V_1)| \ge \min\{|V_1|, r\}$.

(B) For $i = 1, 2, \ldots, \ell$, choose $j = j(i)$ with $r+1 \le j \le 2r$ and $i \equiv j \pmod r$, and define $c(y_i) = j$. Thus $c(V_2) \subseteq [2r]-[r]$ and $|c(V_2)| \ge \min\{|V_2|, r\}$.

To see that c is a $(2r, 1)$-coloring of G, we take any edge $xy \in E(G)$. Since G is a bipartite graph with vertex bipartition (V_1, V_2), we may assume that $x \in V_1$ and $y \in V_2$. Then by (A) and (B), we have $c(x) \ne c(y)$ and so c is a $(2r, 1)$-coloring of G. To see that c is indeed a $(2r, r)$-coloring of G, we pick an arbitrary vertex $z \in V(G)$. If $z \in V_1$, then $z = x_i$ for some i with $1 \le i \le h$. By (A), either $n_i \le r$ and $|c(N_G(x_i))| = |N_G(x_i)|$, or $n_i \ge r$ and $|c(N_G(x_i))| \ge r$. Similarly, if $z \in V_2$, then using (B), we also conclude that $|c(N_G(z))| \ge \min\{|N_G(z)|, r\}$. Thus c is a $(2r, r)$-coloring of G satisfying $c(U) \subseteq [r]$ and $c(V) \subseteq [2r] - [r]$. This completes the proof of Lemma 3.11. ■

Lemma 3.12. *Let G be a connected P_5-free bipartite graph. If G has a dominating clique K_2, then $\chi_r(G) \le 2r$.*

Proof. Throughout the proof of this lemma, let (U, V) denote the vertex bipartition of G. We shall prove the lemma arguing by induction on r. Since G is bipartite, Lemma 3.12 holds for $r = 1$. We assume that $r > 1$ and that Lemma 3.12 holds for smaller values of r.

Since G has a dominating K_2, there exists a pair of adjacent vertices u_0, v_0 such that $V(G) = N_G(u_0) \cup N_G(v_0)$ and such that $N_G(u_0) = V$ and $N_G(v_0) = U$. Define $G' = G - \{u_0, v_0\}$. Then G' is also a P_5-free bipartite graph. Let $H_1, H_2, \ldots, H_t$ be the connected components of G'. Then each H_i is a connected P_5-free bipartite graph. By Theorem 3.4 and since H_i is bipartite, if $|E(H_i)| > 0$, then H_i has a dominating P_2 or a dominating P_3. Thus by induction and by Lemma 3.11, G' has a $(2r-2, r-1)$-coloring $c' : V(G') \mapsto [2r-2]$ satisfying the following properties:

(A) For each i with $1 \le i \le t$, if $|E(H_i)| > 0$, then $c'(U \cap V(H_i)) \subseteq [r-1]$ and $c'(V \cap V(H_i)) \subseteq [2r-2] - [r-1]$.

(B) Both $|c'(U)| \ge \min\{|U - \{u_0\}|, r-1\}$ and $|c'(V)| \ge \min\{|V - \{v_0\}|, r-1\}$.

We extend c' to $c : V(G) \mapsto [2r]$ by coloring u_0, v_0 with two new colors $\{2r-1, 2r\}$. Since c' is a $(2r-2, r-1)$-coloring of G' satisfying (A) and (B), and since $N_G(u_0) = V$ and $N_G(v_0) = U$, it follows by the definition of c that c is a $(2r, r)$-coloring of G. ■

Theorem 3.13. *If G is a bipartite P_5-free graph. Then for any $r \ge 2$,*

$$\chi_r(G) \le 2r.$$

Proof. By Theorem 3.4 and since G is bipartite, G has a dominating path P_t with $t = 2$ or 3. Therefore, Theorem 3.13 follows from Lemmas 3.11 and 3.12. ■

3.2. 2-hued colorings of P_5-free graphs

In this section, we shall prove Theorem 1.8. It suffices to prove Theorem 1.8 for connected P_5-free graphs. By Theorem 3.4, G has a dominating clique K_s for some $s \ge 1$ or a dominating path P_3. Let J be a dominating maximal clique K_s or a dominating P_3 of G. By Theorem 3.13, we may assume that G is not bipartite. If $J = K_1$, then $E(G) = \emptyset$ and $G = K_1$, and so nothing needs to be proved. If $J = K_s$ for some $s \ge 3$, then by Corollary 3.6, we have $\chi_2(G) \le 2\chi(G)$. Hence we assume that $J \in \{K_2, P_3\}$.

Let $k = \chi(G)$ and let $c_1 : V(G) \mapsto [k]$ be a $(k, 1)$-coloring of G. We also use $c_1 : V(G) - V(J) \mapsto [k]$ be the restriction of c_1. Let $|V(J)| = \ell$ and $V(J) = \{w_1, w_2, \ldots, w_\ell\}$. Define $c : V(G) \mapsto [k + \ell]$ as follows.

$$c(v) = \begin{cases} c_1(v) & \text{if } v \in V(G) - V(J) \\ k + j & \text{if } v = w_j \in V(J), \quad 1 \le j \le \ell. \end{cases}$$

Since c_1 is a $(k, 1)$-coloring of G, we conclude that c is also a $(k, 1)$-coloring of G. By the definition of a dominating subgraph, if $v \in V(G) - V(J)$, then either v is of degree one in G, or v is adjacent to at least one vertex in $V(G) - V(J)$, or v is adjacent to at least two vertices of $V(J)$. In any case, $|c(N_G(v))| \ge \min\{d_G(v), 2\}$. Similarly, for any $v \in V(J)$, we also have $|c(N_G(v))| \ge \min\{d_G(v), 2\}$. It follows by Definition 1.1 that c is a $(k + \ell, 2)$-coloring of G, and so $\chi_2(G) \le \chi(G) + \ell$. Since G is not bipartite, we have $\chi(G) \ge 3 \ge \ell$, and so $\chi_2(G) \le \chi(G) + \ell \le 2\chi(G)$. ■

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